



Stability of two variable pexiderized quadratic functional equation in intuitionistic fuzzy Banach spaces

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Abstract:

The present work is about the stability of a Pexiderised quadratic functional equation. The study is in the framework of intuitionistic fuzzy Banach spaces. The approach is through a fixed point method. The stability studied is Hyers-Ulam-Rassias stability type.

Keywords: Hyers-Ulam stability; Pexider type functional equation; Intuitionistic fuzzy norm; Alternative fixed point theorem.

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1. Introduction

The study of Hyers-Ulam-Rassias stability for functional equations has a large literature. The line of research originated in the works of Ulam [21], Hyers [3] and Rassias [27]. It has been applied to a number of areas in mathematics like differential equation [18], functional equation [1] isometries [29] etc. Particularly, in the present context we work with certain functional equations in intuitionistic fuzzy Banach spaces and investigate the Hyers-Ulam-Rassias type stability for them.

The Hyers-Ulam-Rassias stability studies for several types of functional equations on Banach spaces are studied in works like [12, 13, 17, 23]. In fuzzy linear spaces, which are fuzzy extensions of linear spaces, such studies have been performed in works like [2, 24, 25]. In intuitionistic fuzzy Banach spaces, which are further extensions of fuzzy Banach spaces, such problems have appeared in works like [6, 9, 11, 14, 16].

Our main result is in the framework of intuitionistic fuzzy Banach spaces. It is well known that in 1965 the work of Zadeh [8] created a new tenet in mathematics which is the concept of fuzzy sets. Like most other branches of mathematics, fuzzy mathematics was introduced in linear algebra and functional analysis creating in the sequel many new concepts of which an instance in the fuzzy Banach spaces. Intuitionistic fuzzy sets are further extensions of fuzzy sets. This has led to a further generalization of fuzzy Banach spaces into the concept of intuitionistic fuzzy Banach spaces which is our choice of the mathematical framework to work upon in the present paper.

The functional equation we consider here is a two variable Pexiderized quadratic function equation. It is an extension of the ordinary quadratic functional equation. These equations have been considered for the purpose of investigating the Hyers-Ulam-Rassias stability in works like [10, 22, 26, 30].

We obtain our result by an application of a fixed point theorem on generalized metric space obtained by allowing the metric function to assume infinite value. That is, by extending the ordinary metric function to assume values in the extended number system [6]. The fixed point result which we apply is obtainable in [4, 9, 28].

2. Mathematical Background

A mapping $f : R \times R \rightarrow R$ is said to be a quadratic form in two variables if $f(x) = ax^2 + bxy + cy^2$ for all $a, b, c, x, y \in R$. If we consider X and Y are to be a real vector space and a Banach space respectively then for a mapping $f : X \times X \rightarrow Y$, consider the functional equation

$$f(x + y, z + w) + f(x - y, z - w) = 2f(x, z) + 2f(y, w). \quad (2.1)$$

Then a solution of (2.1) is called as quadratic mapping in two variables. Particularly, if $X = Y = R$, the quadratic form $f(x) = ax^2 + bxy + cy^2$ is a solution of (2.1). The form

$$f(x + y, z + w) + f(x - y, z - w) = 2g(x, z) + 2h(y, w) \quad (2.2)$$

is known as Pexiderized quadratic [19, 20] functional equation in two variables which is an extension of the above definition of quadratic functional equation.

Definition 2.1. [5] Consider the set L^* and the order relation \leq_{L^*} defined by

$$L^* = \left\{ (x_1, x_2) : (x_1, x_2) \in [0, 1]^2 \quad \text{and} \quad x_1 + x_2 \leq 1 \right\},$$

$$(x_1, x_2) \leq_{L^*} (y_1, y_2) \Leftrightarrow x_1 \leq y_1, x_2 \geq y_2, \forall (x_1, x_2), (y_1, y_2) \in L^*.$$

Then (L^*, \leq_{L^*}) is a complete lattice.

The elements $0_{L^*} = (0, 1)$ and $1_{L^*} = (1, 0)$ are its units.

Definition 2.2. [8] A fuzzy set A of a non-empty set X is characterized by a membership function μ_A which associates each point of X to a real number in the interval $[0, 1]$. With the value of $\mu_A(x)$ at x representing the grade of membership of x in A .

Definition 2.3. [7] Let E be any nonempty set. An intuitionistic fuzzy set A of E is an object of the form $A = \{(x, \mu_A(x), \nu_A(x)) : x \in E\}$, where the functions $\mu_A : E \rightarrow [0, 1]$ and $\nu_A : E \rightarrow [0, 1]$ denote the degree of membership and the degree of non-membership of the element $x \in E$ respectively and for every $x \in E$,

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1.$$

For our notational purposes we denote an intuitionistic fuzzy set on X by any function $A_{\mu, \nu} = X \rightarrow L^*$ given by $A_{\mu, \nu}(x) = (\mu_A(x), \nu_A(x))$ with $\mu_A, \nu_A : X \rightarrow [0, 1]$ satisfying $0 \leq \mu_A(x) + \nu_A(x) \leq 1$.

