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# A transmuted version of the generalized half-normal distribution

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## Abstract:

*An extension of the generalized half-normal distribution, given by Cooray and Ananda [5], is proposed and studied. We use the quadratic rank transmutation map to generate a transmuted version of the generalized half-normal distribution. We study some probability properties, discuss maximum likelihood estimation and present real data application indicating that the new distribution can improve the generalized half-normal distribution in fitting real data.*

**Keywords:** Generalized half-normal distribution; Half-normal distribution; Maximum likelihood; Quadratic rank transmutation map; Transmuted distribution.

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## 1. Introduction

Cooray and Ananda [5] study the generalized half-normal (GHN) distribution. A random variable $X$ is said to have the generalized half-normal distribution, denoted as $X \sim \mathrm{GHN}(\sigma, \alpha)$, if its probability density function (pdf) is given by

$$f(x; \sigma, \alpha) = \frac{2\alpha}{\sigma^{\alpha}} x^{\alpha-1} \phi\left[\left(\frac{x}{\sigma}\right)^{\alpha}\right], \quad x > 0, \tag{1.1}$$

and its cumulative distribution function (cdf) is

$$F(x; \sigma, \alpha) = 2\Phi\left[\left(\frac{x}{\sigma}\right)^{\alpha}\right] - 1, \tag{1.2}$$

where $\sigma > 0$ is a scale parameter, $\alpha > 0$ is a shape parameter and $\phi(\cdot)$ and $\Phi(\cdot)$ are the pdf and cdf of the standard normal distribution. The classical half-normal distribution (Hogg and Tanis, [7]) is obtained as a special case when $\alpha = 1$.

In this article we use the transmutation map approach suggested by Shaw and Buckley [13] to define a new distribution that extends the generalized half-normal distribution. We will call the generalized distribution as the transmuted generalized half-normal distribution.

**Definition 1.1.** *A random variable $X$ is said to have transmuted distribution if its pdf is given by*

$$g(x) = f(x)\left[1 + \lambda - 2\lambda F(x)\right] \tag{1.3}$$

*and its cdf is given by*

$$G(x) = (1 + \lambda)F(x) - \lambda F^2(x), \tag{1.4}$$

*where $|\lambda| \leq 1$ is a shape parameter and $f(x)$ and $F(x)$ are the pdf and cdf of the base distribution, respectively.*

Observe that at $\lambda = 0$ we have the distribution of the base random variable. Aryal and Tsokos [2] studied the transmuted Gumbel distribution; Aryal and Tsokos [9] studied the transmuted Weibull distribution; Merovci [10] studied the transmuted Rayleigh distribution; Merovci [10] studied the transmuted generalized Rayleigh distribution.

The rest of the paper is organized as follows: In Section 2 we propose the new distribution and investigate its properties. Section 3 discusses the maximum likelihood estimation for the parameters. In addition, simulation studies are performed. Section 4 gives a real data application and reports the results. Section 5 concludes our work.

## 2. Transmuted generalized half-normal distribution

In this section, we present the pdf and cdf of the new distribution. In addition, we derive an analytical expression for distributional moments and use this result to calculate the skewness and kurtosis coefficients.

### 2.1. Pdf and cdf

In this subsection we present the pdf and cdf of the transmuted generalized half-normal distribution. We replace (1) and (2) into (3) to obtain the pdf of the new distribution. The respective cdf is obtained replacing (2) into (4).

**Definition 2.1.** *A random variable $X$ follows a transmuted generalized half-normal (*TGHN*) distribution, denoted as $X \sim \mathrm{TGHN}(\sigma, \alpha, \lambda)$, if its probability density function (pdf) is given by*

$$f(x;\sigma,\alpha,\lambda) = \frac{2\alpha}{\sigma^\alpha} x^{\alpha-1} \phi\left[\left(\frac{x}{\sigma}\right)^\alpha\right]\left\{1+3\lambda-4\lambda\Phi\left[\left(\frac{x}{\sigma}\right)^\alpha\right]\right\}, \quad x>0, \tag{2.1}$$

*and its cumulative distribution function (cdf) is given by*

$$F(x;\sigma,\alpha,\lambda) = \left\{2\Phi\left[\left(\frac{x}{\sigma}\right)^\alpha\right]-1\right\}\left\{1+2\lambda-2\lambda\Phi\left[\left(\frac{x}{\sigma}\right)^\alpha\right]\right\}, \tag{2.2}$$

*where $\sigma > 0$ is a scale parameter and $\alpha > 0$ and $|\lambda| < 1$ are shape parameters and $\phi(\cdot)$ and $\Phi(\cdot)$ are the pdf and cdf of the standard normal distribution.*

Note that the classical half-normal distribution and the generalized half-normal distribution are special cases of the transmuted generalized half-normal distribution. More specifically, if $X \sim \mathrm{TGH}(\sigma, \alpha, \lambda)$, then

1. For $\lambda = 0$ we obtain

$$f(x;\sigma,\alpha) = \frac{2\alpha}{\sigma^\alpha} x^{\alpha-1} \phi\left(\left(\frac{x}{\sigma}\right)^\alpha\right), \tag{2.3}$$

   which is the generalized half-normal law (Cooray and Ananda [5]).

2. For $\lambda = 0$ and $\alpha = 1$ we obtain

$$f(x;\sigma) = \frac{2}{\sigma}\phi\left(\frac{x}{\sigma}\right), \tag{2.4}$$

which is the half-normal law (Hogg and Tanis [7]).

