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Radius problem for the class of analytic functions based on Ruscheweyh derivative

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Abstract:

Let $\mathcal{A}$ be the class of analytic functions $f(z)$ with the normalized condition $f(0) = f'(0) - 1 = 0$ in the open unit disk $\mathbf{U}$. By making use of Ruscheweyh derivative operator, a new subclass $\mathcal{A}(\beta_1, \beta_2, \beta_3, \beta_4; \lambda)$ of $f(z) \in \mathcal{A}$ satisfying the inequality

$$\left|\beta_1 z \left(\frac{D^2 f(z)}{z}\right)' + \beta_2 z^2 \left(\frac{D^2 f(z)}{z}\right)'' + \beta_3 z^3 \left(\frac{D^2 f(z)}{z}\right)''' + \beta_4 z^4 \left(\frac{D^2 f(z)}{z}\right)''''\right| \leq \lambda,$$

for some complex numbers $\beta_1, \beta_2, \beta_3, \beta_4$ and for some real $\lambda > 0$ is introduced. The object of the present paper is to obtain some properties of the function class $\mathcal{A}(\beta_1, \beta_2, \beta_3, \beta_4; \lambda)$. Also the radius problems of $\frac{1}{\delta} f(\delta z) \in \mathcal{A}(\beta_1, \beta_2, \beta_3, \beta_4; \lambda)$ where $f(z)$ satisfies the condition $\Re\{\frac{D^2 f(z)}{z}\} > \alpha$ is considered.

Keywords: Analytic function; Univalent function; Ruscheweyh derivative; Cauchy-Schwarz inequality; Radius problema; Hölder inequality.
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1. Introduction and Motivation

Let $\mathcal{A}_0$ be the class of functions $f(z)$ of the form:

$$f(z) = a_0 + a_1 z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk
$\mathbf{U}:= \{z\colon z \in \mathbf{C} \ \ \textit{and} \ \ |z| < 1\}$.

If $f(z) \in \mathcal{A}_0$ is given by (1.1), together with following normalization: $a_0 = 0$ and $a_1 = 1$, then we say that $f(z) \in \mathcal{A}$. Let $\mathcal{S}$ be the subclass of all functions in $\mathcal{A}$, which are univalent in $\mathbf{U}$.

For $f \in \mathcal{A}$, $\ k \in \mathbf{N}_0 := \mathbf{N} \cup \{0\}$, Ruscheweyh [3] defined the operator $D^k : \mathcal{A} \longrightarrow \mathcal{A}$ by

$$D^0 f(z) = f(z)$$

$$D^1 f(z) = z f'(z),$$

$$\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots$$

$$(k+1) D^{k+1} f(z) = z(D^k f(z))' + k(D^k f(z)) \quad (z \in \mathbf{U}). \tag{1.2}$$

Thus, if $f(z) \in \mathcal{A}$, $\ f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, then (see [5])

$$D^k f(z) = z + \sum_{n=2}^{\infty} \frac{(k+n-1)!}{k!(n-1)!} a_n z^n \quad (z \in \mathbf{U}). \tag{1.3}$$

The Ruscheweyh operator has gained much aclaim as a unifying factor in the study of many classes of functions in geometric function theory.

Let $\mathcal{M}(\alpha)$ denote the subclass of $\mathcal{A}$ consisting of all $f(z)$ which satisfy the inequality

$$\Re \left\{ \frac{D^2 f(z)}{z} \right\} > \alpha \quad (z \in \mathbf{U}). \tag{1.4}$$

For complex parameters β_1, β_2 and some real $\lambda > 0$, Uyanìk et al.[6] introduced the class $\mathcal{A}(\beta_1, \beta_2; \lambda)$, a subclass of $\mathcal{A}$ as

$$\left| \beta_1 z \left(\frac{f(z)}{z} \right)' + \beta_2 z^2 \left(\frac{f(z)}{z} \right)'' \right| \leq \lambda \quad (z \in \mathbf{U}), \tag{1.5}$$

and discussed some of its properties. Later on in 2011, Uyanìk and Owa [7] introduced the third complex parameter β_3 and redefined the above class as

$$\left|\beta_1 z\left(\frac{f(z)}{z}\right)' + \beta_2 z^2\left(\frac{f(z)}{z}\right)'' + \beta_3 z^3\left(\frac{f(z)}{z}\right)'''\right| \le \lambda \quad (z \in \mathbf{U}), \tag{1.6}$$

and denoted the class by $\mathcal{A}(\beta_1, \beta_2, \beta_3; \lambda)$.

Motivated by the aforementioned work (also; see [1, 2]), we define the subclass of $\mathcal{A}$ as follows.

Definition 1.1. *A function $f(z) \in \mathcal{A}$ is said to be in the class $\mathcal{A}(\beta_1, \beta_2, \beta_3, \beta_4; \lambda)$ if it satisfies the inequality*

$$\left|\beta_1 z\left(\tfrac{D^2 f(z)}{z}\right)' + \beta_2 z^2\left(\tfrac{D^2 f(z)}{z}\right)'' + \beta_3 z^3\left(\tfrac{D^2 f(z)}{z}\right)''' + \beta_4 z^4\left(\tfrac{D^2 f(z)}{z}\right)''''\right| \le \lambda, \tag{1.7}$$

for some complex parameters $\beta_1, \beta_2, \beta_3, \beta_4$ and for some real number $\lambda > 0$.

