Odd harmonious labeling of grid graph

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Abstract:

A graph $G(p, q)$ is said to be odd harmonious if there exists an injection $f: V(G) \rightarrow \{0, 1, 2, \dots, 2q - 1\}$ such that the induced function $f^: E(G) \rightarrow \{1, 3, \dots, 2q - 1\}$ defined by $f^*(uv) = f(u) + f(v)$ is a bijection. In this paper we prove that path union of t copies of $P_m \times P_n$, path union of t different copies of $P_{m_i} \times P_{n_i}$ where $1 \leq i \leq t$, vertex union of t copies of $P_m \times P_n$, vertex union of t different copies of $P_{m_i} \times P_{n_i}$ where $1 \leq i \leq t$, one point union of path of P_n ($t.n.P_m \times P_n$), t super subdivision of grid graph $P_m \times P_n$ are odd harmonious graphs.*

Keywords: Harmonious labeling; Odd harmonious labeling; Grid graph; Path union of graphs; One point union of path of graphs; t -super subdivision of graphs.

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1. Introduction

Throughout this paper by a graph we mean a finite, simple and undirected one. For standard terminology and notation we follow Harary [3]. A graph $G(V, E)$ with p vertices and q edges is called a (p, q) – graph. The graph labeling is an assignment of integers to the set of vertices or edges or both, subject to certain conditions. An extensive survey of various graph labeling problems is available in [1]. Graham and Sloane [2] introduced harmonious labeling during their study of modular versions of additive bases problems stemming from error correcting codes. A graph G is said to be harmonious if there exists an injection $f : V(G) \rightarrow Z_q$ such that the induced function $f^* : E(G) \rightarrow Z_q$ defined by $f^*(uv) = (f(u) + f(v)) \pmod{q}$ is a bijection and f is called harmonious labeling of G . The concept of an odd harmonious labeling was due to Liang and Bai [4]. A graph G is said to be odd harmonious if there exists an injection $f : V(G) \rightarrow \{0, 1, 2, \dots, 2q - 1\}$ such that the induced function $f^* : E(G) \rightarrow \{1, 3, \dots, 2q - 1\}$ defined by $f^*(uv) = f(u) + f(v)$ is a bijection. If $f(V(G)) = \{0, 1, 2, \dots, q\}$ then f is called as strongly odd harmonious labeling and G is called as strongly odd harmonious graph. The odd harmoniousness of graph is useful for the solution of undetermined equations. The following results have been published in [4].

1. If G is an odd harmonious graph, then G is a bipartite graph. Hence any graph that contains an odd cycle is not odd harmonious.
2. If a (p, q) -graph G is odd harmonious, then $2\sqrt{q} \leq p \leq (2q - 1)$.
3. If G is an odd harmonious Eulerian graph with q edges, then $q \equiv 0, 2 \pmod{4}$.

Jeyanthi et al. [5]-[12] proved that the shadow and splitting of graph $K_{2,n}$, C_n for $n \equiv 0 \pmod{4}$, $H_{n,n}$, any two even cycles sharing a common vertex and a common edge, plus graph Pl_n , open star of plus graph $S(t.Pl_n)$, path union of plus graph Pl_n , joining of C_m and plus graph Pl_n with a path, one point union of path of plus graph $P_n^t(t.n.Pl_m)$, super subdivision of any cycle C_m with $m \geq 3$, ladder, cycle C_n for $n \equiv 0 \pmod{4}$ with $K_{1,m}$, m-shadow and m-splitting of the graphs P_n , $H_{n,n}$, $K_{r,s}$, $P_n \oplus \overline{K_2}$ and $Spl_m(C_n)$, $n \equiv 0 \pmod{4}$ subdivided shell graphs, $S_{m,n}$, spiders, m -

shadow and m -splitting graphs are odd harmonious graphs. Selvaraju et al. [13] proved that $K_{1,n}$, $S(K_{1,n})$, $T_{m,n,t}$ and $S(T_{m,n,t})$ are odd harmonious. Vaidya and Shah [14], [15] proved that shadow and splitting graphs of P_n , $K_{1,n}$, $B_{n,n}$ and super subdivision of $H_{n,n}$ are odd harmonious.

We use the following definitions in the subsequent section.

Definition 1. The Cartesian product of graphs G and H denoted as $G \times H$, is the graph with vertex set $V(G) \times V(H) = \{(u, v) / u \in V(G) \text{ and } v \in V(H)\}$ and (u, v) is adjacent to (w, v) if and only if either $u = w$ and edge $(v, v) \in E(H)$ or $v = v$ and edge $(u, w) \in E(G)$. The Cartesian product of two paths P_m and P_n denoted by $P_m \times P_n$ is known as a grid graph on mn vertices and $2mn - (m + n)$ edges.

Definition 2. Let G be a graph and $G_1, G_2, \dots, G_n$, $n \geq 2$ be n copies of graph G . Then the graph obtained by adding an edge from G_i to G_{i+1} ($i = 1, 2, \dots, n - 1$) is called path union of graph G .

Definition 3. A graph G is obtained by replacing each edge of $K_{1,t}$ by a path P_n of length n on $n + 1$ vertices is called one point union for t copies of path P_n , denoted by P_n^t .

Definition 4. A graph G is obtained by replacing each vertices of P_n^t except the central vertex by the graphs $G_1, G_2, \dots, G_n$ is known as one point union for path of graphs, denoted by $P_n^t(G_1, G_2, \dots, G_n)$ where P_n^t is the one point union of t copies of path P_n . If we replace each vertices of P_n^t except the central vertex by the graph H , that is $G_1 = G_2 = G_3 = \dots G_n = H$, such one point union of path graph, denoted by $P_n^t(t.n.H)$.