Figure 1 illustrates some of the possible shape of the pdf and cdf of a transmuted generalized half-normal distribution for selected values of the parameters $\sigma$, $\alpha$ and $\lambda$.

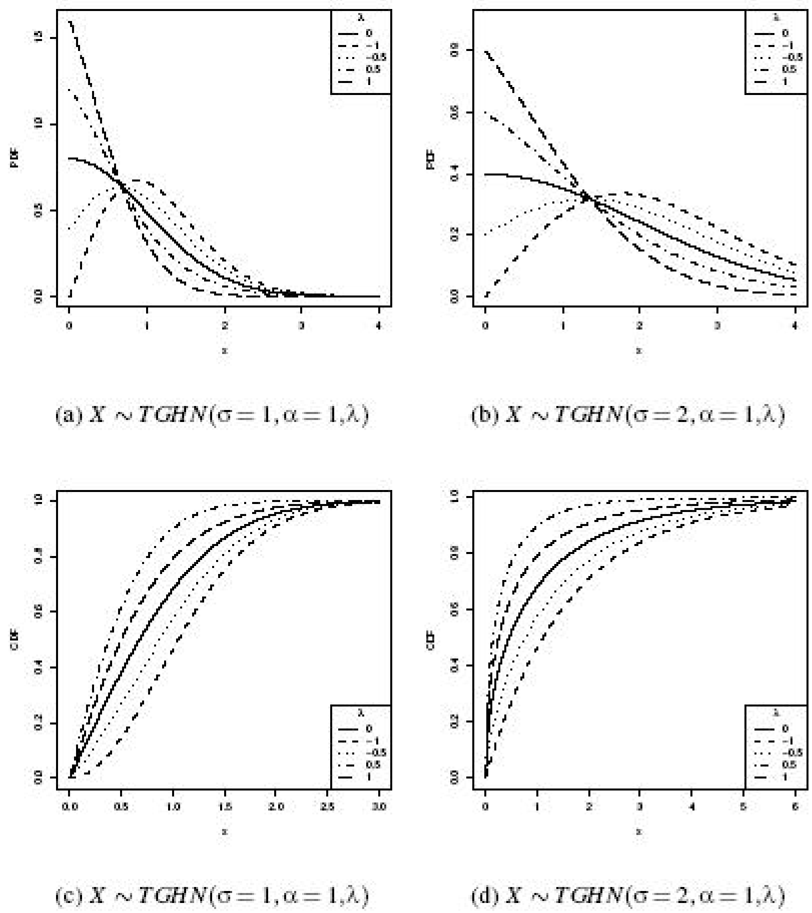



**Figure 1.** Pdf and cdf for a transmuted generalized half-normal distribution.

### 2.2. Transformations

Next, we present some transformations related to TGHN distributions.

**Proposition 2.1.** *Let* $X \sim \mathrm{TGHN}(\sigma, \alpha, \lambda)$. *Then,*

*a)* $W = aX \sim \mathrm{TGHN}(a\sigma, \alpha, \lambda)$ *for all* $a > 0$,

*b) The pdf of* $W = X^{-1}$ *is*

$$f(w; \sigma, \alpha, \lambda) = \frac{2\alpha}{\sigma^\alpha w^{\alpha+1}} \phi\left(\frac{1}{\sigma^\alpha w^\alpha}\right)\left[1 + 3\lambda - 4\lambda\Phi\left(\frac{1}{\sigma^\alpha w^\alpha}\right)\right],$$

*c) The pdf of* $W = \log(X)$ *is given by*

$$f(w; \sigma, \alpha, \lambda) = \frac{2\alpha}{\sigma^\alpha} e^{w(\alpha-1)} \phi\left(\left(\frac{e^w}{\sigma}\right)^\alpha\right)\left[1 + 3\lambda - 4\lambda\Phi\left(\left(\frac{e^w}{\sigma}\right)^\alpha\right)\right].$$

**Proof.** Parts $a) - c)$ are directly obtained from the change-of-variable method. More details in Appendix.

**Remark 1.** *Part a) of Proposition 2.1 indicates that the* TGHN *distributions belong to the scale family, Part b) shows that these distributions are not closed under reciprocation, while the result in Part c) can be used to study regression models in same lines as in the context of regression models for positive random variables; see McDonald and Butler [8]. In addition, Part a) allows us to obtain a two parameter* TGHN *distribution. That is, if* $X \sim \mathrm{TGHN}(\sigma, \alpha, \lambda)$, *then* $X/\sigma \sim \mathrm{TGHN}(1, \alpha, \lambda)$.

### 2.3. Related distribution

The following corollary is a direct consequence of f-tghn and is obtained as an extension of a particular case of the generalized half-normal distribution.

**Corollary 2.1.** *Let* $X \sim TGHN(\sigma, 1, \lambda)$. *Then* $X$ *follows a* $THN(\sigma, \lambda)$ *transmuted half-normal distribution, whose probability density function is given by*

$$f(x; \sigma, \lambda) = \frac{2}{\sigma}\phi\left(\frac{x}{\sigma}\right)\left(1 + 3\lambda - 4\lambda\Phi\left(\frac{x}{\sigma}\right)\right), \quad x > 0, \tag{2.5}$$

*where* $\sigma > 0$ *is a scale parameter,* $|\lambda| \leq 1$ *is a shape parameter and* $\phi$ *and* $\Phi$ *are the pdf and cdf of the standard normal distribution, respectively.*