Definition 2.4. [5] A triangular norm (t -norm) on L^* is a mapping $\Gamma : (L^*)^2 \rightarrow L^*$ satisfying the following conditions :

- (a) $(\forall x \in L^*) (\Gamma(x, 1_{L^*}) = x)$ (boundary condition),
- (b) $(\forall (x, y) \in (L^*)^2) (\Gamma(x, y) = \Gamma(y, x))$ (commutativity),
- (c) $(\forall (x, y, z) \in (L^*)^3) (\Gamma(x, \Gamma(y, z)) = \Gamma(\Gamma(x, y), z))$ (associativity),
- (d) $(\forall (x, x', y, y') \in (L^*)^4) (x \leq_{L^*} x' \text{ and } y \leq_{L^*} y' \Rightarrow \Gamma(x, y) \leq_{L^*} \Gamma(x', y'))$ (monotonicity).

Definition 2.5. [5] A triangular conorm (t -conorm) on L^* is a mapping $S : (L^*)^2 \rightarrow L^*$ satisfying the following conditions :

- (a) $(\forall x \in L^*) (S(x, 0_{L^*}) = x)$ (boundary condition),
- (b) $(\forall (x, y) \in (L^*)^2) (S(x, y) = S(y, x))$ (commutativity),
- (c) $(\forall (x, y, z) \in (L^*)^3) (S(x, S(y, z)) = S(S(x, y), z))$ (associativity),
- (d) $(\forall (x, x', y, y') \in (L^*)^4) (x \leq_{L^*} x' \text{ and } y \leq_{L^*} y' \Rightarrow S(x, y) \leq_{L^*} S(x', y'))$ (monotonicity).

If Γ is continuous then Γ is said to be a continuous t -norm.

Definition 2.6. [5] A continuous t-norm Γ on L^* is said to be continuous t-representable if there exists a continuous t-norm $*$ and there exists a continuous t-conorm \diamond on $[0, 1]$ such that for all $x = (x_1, x_2), y = (y_1, y_2) \in L^*, \Gamma(x, y) = (x_1 * y_1, x_2 \diamond y_2)$

We now define the iterated sequence Γ^n recursively by $\Gamma^1 = \Gamma$ and

$$\Gamma^n(x^{(1)}, x^{(2)}, \dots, x^{(n+1)}) = \Gamma(\Gamma^{(n-1)}(x^{(1)}, x^{(2)}, \dots, x^{(n)}), x^{(n+1)}),$$

$$\forall n \geq 2, x^{(i)} \in L^*.$$

Intuitionistic fuzzy normed linear space was defined by Saadati [15]. Shakeri [22] has stated this definition in more compact form. We state the definition in the form used by Shakeri [22].

Definition 2.7. The triple $(X, P_{\mu, \nu}, \tau)$ is said to be an intuitionistic fuzzy normed space (briefly IFN-space) if X is a vector space, τ is a continuous t-norm and $P_{\mu, \nu}$ is a mapping $X \times (0, \infty) \rightarrow L^*$ which is an intuitionistic fuzzy set satisfying the following conditions :

for all $x, y \in X$ and $t, s > 0$,

$$(i) P_{\mu, \nu}(x, 0) = 0_{L^*};$$

$$(ii) P_{\mu, \nu}(x, t) = 1_{L^*} \text{ if and only if } x = 0;$$

$$(iii) P_{\mu, \nu}(\alpha x, t) = P_{\mu, \nu}\left(x, \frac{t}{|\alpha|}\right) \text{ for all } \alpha \neq 0;$$

$$(iv) P_{\mu, \nu}(x + y, t + s) \geq_{L^*} \Gamma(P_{\mu, \nu}(x, t), P_{\mu, \nu}(y, s)).$$

It can be noted that $P_{\mu, \nu}$ has the form $P_{\mu, \nu}(x, t) = (\mu_x(t), \nu_x(t)) = (\mu(x, t), \nu(x, t))$ such that $0 \leq \mu_x(t) + \nu_x(t) \leq 1$ for all $x \in X$ and $t > 0$. Then with μ and ν the above definition reduces to the more explicit form used in [15].

Example 2.8. Let $(X, \|\cdot\|)$ be a normed linear space. Let

$$M(a, b) = (\min\{a_1, b_1\}, \max\{a_2, b_2\})$$

for all $a = (a_1, a_2), b = (b_1, b_2) \in L^*$ and for $a, b \in [0, 1]$. Then $M(a, b)$ is continuous t-norm.

Definition 2.9. (1) A sequence $\{x_n\}$ in an IFN-space $(X, P_{\mu,\nu}, M)$ is called a Cauchy sequence if for any $\varepsilon > 0$ and $t > 0$, there exists $n_0 \in N$ such that

$$P_{\mu,\nu}(x_n - x_m, t) >_{L^*} (1 - \varepsilon, \varepsilon), \forall n, m \geq n_0.$$

(2) The sequence $\{x_n\}$ is said to be convergent to a point $x \in X$ if

$$P_{\mu,\nu}(x_n - x, t) \rightarrow 1_{L^*} \text{ as } n \rightarrow \infty \text{ for every } t > 0.$$

(3) IFN-space $(X, P_{\mu,\nu}, M)$ is said to be complete if every Cauchy sequence in X is convergent to a point $x \in X$. A complete intuitionistic fuzzy normed space is called an intuitionistic fuzzy Banach space.

We require the following fixed point result in generalized metric spaces to establish our result of stability in this paper.

Definition 2.10. Let X be a nonempty set. A function $d : X \times X \rightarrow [0, \infty]$ is called a generalized metric on X if d satisfies

- (i) $d(x, y) = 0$ if and only if $x = y$;
- (ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
- (iii) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then (X, d) is called a generalized metric space.