Example 1.2. *Let us consider a function $f_\gamma(z)$ defined by*

$$\begin{aligned} f_\gamma(z) &= z(1+z)^\gamma \quad (\gamma \in \mathbf{R}) \\ &= z + \sum_{n=2}^{\infty} \binom{\gamma}{n-1} z^n \end{aligned}$$

where

$$\binom{\gamma}{n-1} = \frac{\gamma(\gamma-1)(\gamma-2)\ldots(\gamma-n+2)}{(n-1)!}\,(n \ge 2)\,.$$

Then we have

$$D^2 f_\gamma(z) = z + \sum_{n=2}^{\infty} \frac{n(n+1)}{2} \binom{\gamma}{n-1} z^n. \tag{1.8}$$

Therefore, it follows from (1.8) that

$$\left|\beta_1 z\left(\tfrac{D^2 f_\gamma(z)}{z}\right)' + \beta_2 z^2\left(\tfrac{D^2 f_\gamma(z)}{z}\right)'' + \beta_3 z^3\left(\tfrac{D^2 f_\gamma(z)}{z}\right)''' + \beta_4 z^4\left(\tfrac{D^2 f_\gamma(z)}{z}\right)''''\right| \tag{1.9}$$

$$= \left|\sum_{n=2}^{\infty} \tfrac{n(n+1)}{2} \binom{\gamma}{n-1}\right.$$

$$\left.[\beta_1 + (n-2)\beta_2 + (n-2)(n-3)\beta_3 + (n-2)(n-3)(n-4)\beta_4]\, z^{n-1}\right| .$$

If $\gamma = 1$, then from (1.9) we have

$$\left|\beta_1 z \left(\frac{D^2 f_1(z)}{z}\right)' + \beta_2 z^2 \left(\frac{D^2 f_1(z)}{z}\right)'' + \beta_3 z^3 \left(\frac{D^2 f_1(z)}{z}\right)''' + \beta_4 z^4 \left(\frac{D^2 f_1(z)}{z}\right)''''\right|$$
$$= |3\beta_1 z| \leq 3|\beta_1|.$$

This implies that $f_1(z) \in \mathcal{A}(\beta_1, \beta_2, \beta_3, \beta_4; \lambda)$ for $\lambda \geq 3|\beta_1| > 0$.
If $\gamma = 2$, then (1.9) yields

$$\left|\beta_1 z \left(\frac{D^2 f_2(z)}{z}\right)' + \beta_2 z^2 \left(\frac{D^2 f_2(z)}{z}\right)'' + \beta_3 z^3 \left(\frac{D^2 f_2(z)}{z}\right)''' + \beta_4 z^4 \left(\frac{D^2 f_2(z)}{z}\right)''''\right|$$
$$= |6\beta_1 z + 12(\beta_1 + \beta_2) z^2| \leq 18|\beta_1| + 12|\beta_2| = 6(3|\beta_1| + 2|\beta_2|),$$

which shows that $f_2(z) \in \mathcal{A}(\beta_1, \beta_2, \beta_3, \beta_4, \lambda)$ for $\lambda \geq 6(3|\beta_1| + 2|\beta_2|) > 0$.

Further, if $\gamma = 3$, then from (1.9) we obtain

$$\left|\beta_1 z \left(\frac{D^2 f_3(z)}{z}\right)' + \beta_2 z^2 \left(\frac{D^2 f_3(z)}{z}\right)'' + \beta_3 z^3 \left(\frac{D^2 f_3(z)}{z}\right)''' + \beta_4 z^4 \left(\frac{D^2 f_3(z)}{z}\right)''''\right|$$
$$= |9\beta_1 z + 36(\beta_1 + \beta_2) z^2 + 30(\beta_1 + 2\beta_2 + 2\beta_3) z^3|$$
$$\leq 75|\beta_1| + 96|\beta_2| + 60|\beta_3| = 3[25|\beta_1| + 32|\beta_2| + 20|\beta_3|].$$

Thus $f_3(z) \in \mathcal{A}(\beta_1, \beta_2, \beta_3, \beta_4; \lambda)$ for $\lambda \geq 3[25|\beta_1| + 32|\beta_2| + 20|\beta_3|]$.

Similarly, for $\gamma = 4$, we have

$$\left|\beta_1 z \left(\frac{D^2 f_4(z)}{z}\right)' + \beta_2 z^2 \left(\frac{D^2 f_4(z)}{z}\right)'' + \beta_3 z^3 \left(\frac{D^2 f_4(z)}{z}\right)''' + \beta_4 z^4 \left(\frac{D^2 f_4(z)}{z}\right)''''\right|$$
$$= \left|12\beta_1 z + 72(\beta_1 + \beta_2) z^2 + 120(\beta_1 + 2\beta_2 + 2\beta_3) z^3 + 60(\beta_1 + 3\beta_2 + 6\beta_3 + 6\beta_4) z^4\right|$$
$$\leq 264|\beta_1| + 492|\beta_2| + 600|\beta_3| + 360|\beta_4| = 12[22|\beta_1| + 41|\beta_2| + 50|\beta_3| + 30|\beta_4|].$$

Thus, $f_4(z) \in \mathcal{A}(\beta_1, \beta_2, \beta_3, \beta_4; \lambda)$ for $\lambda \geq 12[22|\beta_1| + 41|\beta_2| + 50|\beta_3| + 30|\beta_4|]$.

2. Properties of the class $\mathcal{A}(\beta_1, \beta_2, \beta_3, \beta_4; \lambda)$

Our first result provides a sufficient condition for a function $f \in \mathcal{A}$ to be in the class $\mathcal{A}(\beta_1, \beta_2, \beta_3, \beta_4; \lambda)$.

Theorem 2.1. *If $f(z) \in \mathcal{A}$ satisfies*

$$\sum_{n=2}^{\infty} \frac{n(n+1)}{2}(n-1)\Big[|\beta_1| + (n-2)|\beta_2|$$
$$+ (n-2)(n-3)|\beta_3| + (n-2)(n-3)(n-4)|\beta_4|\Big]|a_n| \leq \lambda, \tag{2.1}$$

for some complex numbers $\beta_1, \beta_2, \beta_3, \beta_4$ *and for some real* $\lambda > 0$, *then* $f(z) \in \mathcal{A}(\beta_1, \beta_2, \beta_3, \beta_4; \lambda)$.