### 2.4. Reliability analysis

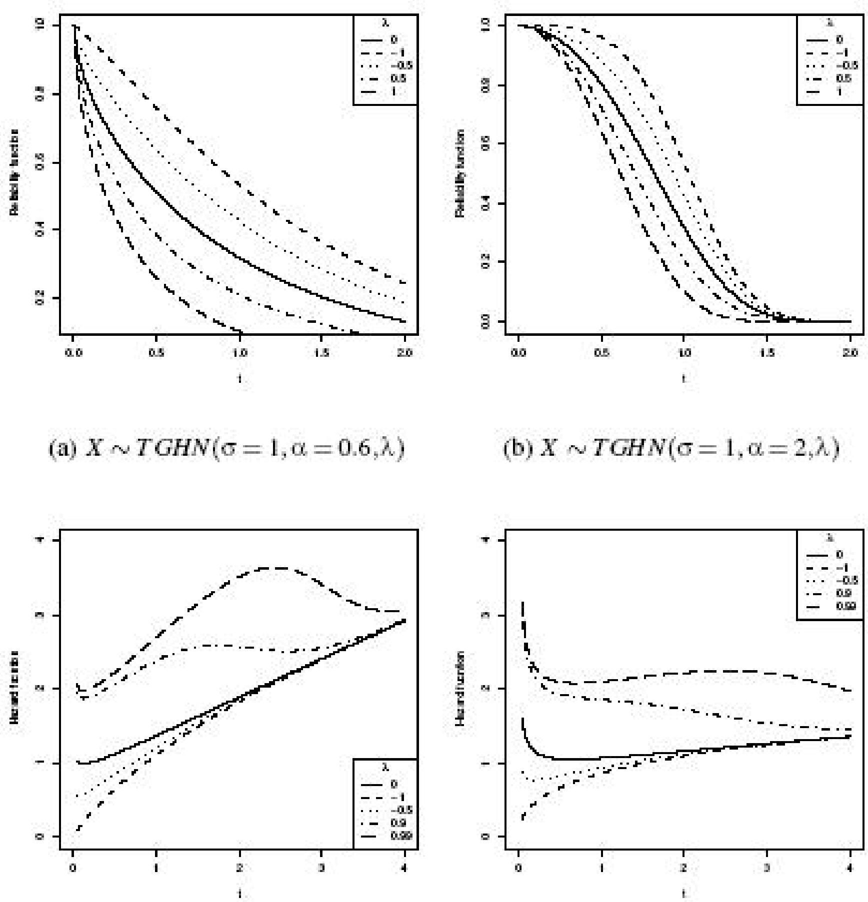
The reliability function $R_T(t)$, which is the probability of an item not failing prior to some time $t$, is defined by $R_T(t) = 1 - F_T(t)$. The reliability function of a transmuted generalized half-normal distribution is given by

$$R_T(t) = 1 - \left\{2\Phi\left[\left(\frac{t}{\sigma}\right)^{\alpha}\right] - 1\right\}\left\{1 + 2\lambda - 2\lambda\Phi\left[\left(\frac{t}{\sigma}\right)^{\alpha}\right]\right\}.$$

An interesting characteristic of a random variable is its hazard rate function defined by $h_T(t) = \frac{f_T(t)}{1-F_T(t)}$ which is an important quantity in the life-time analysis of a certain phenomenon. It can be loosely interpreted as the conditional probability of failure at time $t$, given it has survived to time $t$. The hazard rate function for a transmuted generalized half-normal random variable is given by

$$h_T(t) = \frac{\frac{2\alpha}{\sigma^{\alpha}} t^{\alpha-1} \phi\left[\left(\frac{t}{\sigma}\right)^{\alpha}\right]\left\{1 + 3\lambda - 4\lambda\Phi\left[\left(\frac{t}{\sigma}\right)^{\alpha}\right]\right\}}{1 - \left\{2\Phi\left[\left(\frac{t}{\sigma}\right)^{\alpha}\right] - 1\right\}\left\{1 + 2\lambda - 2\lambda\Phi\left[\left(\frac{t}{\sigma}\right)^{\alpha}\right]\right\}}.$$

Figure 2 displays some plots of the reliability function and the hazard rate function of a TGHN distribution for different values of its parameters.




(a) $X \sim TGHN(\sigma = 1, \alpha = 0.6, \lambda)$ (b) $X \sim TGHN(\sigma = 1, \alpha = 2, \lambda)$

(c) $X \sim TGHN(\sigma = 1, \alpha = 0.9, \lambda)$ (d) $X \sim TGHN(\sigma = 1, \alpha = 0.7, \lambda)$

**Figure 2.** Reliability and hazard rate function for a transmuted generalized half-normal distribution.

### 2.5. Moment and related measures

In this subsection, analytical expression for the $r$-th moment is derived. In addition, we use this result to calculate the mean, variance and the skewness and kurtosis coefficients.

**Proposition 2.2.** *Let $X \sim TGHN(\sigma, \alpha, \lambda)$. Then, for $r = 1, 2, ...$ it follows that $r$-th moment is given by*

$$\mu_r = E(X^r) = 2\sigma^r a_r, \tag{2.6}$$

*where $a_r$ is defined as*

$$a_r = \int_0^\infty u^{r/\alpha}\phi(u)[1 + 3\lambda - 4\lambda\Phi(u)]\, du. \tag{2.7}$$

**Proof.** Using the defining moments, the $r$-th moment is given by

$$E(X^r) = \int_0^\infty x^r \frac{2\alpha}{\sigma^\alpha} x^{\alpha-1}\phi\left[\left(\frac{x}{\sigma}\right)^\alpha\right]\left\{1 + 3\lambda - 4\lambda\Phi\left[\left(\frac{x}{\sigma}\right)^\alpha\right]\right\} dx,$$

and by letting $u = \left(\frac{x}{\sigma}\right)^\alpha$, the result is obtained.