Theorem 2.11. ([4] and [9])

Let (X, d) be a complete generalized metric space and let $J : X \rightarrow X$ be a strictly contractive mapping with Lipschitz constant $0 < L < 1$, that is,

$$d(Jx, Jy) \leq Ld(x, y),$$

for all $x, y \in X$.

Then for each $x \in X$, either

$$d(J^n x, J^{n+1} x) = \infty, \forall n \geq 0$$

or,

$$d(J^n x, J^{n+1} x) < \infty \quad \forall n \geq n_o$$

for some non-negative integers n_0 . Moreover, if the second alternative

holds then

- (1) the sequence $\{J^n x\}$ converges to a fixed point y^* of J ;
- (2) y^* is the unique fixed point of J in the set

$$Y = \{y \in X : d(J^{n_0}x, y) < \infty\};$$

$$(3) d(y, y^*) \leq (\frac{1}{1-L})d(y, Jy) \text{ for all } y \in Y.$$

3. The Hyers-Ulam-Rassias Stability Result

Throughout this section we assumed that X is a real vector space, $(Y, P_{\mu, \nu}, M)$ is a complete IFN-space and $(Z, P'_{\mu, \nu}, M)$ is an IFN-space and denote

$$Df(x, y, z, w) = f(x+y, z+w) + f(x-y, z-w) - 2g(x, z) - 2h(y, w) \quad (3.1)$$

Theorem 3.1. Let $\phi : X \times X \times X \times X \rightarrow Z$ be a function such that

$$P'_{\mu, \nu}(\phi(2x, 2y, 2z, 2w), t) \geq {}_{L^*}P'_{\mu, \nu}(\alpha\phi(x, y, z, w), t)$$

for some real α with $0 < \alpha < 2$ and

$$\lim_{n \rightarrow \infty} P'_{\mu, \nu}(\phi(2^n x, 2^n y, 2^n z, 2^n w), 2^n t) = 1_{L^*}$$

for all $x, y, z, w \in X$ and $t > 0$. If $f, g, h : X \times X \times X \times X \rightarrow Y$ are mappings with $f(-x, -y) = -f(x, y)$ and $g(0, 0) = 0$ such that

$$(3.2) \quad P_{\mu, \nu}(Df(x, y, z, w), t) \geq {}_{L^*}P'_{\mu, \nu}(\phi(x, y, z, w), t)$$

$(x, y, z, w \in X, t > 0)$, where $Df(x, y, z, w)$ is given by (3.1). Then there exists a unique additive mapping $A : X \times X \rightarrow Y$ where $A(x, z) \rightarrow \left(\frac{f(2^n x, 2^n z)}{2^n}\right)$ as $n \rightarrow \infty$ for all $x, z \in X$ satisfying

$$(3.3) \quad P_{\mu, \nu}(f(x, z) - A(x, z), t) \geq {}_{L^*}M_1((x, z), t(2 - \alpha))$$

and

$$(3.4) \quad P_{\mu, \nu}(A(x, z) - g(x, z) - h(x, z), t) \geq {}_{L^*}M_1\left((x, z), \frac{t \times 3(2 - \alpha)}{5 - \alpha}\right)$$

where

$$(3.5) \quad M_1((x, z), t) = M^2 \left\{ P'_{\mu, \nu} \left(\phi(x, x, z, z), \frac{t}{3} \right), \right.$$

$$\left. P'_{\mu, \nu} \left(\phi(x, 0, z, 0), \frac{t}{3} \right), P'_{\mu, \nu} \left(\phi(0, x, 0, z), \frac{t}{3} \right) \right\}.$$

Proof. Putting $x = y = z = w = 0$ in the equation (2.2) we see that $f(0, 0) = g(0, 0) + h(0, 0)$ and since $f(-x, -y) = -f(x, y)$ we have $f(0, 0) = 0$ also by the condition $g(0, 0) = 0$ gives $h(0, 0) = 0$.

Now replacing the role of x, y, z, w by y, x, w, z respectively in (3.2) we get

$$P_{\mu, \nu}(f(x + y, z + w) - f(x - y, z - w) - 2g(x, z) - 2h(y, w), t)$$

$$(3.6) \quad \geq_{L^*} P'_{\mu, \nu}(\phi(y, x, w, z), t)$$

Also using (3.2) and (3.6) we have

$$P_{\mu, \nu}(2f(x + y, z + w) - 2g(x, z) - 2h(y, w) - 2g(y, w) - 2h(x, z), 2t)$$

$$\geq_{L^*} M \{P'_{\mu, \nu}(\phi(x, y, z, w), t), P'_{\mu, \nu}(\phi(y, x, w, z), t)\}$$

that is,

$$P_{\mu, \nu}(f(x + y, z + w) - g(x, z) - h(y, w) - g(y, w) - h(x, z), t)$$

$$(3.7) \quad \geq_{L^*} M \{P'_{\mu, \nu}(\phi(x, y, z, w), t), P'_{\mu, \nu}(\phi(y, x, w, z), t)\}$$

Now putting $y = 0, w = 0$ in (3.7) we have

$$P_{\mu, \nu}(f(x, z) - g(x, z) - g(0, 0) - h(0, 0) - h(x, z), t)$$

$$\geq_{L^*} M \{P'_{\mu, \nu}(\phi(x, 0, z, 0), t), P'_{\mu, \nu}(\phi(0, x, 0, z), t)\}.$$