Proof. Suppose that (2.1) holds true for some real $\lambda > 0$. Then

$$\begin{aligned}
&\left|\beta_1 z\left(\tfrac{D^2 f(z)}{z}\right)' + \beta_2 z^2\left(\tfrac{D^2 f(z)}{z}\right)'' + \beta_3 z^3\left(\tfrac{D^2 f(z)}{z}\right)''' + \beta_4 z^4\left(\tfrac{D^2 f(z)}{z}\right)''''\right| \\
&= \left|\textstyle\sum_{n=2}^{\infty} \tfrac{n(n+1)}{2}(n-1)\Big[\beta_1 + (n-2)\beta_2 + (n-2)(n-3)\beta_3\right. \\
&\left. +(n-2)(n-3)(n-4)\beta_4\Big] a_n z^{n-1}\right| \\
&\leq \textstyle\sum_{n=2}^{\infty} \tfrac{n(n+1)}{2}(n-1)\Big[|\beta_1| + (n-2)\,|\beta_2| \\
&+(n-2)(n-3)|\beta_3| + (n-2)(n-3)(n-4)|\beta_4|\Big]|a_n||z|^{n-1} \\
&< \textstyle\sum_{n=2}^{\infty} \tfrac{n(n+1)}{2}(n-1)\,[|\beta_1| + (n-2)|\beta_2| \\
&+(n-2)(n-3)|\beta_3| + (n-2)(n-3)(n-4)\,|\beta_4|]\,|a_n \\
&\leq \lambda,
\end{aligned}$$

by virtue of (2.1). This shows that

$$f(z) \in A(\beta_1, \beta_2, \beta_3, \beta_4; \lambda).$$

Thus, the proof of Theorem 2.1 is completed.

The next theorem gives the necessary condition for the function $f \in \mathcal{A}$ to be in the class $\mathcal{A}(\beta_1, \beta_2, \beta_3, \beta_4; \lambda)$.

Theorem 2.2. *If* $f(z) \in \mathcal{A}(\beta_1, \beta_2, \beta_3, \beta_4; \lambda)$ *with* $\arg\beta_1 = \arg\beta_2 = \arg\beta_3 = \arg\beta_4 = \psi$ *and* $a_n = |a_n|e^{i((n-1)\beta-\psi)}$ $(n = 2, 3, 4, \cdots)$, *then*

$$\sum_{n=2}^{\infty} \frac{n(n+1)}{2}(n-1)$$

$$[|\beta_1| + (n-2)|\beta_2| + (n-2)(n-3)|\beta_3| + (n-2)(n-3)(n-4)|\beta_4|]\,|a_n| \leq \lambda. \tag{2.2}$$

Proof. If $f(z) \in \mathcal{A}(\beta_1, \beta_2, \beta_3, \beta_4; \lambda)$ with $\arg\beta_1 = \arg\beta_2 = \arg\beta_3 = \arg\beta_4 = \psi$ and $a_n = |a_n|e^{i((n-1)\beta-\psi)}$, then we have

$$\begin{aligned}
&\left|\beta_1 z\left(\tfrac{D^2 f(z)}{z}\right)' + \beta_2 z^2\left(\tfrac{D^2 f(z)}{z}\right)'' + \beta_3 z^3\left(\tfrac{D^2 f(z)}{z}\right)''' + \beta_4 z^4\left(\tfrac{D^2 f(z)}{z}\right)''''\right| \\
&= \left|\textstyle\sum_{n=2}^{\infty} \tfrac{n(n+1)}{2}(n-1)[\beta_1 + (n-2)\beta_2 + (n-2)(n-3)\beta_3\right. \\
&\left. +(n-2)(n-3)(n-4)\beta_4] a_n z^{n-1}\right| \\
&= \left|\textstyle\sum_{n=2}^{\infty} \tfrac{n(n+1)}{2}(n-1)[|\beta_1| + (n-2)|\beta_2| + (n-2)(n-3)|\beta_3|\right. \\
&\left. +(n-2)(n-3)(n-4)|\beta_4|]|a_n|e^{i(n-1)\beta-\psi)} z^{n-1}\right| \leq \lambda,
\end{aligned} \tag{2.3}$$

for all $z \in \mathbf{U}$. Let us consider a point $z \in \mathbf{U}$ such that $z = |z|e^{-i\beta}$. Then it follows from (2.3) that

$$\left|\beta_1 z \left(\frac{D^2 f(z)}{z}\right)' + \beta_2 z^2 \left(\frac{D^2 f(z)}{z}\right)'' + \beta_3 z^3 \left(\frac{D^2 f(z)}{z}\right)''' + \beta_4 z^4 \left(\frac{D^2 f(z)}{z}\right)''''\right|$$
$$= \sum_{n=2}^{\infty} \frac{n(n+1)}{2}(n-1)\,[|\beta_1| + (n-2)|\beta_2|$$
$$+(n-2)(n-3)|\beta_3| + (n-2)(n-3)(n-4)|\beta_4||a_n||z|^{n-1} \leq \lambda.$$

Letting $|z| \longrightarrow 1^-$, we obtain

$$\sum_{n=2}^{\infty} \frac{n(n+1)}{2}(n-1)[|\beta_1| + (n-2)|\beta_2| + (n-2)(n-3)|\beta_3|$$

$$+(n-2)(n-3)(n-4)\ |\beta_4|]|a_n| \leq \lambda.$$

Thus, the proof of Theorem 2.2 is completed.

Corollary 2.3. *If* $f(z) \in \mathcal{A}(\beta_1, \beta_2, \beta_3, \beta_4, \lambda)$ *with* $\arg\beta_1 = \arg\beta_2 = \arg\beta_3 = \arg\beta_4 = \psi$ *and* $a_n = |a_n|e^{i((n-1)\beta-\psi)}$ $(n = 2, 3, \cdots)$ *then*

$$|a_n| \leq \frac{2\lambda}{n(n+1)(n-1)[|\beta_1|+(n-2)|\beta_2|+(n-2)(n-3)|\beta_3|+(n-2)(n-3)(n-4)|\beta_4|]}$$
$$(n = 2, 3, 4, ...).$$

A distortion property for the function f in the class $A(\beta_1, \beta_2, \beta_3, \beta_4; \lambda)$ is considered in the following corollary.