**Corollary 2.2.** *Let $X \sim TGHN(\sigma, \alpha, \lambda)$. Then, the mean and variance are respectively*

$$E(X) = 2\sigma a_1 \quad \text{and} \quad Var(X) = 2\sigma^2\left(a_2 - 2\sigma a_1^2\right).$$

**Corollary 2.3.** *Let $X \sim TGHN(\sigma, \alpha, \lambda)$. Then, the skewness ($\sqrt{\beta_1}$) and kurtosis ($\beta_2$) coefficients are respectively*

$$\sqrt{\beta_1} = \frac{a_3 - 6a_1a_2 + 8a_1^3}{\sqrt{2}\left(a_2 - 2a_1^2\right)^{3/2}} \quad \text{and} \quad \beta_2 = \frac{a_4 - 8a_1a_3 + 24a_1^2a_2 - 24a_1^4}{2\left(a_2 - 2a_1^2\right)^2}.$$

**Remark 2.** *If $\lambda = 0$ and $\alpha = 1$ the skewness and kurtosis coefficients of the* TGHN *distribution take the approximate values* 0.995 *and* 3.869, *respectively, which correspond to those for the half-normal distribution. If $\lambda = 0$ the asymmetry and kurtosis coefficients of the* TGHN *distribution take the values.*

$$\sqrt{\beta_1}_{GHN} = \frac{\Gamma\left(\frac{3}{2\alpha}+\frac{1}{2}\right) - \frac{3}{\sqrt{\pi}}\Gamma\left(\frac{1}{2\alpha}+\frac{1}{2}\right)\Gamma\left(\frac{1}{\alpha}+\frac{1}{2}\right) + \frac{2}{\pi}\Gamma^3\left(\frac{1}{2\alpha}+\frac{1}{2}\right)}{\frac{1}{\pi^3}\left[\Gamma\left(\frac{1}{\alpha}+\frac{1}{2}\right) - \frac{1}{\sqrt{\pi}}\Gamma^2\left(\frac{1}{2\alpha}+\frac{1}{2}\right)\right]^{3/2}}$$

*and*

$$\beta_{2GHN} = \frac{\Gamma\left(\frac{2}{\alpha}+\frac{1}{2}\right) - \frac{4}{\sqrt{\pi}}\Gamma\left(\frac{1}{2\alpha}+\frac{1}{2}\right)\Gamma\left(\frac{3}{2\alpha}+\frac{1}{2}\right) + \frac{6}{\pi}\Gamma^2\left(\frac{1}{2\alpha}+\frac{1}{2}\right)\Gamma\left(\frac{1}{2\alpha}+\frac{1}{2}\right) - \frac{3}{\pi^{3/2}}\Gamma^4\left(\frac{1}{2\alpha}+\frac{1}{2}\right)}{\frac{1}{\sqrt{\pi}}\left[\Gamma\left(\frac{1}{\alpha}+\frac{1}{2}\right) - \frac{1}{\sqrt{\pi}}\Gamma^2\left(\frac{1}{2\alpha}+\frac{1}{2}\right)\right]^2},$$

*respectively, which correspond to those for the generalized half-normal distribution. Figure 3 depict graphic representations for the asymmetry and kurtosis coefficients, respectively, of the* TGHN *distribution for different values of the $\alpha$ and $\lambda$.*

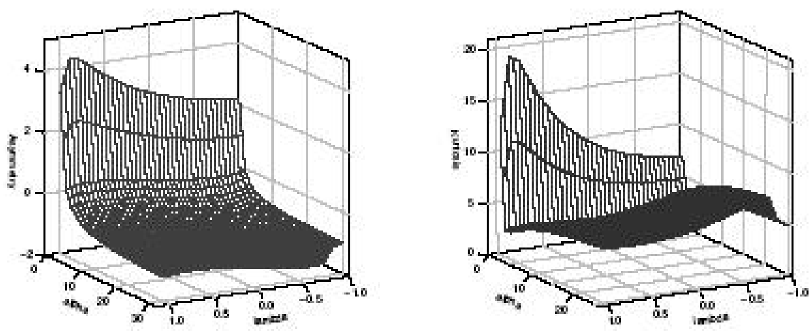


**Figure 3.** Skewness and kurtosis coefficients for a transmuted generalized half-normal distribution.

### 2.6. Order statistics

In statistics, the j-th order statistical of a sample is equal to its j-th-smallest value. Together with rank statistics, order statistics are among the most fundamental tools in non-parametric statistics and inference. For a sample of size $n$, the n-th order statistics (or, the largest order statistic) is its maximum, that is,

$$X_{(n)} = \max\{X_1, X_2, ..., X_n\}.$$

Similarly, $X_{(1)} = \min\{X_1, X_2, ..., X_n\}$ is the minimum of the sample.