That is,

$$P_{\mu, \nu}(f(x, z) - g(x, z) - h(x, z), t)$$

$$\geq_{L^*} M \{ P'_{\mu, \nu} (\phi(x, 0, z, 0), t), P'_{\mu, \nu} (\phi(0, x, 0, z), t) \} \quad (3.8)$$

Again replacing x by y and z by w respectively in (3.8) we get

$$P_{\mu, \nu}(f(y, w) - g(y, w)) - h(y, w), t)$$

$$\geq_{L^*} M \{ P'_{\mu, \nu} (\phi(y, 0, w, 0), t), P'_{\mu, \nu} (\phi(0, y, 0, w), t) \} \quad (3.9)$$

Hence from (3.7), (3.8) and (3.9) we have

$$\begin{aligned} & P_{\mu, \nu}(f(x + y, z + w) - f(x, z) - f(y, w), 3t) \\ & \geq_{L^*} M^5 \{ P'_{\mu, \nu} (\phi(x, y, z, w), t), P'_{\mu, \nu} (\phi(y, x, w, z), t) \\ & \quad P'_{\mu, \nu} (\phi(x, 0, z, 0), t), P'_{\mu, \nu} (\phi(0, x, 0, z), t) \\ & \quad P'_{\mu, \nu} (\phi(y, 0, w, 0), t), P'_{\mu, \nu} (\phi(0, y, 0, w), t) \}. \end{aligned}$$

Therefore

$$\begin{aligned} & P_{\mu, \nu}(f(x + y, z + w) - f(x, z) - f(y, w), t) \\ & \geq_{L^*} M^5 \left\{ P'_{\mu, \nu} \left(\phi(x, y, z, w), \frac{t}{3} \right), P'_{\mu, \nu} \left(\phi(y, x, w, z), \frac{t}{3} \right) \right. \\ & \quad \left. P'_{\mu, \nu} \left(\phi(x, 0, z, 0), \frac{t}{3} \right), P'_{\mu, \nu} \left(\phi(0, x, 0, z), \frac{t}{3} \right) \right. \\ (3.10) \quad & \quad \left. P'_{\mu, \nu} \left(\phi(y, 0, w, 0), \frac{t}{3} \right), P'_{\mu, \nu} \left(\phi(0, y, 0, w), \frac{t}{3} \right) \right\}. \end{aligned}$$

Now putting $y = x$ and $w = z$ in (3.10)

$$\begin{aligned} & P_{\mu, \nu}(f(2x, 2z) - 2f(x, z), t) \\ & \geq_{L^*} M^5 \left\{ P'_{\mu, \nu} \left(\phi(x, x, z, z), \frac{t}{3} \right), P'_{\mu, \nu} \left(\phi(x, x, z, z), \frac{t}{3} \right) \right. \\ & \quad \left. P'_{\mu, \nu} \left(\phi(x, 0, z, 0), \frac{t}{3} \right), P'_{\mu, \nu} \left(\phi(0, x, 0, z), \frac{t}{3} \right) \right. \\ & \quad \left. P'_{\mu, \nu} \left(\phi(x, 0, z, 0), \frac{t}{3} \right), P'_{\mu, \nu} \left(\phi(0, x, 0, z), \frac{t}{3} \right) \right\} \end{aligned}$$

$$\begin{aligned}
&= M^2 \left\{ P'_{\mu, \nu} \left(\phi(x, x, z, z), \frac{t}{3} \right), P'_{\mu, \nu} \left(\phi(x, 0, z, 0), \frac{t}{3} \right), \right. \\
&\quad \left. P'_{\mu, \nu} \left(\phi(0, x, 0, z), \frac{t}{3} \right) \right\} \\
&= M_1((x, z), t)
\end{aligned} \tag{3.11}$$

Now consider the set $E := \{g : X \times X \rightarrow Y\}$ and introduce a complete generalized metric on E by

$$\begin{aligned}
d(g, h) &= \inf \{k \in R^+ : P_{\mu, \nu}(g(x, z) - h(x, z), kt) \geq_{L^*} M_1((x, z), t) \forall x, z \in X, t > 0\} \text{ and } g, h \in E \text{ and a mapping } J : E \rightarrow E \text{ by } Jg(x, z) = \frac{1}{2}g(2x, 2z) \text{ for all } g \in E \text{ and } x, z \in X.
\end{aligned}$$

Now we examine that J is a strictly contracting mapping of E with the Lipschitz constant $\frac{\alpha}{2}$.

Let $g, h \in E$ and $\epsilon > 0$. Then there exists $k' \in R^+$ satisfying $P_{\mu, \nu}(g(x, z) - h(x, z), k't) \geq_{L^*} M_1((x, z), t)$ such that $d(g, h) \leq k' < d(g, h) + \epsilon$