Corollary 2.4. *If* $f(z) \in \mathcal{A}(\beta_1, \beta_2, \beta_3, \beta_4; \lambda)$ *with* $\arg\beta_1 = \arg\beta_2 = \arg\beta_3 = \arg\beta_4 = \psi$ *and* $a_n = |a_n|e^{i((n-1)\beta-\psi)}$ $(n = 2, 3, 4, \cdots)$, *then we have*

$$|z| - \sum_{n=2}^{\infty} |a_n||z|^n - C_j|z|^{j+1} \leq |f(z)| \leq |z| + \sum_{n=2}^{\infty} |a_n||z|^n + C_j|z|^{j+1},$$

(2.4)

with

$$C_j = \frac{\lambda - \sum_{n=2}^{\infty} \frac{n(n+1)}{2}(n-1)[|\beta_1|+(n-2)|\beta_2|+(n-2)(n-3)|\beta_3|+(n-2)(n-3)(n-4)|\beta_4|]|a_n|}{\frac{(j+1)(j+2)}{2} j[|\beta_1|+(j-1)|\beta_2|+(j-1)(j-2)|\beta_3|+(j-1)(j-2)(j-3)|\beta_4|]}$$

(2.5)

and

$$1-\sum_{n=2}^{j} n|a_n||z|^{n-1}-Q_j|z|^j \leq |f'(z)| \leq 1+\sum_{n=2}^{j} n|a_n||z|^{n-1}+Q_j|z|^j \quad (z \in \mathbf{U}) \tag{2.6}$$

with

$$Q_j = \frac{2\left[\lambda-\sum_{n=2}^{\infty}\frac{n(n+1)(n-1)}{2}\{|\beta_1|+(n-2)|\beta_2|+(n-2)(n-3)|\beta_3|+(n-2)(n-3)(n-4)|\beta_4|\}|a_n|\right]}{j(j+2)[|\beta_1|+(j-1)|\beta_2|+(j-1)(j-2)|\beta_3|+(j-1)(j-2)(j-3)|\beta_4|]}. \tag{2.7}$$

Proof. Since $f(z) \in \mathcal{A}(\beta_1,\beta_2,\beta_3,\beta_4;\lambda)$, hence it follows from Theorem 2.2 that

$$\sum_{n=j+1}^{\infty}\frac{n(n+1)}{2}(n-1)[|\beta_1|+(n-2)|\beta_2|+(n-2)(n-3)|\beta_3|+(n-2)(n-3)(n-4)|\beta_4|]|a_n|$$

$$\leq \lambda-\sum_{n=2}^{j}\frac{n(n+1)}{2}(n-1)[|\beta_1|+(n-2)|\beta_2|+(n-2)(n-3)|\beta_3|+(n-2)(n-3)(n-4)|\beta_4|]|a_n|. \tag{2.8}$$

Further, we note that

$$\begin{aligned}&\tfrac{j(j+1)(j+2)}{2}\left[|\beta_1|+(j-1)|\beta_2|+(j-1)(j-2)|\beta_3|\right.\\&+(j-1)(j-2)(j-3)|\beta_4|\textstyle\sum_{n=j+1}^{\infty}|a_n|\\&\leq \textstyle\sum_{n=j+1}^{\infty}\frac{n(n+1)(n-1)}{2}\left[|\beta_1|++(n-2)|\beta_2|+(n-2)(n-3)|\beta_3|\right.\\&\qquad\qquad+(n-2)(n-3)(n-4)|\beta_4||a_n|.\end{aligned} \tag{2.9}$$

It follows from (2.8) and (2.9) that

$$\begin{aligned}&\textstyle\sum_{n=j+1}^{\infty}|a_n|\\&\leq \frac{\lambda-\sum_{n=2}^{j}\frac{n(n+1)(n-1)}{2}[|\beta_1|+(n-2)|\beta_2|+(n-2)(n-3)|\beta_3|+(n-2)(n-3)(n-4)|\beta_4|]|a_n|}{\frac{j(j+1)(j+2)}{2}[|\beta_1|+(j-1)|\beta_2|+(j-1)(j-2)|\beta_3|+(j-1)(j-2)(j-3)|\beta_4|]}\\&=C_j,\end{aligned}$$

where C_j is given in (2.5).

Since $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$, which implies

$$|f(z)| \leq |z|+\sum_{n=2}^{j}|a_n||z|^n+\sum_{n=j+1}^{\infty}|a_n||z|^n$$

$$\leq |z| + \sum_{n=2}^{j} |a_n||z|^n + C_j|z|^{j+1}, \tag{2.10}$$

and

$$|f(z)| \geq |z| - \sum_{n=2}^{j} |a_n||z|^n - \sum_{n=j+1}^{\infty} |a_n||z|^n$$

$$\geq |z| - \sum_{n=2}^{j} |a_n||z|^n - C_j|z|^{j+1}. \tag{2.11}$$

Thus, the assertion (2.4) follows from (2.10) and (2.11). Further, we note that

$$\begin{aligned}&\tfrac{(j+2)}{2} j[|\beta_1| + (j-1)|\beta_2| + (j-1)(j-2)|\beta_3| + (j-1)(j-2)(j-3)|\beta_4|] \textstyle\sum_{n=j+1}^{\infty} n|a_n| \\ &\leq \textstyle\sum_{n=j+1}^{\infty} \frac{n(n+1)}{2}(n-1)[|\beta_1| + (n-2)|\beta_2| + (n-2)(n-3)|\beta_3| \\ &+(n-2)(n-3)(n-4)|\beta_4|]|a_n| \\ &\leq \lambda - \textstyle\sum_{n=2}^{j} \frac{n(n+1)}{2}(n-1)[|\beta_1| + (n-2)|\beta_2| + (n-2)(n-3)|\beta_3| \\ &+(n-2)(n-3)(n-4)|\beta_4|]|a_n|,\end{aligned}$$