The sample range is the difference between the maximum and the minimum in the sample. It is clearly a function of the order statistics:

$$\text{Range}\{X_1, X_2, ..., X_n\} = X_{(n)} - X_{(1)}.$$

It is well known that if $X_{(1)} \leq X_{(2)} \leq ... \leq X_{(n)}$ denotes the order statistics of a random sample $X_1, X_2, ..., X_n$ from a continuous population with cdf $F_X(x)$ and pdf $f_X(x)$ then, the pdf of $X_{(j)}$ is given by

$$f_{X_{(j)}}(x) = \frac{n!}{(j-1)!(n-j)!} f(x) \left[F(x)\right]^{j-1} \left[1 - F(x)\right]^{n-j},$$

for $j = 1, 2, ..., n$. Therefore the pdf of the j-th order statistics for a transmuted half-normal distribution is given by

$$
\left.
\begin{aligned}
f_{X_{(j)}}(x) &= \frac{2n!\alpha}{(j-1)!(n-j)!\sigma^{\alpha}}\, x^{\alpha-1}\phi\left[\left(\frac{x}{\sigma}\right)^{\alpha}\right]\left\{1+3\lambda-4\lambda\Phi\left[\left(\frac{x}{\sigma}\right)^{\alpha}\right]\right\} \\
&\times \left\{2\Phi\left[\left(\frac{x}{\sigma}\right)^{\alpha}\right]-1\right\}^{j-1}\left\{1+2\lambda-2\lambda\Phi\left[\left(\frac{x}{\sigma}\right)^{\alpha}\right]\right\}^{j-1} \\
&\times \left[1-\left\{2\Phi\left(\left(\frac{x}{\sigma}\right)^{\alpha}\right)-1\right\}\left\{1+2\lambda-2\lambda\Phi\left(\left(\frac{x}{\sigma}\right)^{\alpha}\right)\right\}\right]^{n-j}
\end{aligned}
\right\}. \tag{2.8}
$$

Thus, the pdf the largest order statistics $X_{(n)}$ is given by

$$
\begin{aligned}
f_{X_{(n)}}(x) &= \frac{2n\alpha}{\sigma^{\alpha}}\, x^{\alpha-1}\phi\left[\left(\frac{x}{\sigma}\right)^{\alpha}\right]\left\{1+3\lambda-4\lambda\Phi\left[\left(\frac{x}{\sigma}\right)^{\alpha}\right]\right\} \\
&\times \left\{2\Phi\left[\left(\frac{x}{\sigma}\right)^{\alpha}\right]-1\right\}^{n-1}\left\{1+2\lambda-2\lambda\Phi\left[\left(\frac{x}{\sigma}\right)^{\alpha}\right]\right\}^{n-1}
\end{aligned}
$$

and the pdf of the smallest order statistics $X_{(1)}$ is given by

$$
\begin{aligned}
f_{X_{(1)}}(x) &= \frac{2n\alpha}{\sigma^{\alpha}}\, x^{\alpha-1}\phi\left[\left(\frac{x}{\sigma}\right)^{\alpha}\right]\left\{1+3\lambda-4\lambda\Phi\left[\left(\frac{x}{\sigma}\right)^{\alpha}\right]\right\} \\
&\times \left[1-\left\{2\Phi\left(\left(\frac{x}{\sigma}\right)^{\alpha}\right)-1\right\}\left\{1+2\lambda-2\lambda\Phi\left(\left(\frac{x}{\sigma}\right)^{\alpha}\right)\right\}\right]^{n-1}
\end{aligned}
$$

## 3. Inference

In this section we discuss moment and maximum likelihood estimations for the parameters $\sigma$, $\alpha$ and $\lambda$ of the TGHN distribution. In addition, we present the observed information matrix for the TGHN distribution and conduct a simulation study toillustrate the behavior of maximum likelihood estimates (MLE).

### 3.1. Moment estimation

The following proposition presents an analytical expression for the moment estimators for parameters $\sigma$, $\alpha$ and $\lambda$.

**Proposition 3.1.** *Let $X_1, \ldots, X_n$ be a random sample of size $n$ from the $T \sim \mathrm{TGHN}(\sigma, \alpha, \lambda)$ distribution. Then the moment estimate for $\sigma$, $\alpha$ and $\lambda$ are given by*

$$\left.\begin{array}{rcl} \widehat{\sigma}_M &=& \frac{\overline{X}}{2a_1} \\ \frac{a_2}{a_1^2} &=& \frac{2\overline{X^2}}{\overline{X}^2} \\ \frac{a_3}{a_1^3} &=& \frac{4\overline{X^3}}{\overline{X}^3} \end{array}\right\} \tag{3.1}$$

*where $\overline{X^k} = (1/n)\sum_{i=1}^n X_i^k$, $k = 1, 2, 3$ and $a_i$, $i = 1, 2, 3$ are given in (2.7).*

**Proof.** From Proposition 3.1 and considering the first three equations in the moments method, we have

$$\overline{X} = 2\sigma a_1, \quad \overline{X^2} = 2\sigma^2 a_2 \quad \text{and} \quad \overline{X^3} = 2\sigma^2 a_3. \tag{3.2}$$

Solving the first equation above for $\sigma$ yields $\widehat{\sigma}_M$. Replacing $\widehat{\sigma}_M$ in the other equation above, the second and third equations in (3.1) are obtained.

### 3.2. Maximum Likelihood estimation

In this subsection, we consider the MLE of the parameters $\theta = (\sigma, \alpha, \lambda)$ of the TGHN model. Suppose that $x_1, x_2, ..., x_n$ is a random sample of size $n$ from the transmuted generalized half-normal distribution $TGHN(\theta)$. Then the likelihood function is given by

$$L(\theta) = \frac{2^n \alpha^n}{\sigma^{n\alpha}} \prod_{i=1}^n x_i^{\alpha-1} \phi\left(\left(\frac{x_i}{\sigma}\right)^\alpha\right)\left[1 + 3\lambda - 4\lambda\Phi\left(\left(\frac{x_i}{\sigma}\right)^\alpha\right)\right], \tag{3.3}$$