Then

$$\begin{aligned}
&\inf \{k \in R^+ : P_{\mu, \nu}(g(x, z) - h(x, z), kt) \geq_{L^*} M_1((x, z), t)\} \\
&\leq k' < d(g, h) + \epsilon.
\end{aligned}$$

that is,

$$\begin{aligned}
&\inf \left\{ k \in R^+ : P_{\mu, \nu} \left(\frac{g(2x, 2z)}{2} - \frac{h(2x, 2z)}{2}, \frac{kt}{2} \right) \geq_{L^*} M_1((2x, 2z), t) \right\} \\
&\leq d(g, h) + \epsilon
\end{aligned}$$

that is,

$$\begin{aligned}
&\inf \left\{ k \in R^+ : P_{\mu, \nu}(Jg(x, z) - Jh(x, z), \frac{kt}{2}) \geq_{L^*} M_1(2x, 2z), t \right\} \\
&\leq d(g, h) + \epsilon
\end{aligned}$$

that is,

$$\inf \left\{ k \in R^+ : P_{\mu, \nu}(Jg(x, z) - Jh(x, z), \frac{k \alpha t}{2}) \geq_{L^*} M_1((x, z), t) \right\} \leq d(g, h) + \epsilon$$

$$\text{as } M_1((2^n x, 2^n z), t) = M_1((x, z), \frac{t}{\alpha^n})$$

$$\text{or, } d\left(\frac{2}{\alpha}(Jg, Jh)\right) < d(g, h) + \epsilon$$

$$\text{or, } d\{(Jg, Jh)\} < \frac{\alpha}{2}\{d(g, h) + \epsilon\}.$$

$$\text{Taking } \epsilon \rightarrow 0 \text{ we get } d\{(Jg, Jh)\} < \frac{\alpha}{2}\{d(g, h)\}.$$

Hence we see that J is strictly contractive mapping with Lipschitz constant $\frac{\alpha}{2}$.

$$\text{Now from (3.11) } d(f, Jf) \leq \frac{1}{2},$$

$$\text{and } d(Jf, J^2 f) \leq \frac{\alpha}{2} d(f, Jf) < \infty.$$

Replacing x and z by $2^n x$ and $2^n z$ respectively in (3.11) we get

$$P_{\mu, \nu}(f(2^{n+1}x, 2^{n+1}z) - 2f(2^n x, 2^n z), t) \geq_{L^*} M_1((2^n x, 2^n z), t)$$

$$\text{or, } P_{\mu, \nu}\left(\frac{F(2^{n+1}x, 2^{n+1}z)}{2^{n+1}} - \frac{F(2^n x, 2^n z)}{2^n}, \frac{t}{2^{n+1}}\right) \geq_{L^*} M_1((2^n x, 2^n z), t)$$

$$\geq_{L^*} M_1((x, z), \frac{t}{\alpha^n})$$

$$\text{or, } P_{\mu, \nu}\left(J^{n+1}f(x, z) - J^n f(x, z), t \frac{(\frac{\alpha}{2})^n}{2}\right) \geq_{L^*} M_1((x, z), t)$$

Thus $d(J^{n+1}f, J^n f) \leq \frac{1}{2}(\frac{\alpha}{2})^n < \infty$ as Lipschitz constant $\frac{\alpha}{2} < 1$ for $n \geq n_0 = 1$.

Hence by the fixed point result in Theorem 2.11 there exists a mapping $A : X \times X \rightarrow Y$ such that the following holds

1. A is a fixed point of J ,
- that is, $A(2x, 2z) = 2A(x, z)$ for all $x, z \in X$.

Therefore the mapping A is a unique fixed point of J in the set

$$E_1 = \{g \in E : d(J^{n_0} f, g) = d(J f, g) < \infty\}.$$

Thus $d(J f, A) < \infty$.

Again from (3.11) $d(J f, f) \leq \frac{1}{2} < \infty$.

Hence $F \in E_1$.

Now, $d(f, A) \leq d(f, J f) + d(J f, A) < \infty$.

Thus there exists $k \in (0, \infty)$ satisfying

$$P_{\mu, \nu}(f(x, z) - A(x, z), kt) \geq_{L^*} M_1((x, z), t)$$

for all $x, z \in X, t > 0$;
that is,

$$P_{\mu, \nu}(f(x, z) - A(x, z), kt) \geq_{L^*} M_1((x, z), t).$$

2. $d(J^n f, A)$

$$= \inf \{k \in R^+ : P_{\mu, \nu}(J^n f(x, z) - A(x, z), (\alpha/2)^n kt) \geq_{L^*} M_1((x, z), t)\}$$

since, $P_{\mu, \nu}(f(2^n x, 2^n z) - A(2^n x, 2^n z), kt) \geq_{L^*} M_1((x, z), (\frac{1}{\alpha})^n t)$
and $A(2^n x, 2^n z) = 2^n A(x, z)$.

Therefore $d(J^n f, A) \leq k(\frac{\alpha}{2})^n \rightarrow 0$ as $n \rightarrow \infty$. This implies the equality

$$(3.12) \quad A(x, z) = \lim_{n \rightarrow \infty} J^n f(x, z) = \lim_{n \rightarrow \infty} \frac{f(2^n x, 2^n z)}{2^n}$$

for all $x, z \in X$.

3. $d(f, A) \frac{1}{1-L} d(f, J f)$ with $f \in E_1$ which implies the inequality

$$d(f, A) \leq \frac{1}{1-\frac{\alpha}{2}} \times \frac{1}{2} = \frac{1}{2-\alpha}$$

then it follows that

$$P_{\mu, \nu}(A(x, z) - f(x, z), \frac{1}{2-\alpha} t) \geq_{L^*} M_1((x, z), t).$$

It implies that

$$P_{\mu,\nu}(A(x, z) - f(x, z), t) \geq_{L^*} M_1((x, z), (2 - \alpha)t).$$

That is,

$$(3.13) \quad P_{\mu,\nu}(A(x, z) - f(x, z), t) \geq_{L^*} M_1((x, z), (2 - \alpha)t)$$

for all $x, z \in X; t > 0$.