which yields

$$\begin{aligned}&\textstyle\sum_{n=j+1}^{\infty} n|a_n| \\ &\leq \frac{2[\lambda - \sum_{n=2}^{\infty} \frac{n(n+1)}{2}(n-1)\{|\beta_1|+(n-2)|\beta_2|+(n-2)(n-3)|\beta_3|+(n-2)(n-3)(n-4)|\beta_4|\}|a_n|]}{(j+2)j[|\beta_1|+(j-1)|\beta_2|+(j-1)(j-2)|\beta_3|+(j-1)(j-2)(j-3)|\beta_4|]} \\ &= Q_j\end{aligned}$$

where Q_j is given by (2.7). Now

$$|f'(z)| \leq 1+\sum_{n=2}^{j} n|a_n||z|^{n-1}+\sum_{n=j+1}^{\infty} n|a_n||z|^{n-1} \leq 1+\sum_{n=2}^{j} n|a_n||z|^{n-1}+Q_j|z|^j, \tag{2.12}$$

and

$$|f'(z)| \geq 1 - \sum_{n=2}^{j} n|a_n||z|^{n-1} - \sum_{n=j+1}^{\infty} n|a_n||z|^{n-1}$$

$$\geq 1 - \sum_{n=2}^{j} n|a_n||z|^{n-1} - Q_j|z|^j. \tag{2.13}$$

Combining (2.12) and (2.13) we obtain the inequality (2.6). This completes the proof of Corollary 2.4.

3. Radius Problem for the class $M(\beta,\alpha)$

Let $\mathcal{A}_\beta$ denote the subclass of $\mathcal{A}$ consisting of all functions $f(z)$ with $a_n = |a_n|e^{i((n-1)\beta+\pi)}$ $(n = 2, 3, \cdots)$. If $\beta = 0$, then a function $f(z) \in \mathcal{A}_\beta$ becomes an analytic function with negative coefficients in $\mathbf{U}$ considered by Silverman [4].

For $f(z) \in \mathcal{A}_\beta$, define the class $\mathcal{M}(\beta,\alpha)$ by

$$\mathcal{M}(\beta,\alpha) = \mathcal{A}_\beta \cap \mathcal{M}(\alpha) \quad (0 \le \alpha < 1).$$

To obtain the radius problem for the class $\mathcal{M}(\beta,\alpha)$, we need the following lemma.

Lemma 3.1. *If $f(z) \in \mathcal{M}(\beta,\alpha)$ then*

$$\sum_{n=2}^{\infty} \frac{n(n+1)}{2}|a_n| \le 1-\alpha \quad (0 \le \alpha < 1). \tag{3.1}$$

Proof. Let $f(z) \in \mathcal{M}(\beta,\alpha)$. Then we have

$$\begin{aligned}&\Re\left[\tfrac{D^2 f(z)}{z}\right] = \Re\left[1+\textstyle\sum_{n=2}^{\infty}\tfrac{n(n+1)}{2}a_n z^{n-1}\right]\\ &= \Re\left[1+\textstyle\sum_{n=2}^{\infty}\tfrac{n(n+1)}{2}|a_n|e^{i((n-1)\beta+\pi)}z^{n-1}\right]\\ &= \Re\left[1-\textstyle\sum_{n=2}^{\infty}\tfrac{n(n+1)}{2}|a_n|e^{i(n-1)\beta}z^{n-1}\right] > \alpha,\end{aligned}$$

for all $z \in \mathbf{U}$. Let us consider a point $z \in \mathbf{U}$ such that $z = |z|e^{-i\beta}$. Then we have

$$1-\sum_{n=2}^{\infty}\frac{n(n+1)}{2}|a_n||z|^{n-1} > \alpha.$$

Therefore, letting $|z| \longrightarrow 1^{-1}$, we obtain

$$\sum_{n=2}^{\infty}\frac{n(n+1)}{2}|a_n| \le 1-\alpha.$$

Corollary 3.2. *If $f(z) \in \mathcal{M}(\beta,\alpha)$, then*

$$|a_n| \le \frac{2(1-\alpha)}{n(n+1)} \quad (n = 2, 3, \cdots).$$

Remark 3.3. *By Lemma 3.1, we observe that if $f(z) \in \mathcal{M}(\beta,\alpha)$, then*

$$\sum_{n=2}^{\infty} \frac{(n-1)(n+1)}{2}|a_n| \leq \sum_{n=2}^{\infty} \frac{n(n+1)}{2}|a_n| \leq 1-\alpha. \tag{3.2}$$

Now, we derive the following:

Theorem 3.4. *If $f(z) \in \mathcal{M}(\beta,\alpha)$ and $\delta \in \mathbf{C}$ $(0 < |\delta| < 1)$. Then the function $\frac{1}{\delta}f(\delta z) \in A(\beta_1,\beta_2,\beta_3,\beta_4,\lambda)$ for $0 < |\delta| \leq |\delta_0(\lambda)|$ where $|\delta_0(\lambda)|$ is the smallest positive root of the equation*

$$\begin{aligned} &\tfrac{|\beta_1||\delta|}{(1-|\delta|^2)^{\frac{5}{2}}}\sqrt{6(1+|\delta|^2)}\sqrt{(1-\alpha)} + \tfrac{2|\beta_2||\delta|^2}{(1-|\delta|^2)^{\frac{7}{2}}}\sqrt{9+57|\delta|^2+24|\delta|^4} \\ &\sqrt{1-\alpha-\tfrac{3}{2}|a_2|^2} \\ &+\tfrac{4|\beta_3||\delta|^3}{(1-|\delta|^2)^{\frac{9}{2}}}\sqrt{30+405|\delta|^2+675|\delta|^4+150|\delta|^6}\sqrt{1-\alpha-\tfrac{3}{2}|a_2|^2-4|a_3|^2} \\ &+\tfrac{12|\beta_4||\delta|^4}{(1-|\delta|^2)^{\frac{11}{2}}}\sqrt{75+1695|\delta|^2+805|\delta|^4+4425|\delta|^6+1000|\delta|^8} \\ &\sqrt{1-\alpha-\frac{3}{2}|a_2|^2-4|a_3|^2-\frac{15}{2}|a_4|^2} = \lambda. \end{aligned} \tag{3.3}$$