with respective sample log-likelihood function

$$\begin{array}{rcl} l(\theta) = \log L(\theta) &=& n\log 2 + n\log\alpha - n\alpha\log\sigma + (\alpha-1)\sum_{i=1}^n \log(x_i) \\ &+& \frac{1}{\sigma^\alpha}\sum_{i=1}^n x_i^\alpha \\ &+& \sum_{i=1}^n \log\left[1 + 3\lambda - 4\lambda\Phi\left(\left(\frac{x_i}{\sigma}\right)^\alpha\right)\right], \end{array}$$

so that the maximum likelihood equations are given by

$$\left.\begin{array}{rcl} -\frac{n\alpha}{\sigma} - \frac{\alpha}{\sigma^{\alpha+1}}\sum_{i=1}^n x_i^\alpha + \sum_{i=1}^n \frac{H_1(x_i)}{H(x_i)} &=& 0, \\ \frac{n}{\alpha} - n\log\sigma + \sum_{i=1}^n \log x_i + \frac{1}{\sigma^\alpha}\sum_{i=1}^n x_i^\alpha \log\left(\frac{x_i}{\sigma}\right) + \sum_{i=1}^n \frac{H_2(x_i)}{H(x_i)} &=& 0, \\ \sum_{i=1}^n \frac{H_3(x_i)}{H(x_i)} &=& 0, \end{array}\right\} \tag{3.4}$$

where $H(x_i) = 1+3\lambda-4\lambda\Phi((x_i/\sigma)^{\alpha})$, $H_1(x_i) = \frac{d}{d\sigma}H(x_i)$, $H_2(x) = \frac{d}{d\alpha}H(x_i)$ and $H_3(x_i) = \frac{d}{d\lambda}H(x_i)$. The solution for the equations lik can be obtained by using the optim function available in software R Development Core Team [11], the specific method is the L-BFGS-B developed by Byrd, Lu, Nocedal and Zhu [4] which allows box constraint. This uses a limited-memory modification of the quasi-Newton method.

### 3.3. Observed information matrix

In this subsection, we consider the observed information matrix of the $TGHN$ model. Given $X \sim TGHN(\theta)$, the observed information matrix is

$$I_n(\theta) = \begin{pmatrix} \frac{\partial^2 l(\theta)}{\partial\sigma^2} & \frac{\partial^2 l(\theta)}{\partial\alpha\partial\sigma} & \frac{\partial^2 l(\theta)}{\partial\lambda\partial\sigma} \\ & \frac{\partial^2 l(\theta)}{\partial\alpha^2} & \frac{\partial^2 l(\theta)}{\partial\lambda\partial\alpha} \\ & & \frac{\partial^2 l(\theta)}{\partial\lambda^2} \end{pmatrix},$$

where

$$\frac{\partial^2 l(\theta)}{\partial\sigma^2} = \frac{n\alpha}{\sigma^2} + \frac{\alpha(\alpha+1)}{\sigma^{\alpha+2}}\sum_{i=1}^n x_i^{\alpha} + \sum_{i=1}^n \frac{\partial}{\partial\sigma}\frac{H_1(x_i)}{H(x_i)},$$

$$\frac{\partial^2 l(\theta)}{\partial\alpha\partial\sigma} = -\frac{n}{\sigma} - \frac{1-\alpha\log(\sigma)}{\sigma^{\alpha+1}}\sum_{i=1}^n x_i^{\alpha} - \frac{\alpha}{\sigma^{\alpha+1}}\sum_{i=1}^n x_i^{\alpha}\log(x_i) + \sum_{i=1}^n \frac{\partial}{\partial\alpha}\frac{H_1(x_i)}{H(x_i)},$$

$$\frac{\partial^2 l(\theta)}{\partial\lambda\partial\sigma} = \sum_{i=1}^n \frac{\partial}{\partial\lambda}\frac{H_1(x_i)}{H(x_i)},$$

$$\frac{\partial^2 l(\theta)}{\partial\sigma\partial\alpha} = -\frac{n}{\sigma} - \frac{\alpha}{\sigma^{\alpha+1}}\sum_{i=1}^n x_i^{\alpha}\log\left(\frac{x_i}{\sigma}\right) - \frac{1}{\sigma^{\alpha+1}}\sum_{i=1}^n x_i^{\alpha} + \sum_{i=1}^n \frac{\partial}{\partial\sigma}\frac{H_2(x_i)}{H(x_i)}$$

$$\frac{\partial^2 l(\theta)}{\partial\alpha^2} = -\frac{n}{\alpha^2} + \frac{1}{\sigma^{\alpha}}\sum_{i=1}^n x_i^{\alpha}\log^2\left(\frac{x_i}{\sigma}\right) + \sum_{i=1}^n \frac{\partial}{\partial\alpha}\frac{H_2(x_i)}{H(x_i)},$$

$$\frac{\partial^2 l(\theta)}{\partial\lambda\partial\alpha} = \sum_{i=1}^n \frac{\partial}{\partial\lambda}\frac{H_2(x_i)}{H(x_i)},$$

$$\frac{\partial^2 l(\theta)}{\partial\sigma\partial\lambda} = \sum_{i=1}^n \frac{\partial}{\partial\sigma}\frac{H_3(x_i)}{H(x_i)}, \quad \frac{\partial^2 l(\theta)}{\partial\alpha\partial\lambda} = \sum_{i=1}^n \frac{\partial}{\partial\alpha}\frac{H_3(x_i)}{H(x_i)}$$

and $\frac{\partial^2 l(\theta)}{\partial\lambda^2} = \sum_{i=1}^n \frac{\partial}{\partial\lambda}\frac{H_3(x_i)}{H(x_i)}$.