Replacing x, y, z and w by $2^n x, 2^n y, 2^n z$ and $2^n w$ respectively in (3.10) we have

$$\begin{aligned} P_{\mu,\nu} \left(\frac{f(2^n(x+y), 2^n(z+w))}{2^n} - \frac{f(2^n x, 2^n z)}{2^n} - \frac{f(2^n y, 2^n w)}{2^n}, t \right) \\ \geq_{L^*} M^5 \left\{ P'_{\mu,\nu} \left(\phi(2^n x, 2^n y, 2^n z, 2^n w), \frac{2^n t}{3} \right) \right. \\ \left. P'_{\mu,\nu} \left(\phi(2^n y, 2^n x, 2^n w, 2^n z), \frac{2^n t}{3} \right), \right. \\ \left. P'_{\mu,\nu} \left(\phi(2^n x, 0, 2^n z, 0), \frac{2^n t}{3} \right), P'_{\mu,\nu} \left(\phi(0, 2^n x, 0, 2^n z), \frac{2^n t}{3} \right) \right., \\ \left. P'_{\mu,\nu} \left(\phi(2^n y, 0, 2^n w, 0), \frac{2^n t}{3} \right), P'_{\mu,\nu} \left(\phi(0, 2^n y, 0, 2^n w), \frac{2^n t}{3} \right) \right\} \end{aligned} \quad (3.14)$$

Now by the condition

$$\lim_{n \rightarrow \infty} P'_{\mu,\nu}(\phi(2^n x, 2^n y, 2^n z, 2^n w), 2^n t) = 1_{L^*}$$

for all $x, y, z, w \in X, t > 0$ and taking the limit as $n \rightarrow \infty$ in (3.14) we obtain

$$P_{\mu,\nu}(A(x+y, z+w) - A(x, z) - A(y, w), t) = 1_{L^*}$$

This implies that

$$(3.15) \quad A(x+y, z+w) = A(x, z) + A(y, w).$$

Therefore A is additive.

Now using (3.8) and (3.13) we have

$$\begin{aligned}
& P_{\mu, \nu}(A(x, z) - g(x, z) - h(x, z), t \frac{5-\alpha}{3}) \\
&= P_{\mu, \nu}(A(x, z) - f(x, z) + f(x, z) - g(x, z) - h(x, z), t + \frac{2-\alpha}{3}t) \\
&\geq {}_{L^*} M(P_{\mu, \nu}(A(x, z) - f(x, z), t), \\
&P_{\mu, \nu}\left(f(x, z) - g(x, z) - h(x, z), \frac{2-\alpha}{3}t\right) \\
&\geq {}_{L^*} M(M_1((x, z), (2-\alpha)t), M\left(P'_{\mu, \nu}\left(\phi(x, 0, z, 0), \frac{2-\alpha}{3}t\right)\right), \\
&P'_{\mu, \nu}\left(\phi(0, x, 0, z), \frac{2-\alpha}{3}t\right) \geq {}_{L^*} M(M_1((x, z), (2-\alpha)t), \\
&M_1((x, z), (2-\alpha)t) \geq {}_{L^*} M_1((x, z), (2-\alpha)t).
\end{aligned}$$

Therefore

$$P_{\mu, \nu}(A(x, z) - g(x, z) - h(x, z), t) \geq {}_{L^*} M_1\left((x, z), \frac{t \times 3(2-\alpha)}{5-\alpha}\right).$$

The uniqueness of A follows from the fixed point theorem. Hence completes the proof of the theorem. —

Corollary 3.2. Let X be norm linear space with norm $\|\cdot\|$, $(Z, P'_{\mu, \nu}, M)$ be an IFN-space, $(Y, P_{\mu, \nu}, M)$ be a complete IFN-space and $z_0 \in Z$. If $p < 1$ and $f, g, h : X \rightarrow Y$ are mappings with $f(-x, -y) = -f(x, y)$ and $g(0, 0) = 0$ such that

$$\begin{aligned}
& P_{\mu, \nu}(f(x+y, z+w) + f(x-y, z-w) - 2g(x, z) - 2h(y, w), t) \\
&\geq {}_{L^*} P'_{\mu, \nu}(z_0(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p), t) \\
&\quad (x, y, z, w \in X, t > 0, z_0 \in Z)
\end{aligned}$$

then there exists a unique additive mapping $A : X \times X \rightarrow Y$ such that $P_{\mu, \nu}(f(x, z) - A(x, z), t) \geq {}_{L^*} P'_{\mu, \nu}(z_0(\|x\|^p + \|z\|^p), \frac{t}{6}(2 - 2^p))$

and

$$P_{\mu, \nu}(A(x, z) - g(x, z) - h(x, z), t) \geq {}_{L^*} P'_{\mu, \nu}(z_0(\|x\|^p + \|z\|^p), \frac{(2-2^p)}{10-2^{p+1}}t) \text{ for all } x, z \in X \text{ and } t > 0, z_0 \in Z.$$

Proof : Define $\phi(x, y, z, w) = z_0(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p)$ and it can be proved by similar way as Theorem 3.1 by $\alpha = 2^p$.