Proof. For $f(z) \in \mathcal{M}(\beta,\alpha)$, we see that

$$\frac{1}{\delta}f(\delta z) = z + \sum_{n=2}^{\infty} \delta^{n-1}a_n z^n \quad (0 < |\delta| < 1).$$

To show $\frac{1}{\delta}f(\delta z) \in A(\beta_1,\beta_2,\beta_3,\beta_4;\lambda)$. In view of Theorem 2.1, it is sufficient to prove

$$\begin{aligned} &\sum_{n=2}^{\infty} \frac{n(n+1)}{2}(n-1)[|\beta_1| + (n-2)|\beta_2| \\ &+ (n-2)(n-3)|\beta_3| + (n-2)(n-3)(n-4)|\beta_4|]|\delta|^{n-1}|a_n| \leq \lambda. \end{aligned} \tag{3.4}$$

We note that

$$\sum_{n=2}^{\infty} \frac{(n+1)(n-1)}{2}|a_n|^2 \leq 1-\alpha.$$

Applying Cauchy-Schwarz inequality to the left hand side of (3.4), we observe that

$$\begin{aligned} &\textstyle\sum_{n=2}^{\infty} \frac{n(n+1)}{2}(n-1)[|\beta_1| + (n-2)|\beta_2| \\ &+(n-2)(n-3)|\beta_3| + (n-2)(n-3)(n-4)|\beta_4|]|\delta|^{n-1}|a_n| \end{aligned}$$

$$\begin{aligned}
&\le \tfrac{|\beta_1|}{|\delta|}\left(\textstyle\sum_{n=2}^{\infty}\tfrac{n^2(n+1)(n-1)}{2}|\delta|^{2n}\right)^{\frac12}\left(\textstyle\sum_{n=2}^{\infty}\tfrac{(n+1)(n-1)}{2}|a_n|^2\right)^{\frac12}\\
&+\tfrac{|\beta_2|}{|\delta|}\left(\textstyle\sum_{n=3}^{\infty}\tfrac{n^2(n+1)(n-1)(n-2)^2}{2}|\delta|^{2n}\right)^{\frac12}\left(\textstyle\sum_{n=3}^{\infty}\tfrac{(n+1)(n-1)}{2}|a_n|^2\right)^{\frac12}\\
&+\tfrac{|\beta_3|}{|\delta|}\left(\textstyle\sum_{n=4}^{\infty}\tfrac{n^2(n+1)(n-1)(n-2)^2(n-3)^2}{2}|\delta|^{2n}\right)^{\frac12}\left(\textstyle\sum_{n=4}^{\infty}\tfrac{(n+1)(n-1)}{2}|a_n|^2\right)^{\frac12}\\
&+\tfrac{|\beta_4|}{|\delta|}\left(\textstyle\sum_{n=5}^{\infty}\tfrac{n^2(n+1)(n-1)(n-2)^2(n-3)^2(n-4)^2}{2}|\delta|^{2n}\right)^{\frac12}\left(\textstyle\sum_{n=5}^{\infty}\tfrac{(n+1)(n-1)}{2}|a_n|^2\right)^{\frac12}\\
&\le \tfrac{|\beta_1|}{|\delta|}\left(\textstyle\sum_{n=2}^{\infty}\tfrac{n^2(n+1)(n-1)}{2}|\delta|^{2n}\right)^{\frac12}\sqrt{1-\alpha}\\
&+\tfrac{|\beta_2|}{|\delta|}\left(\textstyle\sum_{n=3}^{\infty}\tfrac{n^2(n+1)(n-1)(n-2)^2}{2}|\delta|^{2n}\right)^{\frac12}\sqrt{1-\alpha-\tfrac32|a_2|^2}\\
&+\tfrac{|\beta_3|}{|\delta|}\left(\textstyle\sum_{n=4}^{\infty}\tfrac{n^2(n+1)(n-1)(n-2)^2(n-3)^2}{2}|\delta|^{2n}\right)^{\frac12}\sqrt{1-\alpha-\tfrac32|a_2|^2-4|a_3|^2}\\
&+\tfrac{|\beta_4|}{|\delta|}\left(\textstyle\sum_{n=5}^{\infty}\tfrac{n^2(n+1)(n-1)(n-2)^2(n-3)^2(n-4)^2}{2}|\delta|^{2n}\right)^{\frac12}
\end{aligned}$$

$$\sqrt{1-\alpha-\frac{3}{2}|a_2|^2-4|a_3|^2-\frac{15}{2}|a_4|^2}. \tag{3.5}$$

Making use of

$$\sum_{n=2}^{\infty} x^n = \frac{x^2}{1-x} \quad (|x|<1),$$

we have

$$\sum_{n=2}^{\infty} nx^{n+1} = x^2\Big(\sum_{n=2}^{\infty} x^n\Big)' = x^2\left(\frac{x^2}{1-x}\right) = \frac{2x^4-x^5}{(1-x)^2}.$$

Thus

$$\begin{aligned}
\frac12\sum_{n=2}^{\infty} n^2(n+1)(n-1)x^n &= \frac{x^2}{2}\left(\sum_{n=2}^{\infty} nx^{n+1}\right)''' \\
&= \frac{6x^2(1+x)}{(1-x)^5}. \qquad (3.6)
\end{aligned}$$

Since

$$\sum_{n=3}^{\infty}(n-2)x^n = \frac{x^3}{(1-x)^2},$$

we have

$$\sum_{n=3}^{\infty} n(n-2)x^{n+1} = x^2\left(\sum_{n=3}^{\infty}(n-2)x^n\right)' = \frac{3x^4-x^5}{(1-x)^3}.$$