### 3.4. Simulation study

In this subsection, a simulation study is performed to illustrate the behavior of the MLE parameter $\sigma$, $\alpha$ and $\lambda$. We generate 1000 random samples of sizes $n = 50$, $n = 150$ and $n = 300$ from the distribution $TGHN(\theta)$ for fixed values of the parameters. Random numbers $X \sim TGHN(\theta)$ can be generated as

1. Generate $U \sim \text{Unif}(0,1)$.
2. Compute $X = \sigma\left[\Phi^{-1}\left(\frac{1+\lambda-\sqrt{(1+\lambda)^2-4\lambda U}}{4\lambda} + \frac{1}{2}\right)\right]^{1/\alpha}$.

where $\Phi^{-1}$ is the quantile function of the standard normal distribution. Measures and empirical standard deviations are presented in Table 3.1. Here, the parameters are well estimated and the estimates are asymptotically unbiased.

| $\sigma$ | $\alpha$ | $\lambda$ | $\widehat{\sigma}$ (SD) | $\widehat{\alpha}$ (SD) | $\widehat{\lambda}$ (SD) |
|---|---|---|---|---|---|
| $n = 50$ | | | | | |
| 1.0 | 0.5 | 1.0 | 0.916856 (0.255959) | 0.520920 (0.065720) | 0.938142 (0.119670) |
| | | 0.5 | 0.914440 (0.360617) | 0.499354 (0.068656) | 0.325292 (0.383286) |
| | | -1.0 | 1.196665 (0.406695) | 0.570780 (0.124729) | -0.819658 (0.363444) |
| 2.0 | 1.0 | 1.0 | 1.896826 (0.258498) | 1.043402 (0.121237) | 0.938109 (0.123406) |
| | | 0.5 | 1.839288 (0.401754) | 0.977326 (0.140665) | 0.284698 (0.422958) |
| | | -1.0 | 2.163885 (0.343411) | 1.144162 (0.258204) | -0.814281 (0.363698) |
| 3.0 | 2.0 | 1.0 | 2.923249 (0.197402) | 2.078352 (0.243693) | 0.945194 (0.114554) |
| | | 0.5 | 2.869020 (0.310812) | 1.945938 (0.274630) | 0.297925 (0.397159) |
| | | -1.0 | 3.126112 (0.518595) | 2.310691 (1.250204) | -0.794375 (0.884442) |
| $n = 150$ | | | | | |
| $\sigma$ | $\alpha$ | $\lambda$ | $\widehat{\sigma}$ (SD) | $\widehat{\alpha}$ (SD) | $\widehat{\lambda}$ (SD) |
| 1.0 | 0.5 | 1.0 | 0.931762 (0.177101) | 0.510188 (0.0356710) | 0.940026 (0.115232) |
| | | 0.5 | 0.950623 (0.330346) | 0.488532 (0.0473573) | 0.372702 (0.373124) |
| | | -1.0 | 1.101251 (0.289657) | 0.531747 (0.0772491) | -0.904767 (0.256996) |
| 2.0 | 1.0 | 1.0 | 1.909465 (0.189369) | 1.022025 (0.0706584) | 0.936197 (0.121109) |
| | | 0.5 | 1.915185 (0.352607) | 0.977040 (0.0969972) | 0.374632 (0.383413) |
| | | -1.0 | 2.073748 (0.225784) | 1.059712 (0.1542691) | -0.913137 (0.249717) |
| 3.0 | 2.0 | 1.0 | 2.927837 (0.150383) | 2.040683 (0.1426299) | 0.938162 (0.117917) |
| | | 0.5 | 2.921038 (0.285649) | 1.964368 (0.1972316) | 0.365500 (0.384122) |
| | | -1.0 | 3.051073 (0.158212) | 2.120309 (0.3036818) | -0.912356 (0.248957) |
| $n = 300$ | | | | | |
| $\sigma$ | $\alpha$ | $\lambda$ | $\widehat{\sigma}$ (SD) | $\widehat{\alpha}$ (SD) | $\widehat{\lambda}$ (SD) |
| 1.0 | 0.5 | 1.0 | 0.942250 (0.145848) | 0.506106 (0.025127) | 0.949547 (0.109387) |
| | | 0.5 | 0.979323 (0.312453) | 0.488000 (0.037587) | 0.410776 (0.338208) |
| | | -1.0 | 1.049801 (0.180932) | 0.515084 (0.047462) | -0.953957 (0.166514) |
| 2.0 | 1.0 | 1.0 | 1.918997 (0.162825) | 1.015543 (0.049621) | 0.942905 (0.114174) |
| | | 0.5 | 1.947923 (0.340752) | 0.972927 (0.081222) | 0.408279 (0.365341) |
| | | -1.0 | 2.043949 (0.166501) | 1.030886 (0.096272) | -0.953172 (0.178910) |
| 3.0 | 2.0 | 1.0 | 2.936010 (0.132371) | 2.023273 (0.097304) | 0.941166 (0.118513) |
| | | 0.5 | 2.950370 (0.258666) | 1.949546 (0.159468) | 0.406133 (0.358088) |
| | | -1.0 | 3.034126 (0.125131) | 2.069454 (0.217579) | -0.944696 (0.201418) |

Table 3.1: MLE for samples generated of size 50, 150 and 300 for different values of $\sigma$, $\alpha$ and $\lambda$.