Theorem 3.3. Let $\phi : X \times X \times X \times X \rightarrow Z$ be a function such that

$$(3.16) \quad P'_{\mu,\nu}(\phi(2x, 2y, 2z, 2w), t) \geq {}_{L^*}P'_{\mu,\nu}(\alpha\phi(x, y, z, w), t)$$

for some real α with $0 < \alpha < 4$ and

$$\lim_{n \rightarrow \infty} P'_{\mu,\nu}(\phi(2^n x, 2^n y, 2^n z, 2^n w), 4^n t) = 1_{L^*}$$

for all $x, y, z, w \in X$ and $t > 0$. If $f, g, h : X \times X \times X \times X \rightarrow Y$ are mappings with $f(-x, -y) = f(x, y)$ and $f(0, 0) = g(0, 0) = 0$ such that

$$(3.17) \quad P_{\mu,\nu}(Df(x, y, z, w), t) \geq {}_{L^*}P'_{\mu,\nu}(\phi(x, y, z, w), t)$$

where $Df(x, y, z, w)$ is given by (3.1). Then there exists a unique quadratic mapping $Q : X \times X \rightarrow Y$ where $Q(x, z) \rightarrow \left(\frac{f(2^n x, 2^n z)}{4^n}\right)$ as $n \rightarrow \infty$ for all $x, z \in X$ satisfying

$$(3.18) \quad P_{\mu,\nu}(f(x, z) - Q(x, z), t) \geq {}_{L^*}M_1((x, z), t(4 - \alpha))$$

and

$$P_{\mu,\nu}(Q(x, z) - g(x, z), t) \geq {}_{L^*}M_1\left((x, z), \frac{t \times 6(4 - \alpha)}{10 - \alpha}\right)$$

(3.19)
also

$$P_{\mu,\nu}(Q(x, z) - h(x, z), t) \geq {}_{L^*}M_1\left((x, z), \frac{t \times 6(4 - \alpha)}{10 - \alpha}\right)$$

where $Df(x, y, z, w)$ and $M_1((x, z), t)$ are given by (3.1) and (3.5) respectively.

Proof. Putting $y = x$ and $w = z$ in (3.17)

$$P_{\mu,\nu}(f(2x, 2z) + f(0, 0) - 2g(x, z) - 2h(x, z), t)$$

$$\geq {}_{L^*}P'_{\mu,\nu}(\phi(x, x, z, z), t).$$

That is,

$$(3.20) \quad P_{\mu,\nu}(f(2x, 2z) - 2g(x, z) - 2h(x, z), t) \geq {}_{L^*}P'_{\mu,\nu}(\phi(x, x, z, z), t)$$

Also putting $x = 0$ and $z = 0$ in (3.17)

$$P_{\mu,\nu}(2f(y, w) - 2h(y, w) - 2g(0, 0), t) \geq_{L^*} P'_{\mu,\nu}(\phi(0, y, 0, w), t).$$

That is,

$$P_{\mu,\nu}(2f(y, w) - 2h(y, w), t) \geq_{L^*} P'_{\mu,\nu}(\phi(0, y, 0, w), t). \quad (3.21)$$

Again putting $y = 0$ and $w = 0$ in (3.17)

$$P_{\mu,\nu}(2f(x, z) - 2g(x, z) - 2h(0, 0), t) \geq_{L^*} P'_{\mu,\nu}(\phi(x, 0, z, 0), t) \quad (3.22)$$

That is,

$$P_{\mu,\nu}(2f(x, z) - 2g(x, z), t) \geq_{L^*} P'_{\mu,\nu}(\phi(x, 0, z, 0), t). \quad (3.23)$$

Now using (3.17), (3.21), (3.23)

$$\begin{aligned} & P_{\mu,\nu}\{f(x+y, z+w) + f(x-y, z-w) - 2f(x, z) - 2f(y, w), 3t\} \\ &= P_{\mu,\nu}\{f(x+y, z+w) + f(x-y, z-w) - 2g(x, z) - 2h(y, w) - \\ &\quad \{2f(x, z) - 2g(x, z)\} - \{2f(y, w) - 2h(y, w)\}, 3t\} \\ &\geq_{L^*} M^2 \{P'_{\mu,\nu}(\phi(x, y, z, w), t), P'_{\mu,\nu}(\phi(x, 0, z, 0), t), \\ &\quad P'_{\mu,\nu}(\phi(0, y, 0, w), t)\}. \end{aligned}$$

Therefore

$$\begin{aligned} & P_{\mu,\nu}\{f(x+y, z+w) + f(x-y, z-w) - 2f(x, z) - 2f(y, w), t\} \\ &\geq_{L^*} M^2 \left\{ P'_{\mu,\nu}\left(\phi(x, y, z, w), \frac{t}{3}\right), P'_{\mu,\nu}\left(\phi(x, 0, z, 0), \frac{t}{3}\right), \right. \\ (3.24) \quad &\quad \left. P'_{\mu,\nu}\left(\phi(0, y, 0, w), \frac{t}{3}\right) \right\}. \end{aligned}$$

Now putting $y = x$ and $w = z$ in (3.24) we get

$$P_{\mu,\nu}(f(2x, 2z) - 4f(x, z), t)$$

$$\begin{aligned} & \geq_{L^*} M^2 \left(P'_{\mu, \nu} \left(\phi(x, x, z, z), \frac{t}{3} \right), P'_{\mu, \nu} \left(\phi(x, 0, z, 0), \frac{t}{3} \right), \right. \\ & \quad \left. P'_{\mu, \nu} \left(\phi(0, x, 0, z), \frac{t}{3} \right) \right) = M_1((x, z), t). \end{aligned}$$

Thus

$$P_{\mu, \nu}(f(2x, 2z) - 4f(x, z), t) \geq_{L^*} M_1((x, z), t)$$

Now we consider the set $E := \{g : X \times X \rightarrow Y\}$ and introduce a complete generalized metric on E and define a mapping $J : E \rightarrow E$ by $Jg(x, z) = \frac{1}{4}g(2x, 2z)$ for all $g \in E$ and $x, z \in X$. By similar process as before J is strictly contractive mapping with Lipschitz constant $\frac{\alpha}{4}$ and $d(f, Jf) \leq \frac{1}{4}$.