Therefore

$$\sum_{n=3}^{\infty} \frac{n^2(n+1)(n-1)(n-2)^2}{2} x^n = \frac{x^3}{2} \left(\sum_{n=3}^{\infty} n(n-2)x^{n+1} \right)''''$$

$$= \frac{36x^3 + 228x^4 + 96x^5}{(1-x)^7}. \tag{3.7}$$

Again, we have

$$\sum_{n=4}^{\infty} (n-2)(n-3)x^n = x^4 \left(\sum_{n=4}^{\infty} x^{n-2} \right)'' = \frac{2x^4}{(1-x)^3},$$

which implies

$$\sum_{n=4}^{\infty} n(n-2)(n-3)x^{n+1} = x^2 \left(\sum_{n=4}^{\infty} (n-2)(n-3)x^n \right)' = \frac{8x^5 - 2x^6}{(1-x)^4}.$$

Therefore

$$\sum_{n=4}^{\infty} \frac{n^2(n+1)(n-1)(n-2)^2(n-3)^2}{2} x^n = \frac{x^4}{2} \left(\sum_{n=4}^{\infty} n(n-2)(n-3)x^{n+1} \right)'''''$$

$$= \frac{480x^4 + 6480x^5 + 10800x^6 + 2400x^7}{(1-x)^9}. \tag{3.8}$$

Furthermore, we have

$$\sum_{n=5}^{\infty} n(n-2)(n-3)(n-4)x^{n+1} = x^2 \left(\sum_{n=5}^{\infty} (n-2)(n-3)(n-4)x^n \right)'.$$

But

$$\sum_{n=5}^{\infty} (n-2)(n-3)(n-4)x^n = x^5 \left(\sum_{n=5}^{\infty} x^{n-2} \right)''' = \frac{6x^5}{(1-x)^4}.$$

Thus, we have

$$\sum_{n=5}^{\infty} n(n-2)(n-3)(n-4)x^{n+1} = x^2 \left(\frac{6x^5}{(1-x)^4} \right)' = \frac{30x^6 - 6x^7}{(1-x)^5}.$$

This gives us

$$\sum_{n=5}^{\infty} \frac{n^2(n+1)(n-1)(n-2)^2(n-3)^2(n-4)^2}{2} x^n$$
$$= \frac{x^5}{2} \left(\sum_{n=5}^{\infty} n(n-2)(n-3)(n-4)x^{n+1} \right)''''''$$

$$(3.9) \quad = \frac{10800x^5 + 244080x^6 + 835920x^7 + 637200x^8 + 144000x^9}{(1-x)^{11}}.$$

By using (3.6)- (3.9) with $|\delta|^2 = x$ in (3.5), we obtain

$$\sum_{n=2}^{\infty} \tfrac{n(n+1)}{2}(n-1)\left[|\beta_1| + (n-2)|\beta_2| + (n-2)(n-3)|\beta_3|\right.$$
$$\left.+(n-2)(n-3)(n-4)|\beta_4||\delta|^{n-1}|a_n|\right.$$
$$\leq \tfrac{|\beta_1||\delta|}{(1-|\delta|^2)^{\frac{5}{2}}}\sqrt{6(1+|\delta|^2)}\sqrt{(1-\alpha)} + \tfrac{2|\beta_2||\delta|^2}{(1-|\delta|^2)^{\frac{7}{2}}}\sqrt{9+57|\delta|^2+24|\delta|^4}$$
$$\sqrt{1-\alpha-\tfrac{3}{2}|a_2|^2}$$
$$+\tfrac{4|\beta_3||\delta|^3}{(1-|\delta|^2)^{\frac{9}{2}}}\sqrt{30+405|\delta|^2+675|\delta|^4+150|\delta|^6}\sqrt{1-\alpha-\tfrac{3}{2}|a_2|^2-4|a_3|^2}$$
$$+\tfrac{12|\beta_4||\delta|^4}{(1-|\delta|^2)^{\frac{11}{2}}}\sqrt{75+1695|\delta|^2+805|\delta|^4+4925|\delta|^6+1000|\delta|^8}$$
$$\sqrt{1-\alpha-\tfrac{3}{2}|a_2|^2-4|a_3|^2-\tfrac{15}{2}|a_4|^2}.$$

Let us consider the complex number $\delta \ \ (0 < |\delta| < 1)$ such that

$$\tfrac{|\beta_1||\delta|}{(1-|\delta|^2)^{\frac{5}{2}}}\sqrt{6(1+|\delta|^2)}\sqrt{(1-\alpha)} + \tfrac{2|\beta_2||\delta|^2}{(1-|\delta|^2)^{\frac{7}{2}}}\sqrt{9+57|\delta|^2+24|\delta|^4}$$
$$\sqrt{1-\alpha-\tfrac{3}{2}|a_2|^2}$$
$$+\tfrac{4|\beta_3||\delta|^3}{(1-|\delta|^2)^{\frac{9}{2}}}\sqrt{30+405|\delta|^2+675|\delta|^4+150|\delta|^6}\sqrt{1-\alpha-\tfrac{3}{2}|a_2|^2-4|a_3|^2}$$
$$+\tfrac{12|\beta_4||\delta|^4}{(1-|\delta|^2)^{\frac{11}{2}}}\sqrt{75+1695|\delta|^2+805|\delta|^4+4925|\delta|^6+1000|\delta|^8}$$
$$\sqrt{1-\alpha-\tfrac{3}{2}|a_2|^2-4|a_3|^2-\tfrac{15}{2}|a_4|^2} = \lambda.$$