## 4. Real data application

Devore [6] presents a data set associated with energy consumption (in BTU) of 90 homes 140 with gas heating. The electric companies consider essential to analyze this type information for respond to energy demands. Table 4.1 presents summary statistics for the energy consumption data where $b_1$ and $b_2$ are the coefficients of asymmetry and kurtosis, respectively.

| sample size | mean | variance | asymmetry | kurtosis |
|---|---|---|---|---|
| 90 | 10.038 | 8.225 | 0.283 | 3.000 |

Table 4.1: Summary statistics for fatigue life data set.

Using results in Subsection 3.1, moment estimators were computed, leading to $\widehat{\sigma}_M = 9.796$, $\widehat{\alpha}_M = 1.877$ and $\widehat{\lambda}_M = -0.994$. These estimates were then used as starting values for the optim algorithm for maximizing the likelihood function. Table 4.2 presents parameter estimates for the GHN and TGHN models, using maximum likelihood approach and the corresponding Akaike Information Criterion (AIC) [1] and Bayesian Information Criterion (BIC) [12] for model choice. For these data, AIC and BIC shows a better fit of the TGHN model.

| MLE | GHN(SD) | TGHN(SD) |
|---|---|---|
| $\widehat{\sigma}$ | 11.719(0.342) | 9.795(0.748) |
| $\widehat{\alpha}$ | 2.829(0.229) | 1.946(0.332) |
| $\widehat{\lambda}$ | | −1.000(0.353) |
| AIC | 455.873 | 450.907 |
| BIC | 460.872 | 458.406 |

Table 4.2: MLE for fitting various models: GHN and TGHN models on the fatigue life data set.

Figure 4 depicts the histogram for the data with the fitted densities, revealing good performance of the TGHN model.

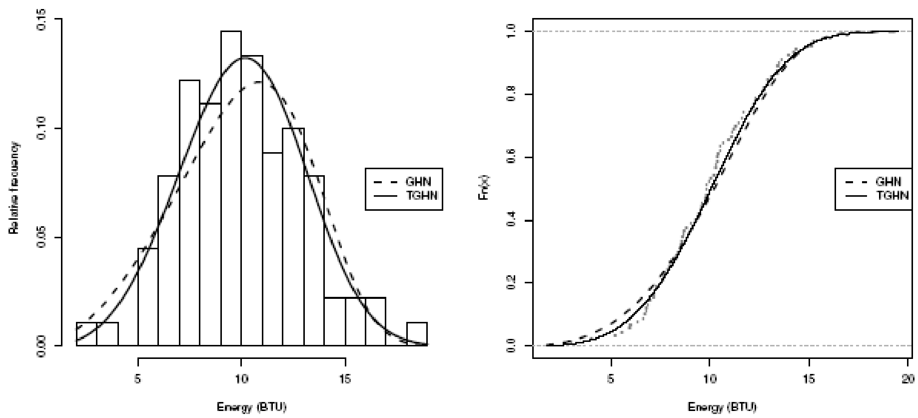



Figure 4: Models fitted by maximum likelihood method for fatigue life data set: TGHN (solid line) and GHN (dashed line).

## 5. Concluding remarks

In this paper we study an extension of the generalized half-normal distribution study by Cooray and Aranda [5]. We use the quadratic rank transmutation map to generate a transmuted generalized half-normal distribution. This extension is more flexible in terms of skewness and kurtosis than the generalized half-normal distribution. MLE for the proposal distribution requires numerical procedures such as the Newton-Raphson algorithm to be computed. Application to real data have demonstrated that the transmuted generalized half-normal distribution can present better fit than generalized half-normal distribution.

## Appendix

### Proof of Proposition 2.1

Let $X \sim TGHN(\sigma, \alpha, \lambda)$, then

a) $W = aX$ and $a > 0$, then

$$\begin{aligned} F_W(w) &= P(W \leq w) = P(aX \leq w) = P\left(X \leq \tfrac{w}{a}\right) = F_X\left(\tfrac{w}{a}\right) \\ &= \left\{2\Phi\left[\left(\tfrac{x}{a\sigma}\right)^{\alpha}\right] - 1\right\}\left\{1 + 2\lambda - 2\lambda\Phi\left[\left(\tfrac{x}{a\sigma}\right)^{\alpha}\right]\right\}. \end{aligned}$$

Therefore $W \sim TGHN(a\sigma, \alpha, \lambda)$.

b) $W = X^{-1}$, then

Therefore

$$\begin{aligned} f_W(w) &= \frac{dF_W(w)}{dw} = \frac{d}{dw}\left[1 - F_X\left(\frac{1}{w}\right)\right] = \frac{1}{w^2} f_X\left(\frac{1}{w}\right) \\ &= \frac{2\alpha}{\sigma^\alpha w^{\alpha+1}} \phi\left(\frac{1}{\sigma^\alpha w^\alpha}\right)\left[1 + 3\lambda - 4\lambda\Phi\left(\frac{1}{\sigma^\alpha w^\alpha}\right)\right]. \end{aligned}$$

c) $W = \log(X)$, then

$$F_W(w) = P(W \le w) = P(\log(X) \le w) = P\left(X \le e^w\right) = F_X(e^w).$$

Therefore

$$\begin{aligned} f_W(w) &= \frac{dF_W(w)}{dw} = \frac{dF_X(e^w)}{dw} = e^w f_X\left(e^w\right) \\ &= \frac{2\alpha}{\sigma^\alpha} e^{w(\alpha-1)} \phi\left(\left(\frac{e^w}{\sigma}\right)^\alpha\right)\left[1 + 3\lambda - 4\lambda\Phi\left(\left(\frac{e^w}{\sigma}\right)^\alpha\right)\right]. \end{aligned}$$

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