Therefore by the alternative fixed point Theorem 2.11 there exists a mapping $Q : X \rightarrow Y$ such that the following holds

1. Q is a fixed point of J , that is, $Q(2x, 2z) = 4Q(x, z)$ for all $x, z \in X$.

2. $d(J^n f, Q) \leq k(\frac{\alpha}{4})^n \rightarrow 0$ as $n \rightarrow \infty$. This implies the equality

$$Q(x, z) = \lim_{n \rightarrow \infty} J^n f(x, z) = \lim_{n \rightarrow \infty} \frac{f(2^n x, 2^n z)}{4^n}.$$

3. $d(f, Q) \leq \frac{1}{1-\frac{\alpha}{4}} d(f, Jf)$ with $f \in E_1$ that implies the inequality

$$d(f, Q) \leq \frac{1}{1-\frac{\alpha}{4}} \times \frac{1}{4} = \frac{1}{4-\alpha}.$$

Thus

$$P_{\mu, \nu}(Q(x, z) - f(x, z), \frac{1}{4-\alpha}t) \geq_{L^*} M_1((x, z), t).$$

That is,

$$P_{\mu, \nu}(Q(x, z) - f(x, z), t) \geq_{L^*} M_1((x, z), (4-\alpha)t)$$

for all $x, z \in X; t > 0$.

Also from (3.24) we obtain

$$P_{\mu, \nu}(Q(x+y, z+w) + Q(x-y, z-w) - 2Q(x, z) - 2Q(y, w), t) = 1_{L^*}$$

which implies that,

$$Q(x + y) + Q(x - y) = 2Q(x) + 2Q(y).$$

Therefore Q is quadratic.

Also from (3.23) we have

$$\begin{aligned} & P_{\mu, \nu}(Q(x, z) - g(x, z), \frac{10-\alpha}{6}t) \\ &= P_{\mu, \nu}(Q(x, z) - f(x, z) + f(x, z) - g(x, z), t + \frac{(4-\alpha)}{6}t) \\ &\geq {}_{L^*}M(P_{\mu, \nu}(Q(x, z) - f(x, z), t), \\ & P_{\mu, \nu}(f(x, z) - g(x, z), \frac{(4-\alpha)}{2.3}t)). \\ &\geq {}_{L^*}M(M_1((x, z), (4-\alpha)t), M(P'_{\mu, \nu}(\phi(x, 0, z, 0), \frac{(4-\alpha)}{3}t))) \\ &\geq {}_{L^*}M(M_1((x, z), (4-\alpha)t), M_1((x, z), (4-\alpha)t)) \\ &\geq {}_{L^*}M_1((x, z), (4-\alpha)t). \end{aligned}$$

Therefore

$$P_{\mu, \nu}(Q(x, z) - g(x, z), t) \geq {}_{L^*}M_1\left((x, z), \frac{t \times 6(4-\alpha)}{10-\alpha}\right).$$

Also we have

$$P_{\mu, \nu}(Q(x, z) - h(x, z), t) \geq {}_{L^*}M_1\left((x, z), \frac{t \times 6(4-\alpha)}{10-\alpha}\right).$$

Corollary 3.4. Let X be norm linear space with norm $\|\cdot\|$, $(Z, P'_{\mu, \nu}, M)$ be an IFN-space, $(Y, P_{\mu, \nu}, M)$ be a complete IFN-space and $z_0 \in Z$. If $p < 2$ and $f, g, h : X \times X \rightarrow Y$ are mappings with $f(-x, -y) = f(x, y)$ and $f(0, 0) = g(0, 0) = 0$ such that

$$\begin{aligned} & P_{\mu, \nu}(f(x + y, z + w) + f(x - y, z - w) - 2g(x, z) - 2h(y, w), t) \\ &\geq {}_{L^*}P'_{\mu, \nu}(z_0 (\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p), t) \\ &\quad (x, y, z, w \in X, t > 0, z_0 \in Z) \end{aligned}$$

then there exists a unique quadratic mapping $Q : X \times X \rightarrow Y$ such that $P_{\mu,\nu}(f(x, z) - Q(x, z), t) \geq_{L^*} P'_{\mu,\nu}(z_0(\|x\|^p + \|z\|^p), \frac{t}{6}(4 - 2^p))$

and

$$P_{\mu,\nu}(Q(x, z) - g(x, z), t) \geq_{L^*} P'_{\mu,\nu}\left(z_0(\|x\|^p + \|z\|^p), \frac{(4-2^p)}{10-2^p}t\right)$$

for all $x, z \in X$ and $t > 0$, $z_0 \in Z$.

Proof : Define $\phi(x, y, z, w) = z_0(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p)$ and it can be proved by similar way as Theorem 3.3 by $\alpha = 2^p$.

4. Conclusion

The findings of the present work can be summarily written as that in intuitionistic fuzzy Banach spaces for an approximate Pexiderized quadratic functional equation in two variables there exist unique linear and quadratic mappings which are appropriate approximations of the mappings involved in the approximate equation. The method used in the proof is a fixed points method. Our approach in this paper may possibly be helpful in dealing with the stabilities of other types of equations as well.

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