If we define the function $h(|\delta|)$ by

$$h(|\delta|) = |\beta_1||\delta|(1-|\delta|^2)^3\sqrt{6(1+|\delta|^2)}\sqrt{(1-\alpha)} + 2|\beta_2||\delta|^2(1-|\delta|^2)^2$$
$$\sqrt{9+57|\delta|^2+24|\delta|^4}$$
$$\sqrt{1-\alpha-\tfrac{3}{2}|a_2|^2} + 4|\beta_3||\delta|^3(1-|\delta|^2)\sqrt{30+405|\delta|^2+675|\delta|^4+150|\delta|^6}$$
$$\sqrt{1-\alpha-\tfrac{3}{2}|a_2|^2-4|a_3|^2}$$
$$+12|\beta_4||\delta|^4\sqrt{75+1695|\delta|^2+805|\delta|^4+4425|\delta|^6+1000|\delta|^8}$$
$$\sqrt{1-\alpha-\tfrac{3}{2}|a_2|^2-4|a_3|^2-\tfrac{15}{2}|a_4|^2} - \lambda(1-|\delta|^2)^{\frac{11}{2}},$$

then we have $h(0) = -\lambda < 0$ and

$$h(1) = 60\sqrt{3}|\beta_4|\sqrt{1-\alpha-\frac{3}{2}|a_2|^2-4|a_3|^2-\frac{15}{2}|a_4|^2} \geq 0.$$

This means that there exists some δ_0 such that $h(|\delta_0|) = 0 \ \ (0 < |\delta_0| < 1)$. This completes the proof of the Theorem 3.1.

Remark 3.5. *If we take $\delta = \frac{1}{2}e^{i\beta}$ in (3.3), then*

$$\lambda = \tfrac{8}{9}\sqrt{10}|\beta_1|\sqrt{1-\alpha} + \tfrac{32\sqrt{11}}{9\sqrt{3}}|\beta_2|\sqrt{1-\alpha-\tfrac{3}{2}|a_2|^2}$$
$$+\tfrac{800\sqrt{2}}{27\sqrt{3}}|\beta_3|\sqrt{1-\alpha-\tfrac{3}{2}|a_2|^2-4|a_3|^2} + \tfrac{64\sqrt{39815}}{81\sqrt{3}}|\beta_4|$$
$$\sqrt{1-\alpha-\tfrac{3}{2}|a_2|^2-4|a_3|^2-\tfrac{15}{2}|a_4|^2}.$$

If we consider $\lambda = 1$ in (3.3) then we obtain

$$|\beta_1||\delta|(1-|\delta|^2)^3\sqrt{6(1+|\delta|^2)}\sqrt{1-\alpha} + 2|\beta_2||\delta|^2(1-|\delta|^2)^2$$
$$\sqrt{9+57|\delta|^2+24|\delta|^4}\sqrt{1-\alpha-\tfrac{3}{2}|a_2|^2}$$
$$+4|\beta_3||\delta|^3(1-|\delta|^2)\sqrt{30+405|\delta|^2+675|\delta|^4+150|\delta|^6}$$
$$\sqrt{1-\alpha-\tfrac{3}{2}|a_2|^2-4|a_3|^2}$$
$$+12|\beta_4||\delta|^4\sqrt{75+1695|\delta|^2+805|\delta|^4+4425|\delta|^6+1000|\delta|^8}$$
$$\sqrt{1-\alpha-\tfrac{3}{2}|a_2|^2-4|a_3|^2-\tfrac{15}{2}|a_4|^2} - (1-|\delta|^2)^{\frac{11}{2}} = 0.$$

4. Open Problem

For the proof of Theorem 3.1 we used Cauchy- Schwarz inequality given by

$$\sum |a_n||b_n| \leq \left(\sum |a_n|^2\right)^{\frac{1}{2}} \left(\sum |b_n|^2\right)^{\frac{1}{2}}.$$

We know that Hölder inequality i.e

$$\sum |a_n||b_n| \leq \left(\sum |a_n|^p\right)^{\frac{1}{p}} \left(\sum |b_n|^q\right)^{\frac{1}{q}} \quad \left(p > 0,\ q > 0;\ \ \frac{1}{p}+\frac{1}{q} \geq 1\right)$$

is the generalization of Cauchy-Schwarz inequality. Therefore, if we find some application of Hölder inequality for the proof of Theorem 3.4 instead of Cauchy-Schwarz inequality, then we derive new result which is the generalization of Theorem 3.4.

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References

[1] O. Kwon, Y. Sim, N. Cho, and H. Srivastava, "Some radius problems related to a certain subclass of analytic functions", *Acta Mathematica Sinica, English Series*, vol. 30, no. 7, pp. 1133–1144, Jul. 2014, doi: 10.1007/s10114-014-3100-0.

[2] S. Maharana, J. Prajapat, and H. Srivastava, "The radius of convexity of partial sums of convex functions in one direction", *Proceedings of the National Academy of Sciences, India Section A: Physical Sciences*, vol. 87, no. 2, pp. 215–219, Feb. 2017, doi: 10.1007/s40010-017-0348-7.

[3] S. Ruscheweyh, "New criteria for univalent functions", *Proceedings of the American Mathematical Society*, vol. 49, no. 1, pp. 109–109, Jan. 1975, doi: 10.1090/S0002-9939-1975-0367176-1.

[4] H. Silverman, "Univalent functions with negative coefficients", *Proceedings of the American Mathematical Society*, vol. 51, no. 1, pp. 109–109, Jan. 1975, doi: 10.1090/S0002-9939-1975-0369678-0.

[5] H. Srivastava, N. Xu, and D. Yang, "Inclusion relations and convolution properties of a certain class of analytic functions associated with the Ruscheweyh derivatives", *Journal of Mathematical Analysis and Applications,* vol. 331, no. 1, pp. 686–700, Jul. 2007, doi: 10.1016/j.jmaa.2006.09.019.

[6] N. Uyanìk, S. Owa, and E. Kadioğlu, "Some properties of functions associated with close-to-convex and starlike of order", *Applied Mathematics and Computation*, vol. 216, no. 2, pp. 381–387, Mar. 2010, doi: 10.1016/j.amc.2010.01.022.

[7] N. Uyanìk and S. Owa, "New extensions for classes of analytic functions associated with close-to-convex and starlike of order α", *Mathematical and Computer Modelling*, vol. 54, no. 1-2, pp. 359–366, Jul. 2011, doi: 10.1016/j.mcm.2011.02.020.