Definition 5. Let $G = (V, E)$ be a graph with p vertices and q edges. A graph H is said to be a t -super subdivision of G if H is obtained from G by replacing every edge e of G by a complete bipartite graph $K_{2,t}$ for some $t \in N$.

2. Main Results

Theorem 2.1. Path union of t copies of $P_m \times P_n$ is odd harmonious, where $m, n, t \geq 2$.

Proof. Let G be a path union of t copies of $P_m \times P_n$, $\forall m, n, t \geq 2$.

Let $x_{k,i,1}, x_{k,i,2}, \dots, x_{k,i,n}$ be the vertices of the k^{th} copy of i^{th} row vertices in G , where $1 \leq k \leq t$, $1 \leq i \leq m$. Now, we join these consecutive copies of the grid graph $P_m \times P_n$ by an edge. Join $x_{k,m,n}$ with $x_{k+1,1,1}$, $\forall 1 \leq k \leq t-1$ by an edge to form a path union of t copies of the grid graph $P_m \times P_n$.

Then $|V(G)| = tmn$ and $|E(G)| = t[2mn - (m + n) + 1] - 1$.

We define a labeling $f : V(G) \rightarrow \{0, 1, 2, \dots, 2\{t[2mn - (m + n) + 1] - 1\} - 1\}$ as follows:

$$f(x_{k,i,j}) = (i-1)(2n-1) + j-1 + (k-1)[m(2n-1) - n + 1], \quad 1 \leq i \leq m, \\ 1 \leq j \leq n, \quad 1 \leq k \leq t.$$

The induced edge labels are

$$f^*(x_{k,i,j}x_{k,i,j+1}) = 2(i-1)(2n-1) + 2j-1 + 2(k-1)[m(2n-1) - n + 1], \\ 1 \leq i \leq m, \quad 1 \leq j \leq n-1, \quad 1 \leq k \leq t;$$

$$f^*(x_{k,i,j}x_{k,i+1,j}) = 2i(2n-1) + 2(j-1) - 2n + 1 + 2(k-1)[m(2n-1) - n + 1], \\ 1 \leq i \leq m-1, \quad 1 \leq j \leq n, \quad 1 \leq k \leq t;$$

$$f^*(x_{k,m,n}x_{k+1,1,1}) = 2k[m(2n-1) - n + 1] - 1, \quad 1 \leq k \leq t-1.$$

In view of the above defined labeling pattern, path union of t copies of $P_m \times P_n$ is odd harmonious, where $m, n, t \geq 2$.

— *An odd harmonious labeling of path union of 3 copies of $P_3 \times P_4$ is shown in Figure 1.*

Figure 1: An odd harmonious labeling of path union of 3 copies of $P_3 \times P_4$.

Theorem 2.2. Path union of t different copies of $P_{m_i} \times P_{n_i}$, $1 \leq i \leq t$ is odd harmonious.

Proof. Let G be a path union of t different copies of the grid graph $P_{m_i} \times P_{n_i}$, $1 \leq i \leq t$. Let $x_{k,i,1}, x_{k,i,2}, \dots, x_{k,i,n_k}$ be the vertices of the k^{th} copy of the i^{th} row vertices in G , where $1 \leq i \leq m_k$, $1 \leq k \leq t$. Now, we join these consecutive copies of the grid graph $P_{m_i} \times P_{n_i}$, $1 \leq i \leq t$ by an edge. Join x_{k,m_k,n_k} with $x_{k+1,1,1}$, $\forall 1 \leq k \leq t-1$ by an edge to form a path union of t copies of the grid graph $P_{m_i} \times P_{n_i}$, $1 \leq i \leq t$.

Then $|V(G)| = n_1m_1 + n_2m_2 + \dots + n_tm_t$ and $|E(G)| = 2(m_1n_1 + m_2n_2 + \dots + m_tn_t) - (m_1 + m_2 + \dots + m_t) - (n_1 + n_2 + \dots + n_t) + (t-1)$.

We define a labeling $f : V(G) \rightarrow \{0, 1, 2, \dots, 2|E(G)| - 1\}$ as follows:

$$f(x_{k,i,j}) = 2[m_1n_1 + m_2n_2 + \dots + m_{k-1}n_{k-1}] - [m_1 + m_2 + \dots + m_{k-1}] - [n_1 + n_2 + \dots + n_{k-1}] + 2n_k(i-1) + j - i + (k-1), \quad 1 \leq k \leq t, \quad 1 \leq i \leq m_k, \quad 1 \leq j \leq n_k.$$

The induced edge labels are

$$f^*(x_{k,i,j}x_{k,i,j+1}) = 4[m_1n_1 + m_2n_2 + \dots + m_{k-1}n_{k-1}] - 2[m_1 + m_2 + \dots + m_{k-1}] - 2[n_1 + n_2 + \dots + n_{k-1}] + 4n_k(i-1) + 2j - 2i + 2k - 1, \quad 1 \leq k \leq t, \quad 1 \leq i \leq m_k, \quad 1 \leq j \leq n_k - 1;$$

$$f^*(x_{k,i,j}x_{k,i+1,j}) = 4[m_1n_1 + m_2n_2 + \cdots + m_{k-1}n_{k-1}] - 2[m_1 + m_2 + \cdots + m_{k-1}] - 2[n_1 + n_2 + \cdots + n_{k-1}] + 2n_k(2i-1) + 2j - 2i + 2k - 3, \quad 1 \leq k \leq t, \quad 1 \leq i \leq m_k - 1, \quad 1 \leq j \leq n_k;$$

$$f^*(x_{k,m_k,n_k}x_{k+1,1,1}) = 4[m_1n_1 + m_2n_2 + \cdots + m_{k-1}n_{k-1}] - 2[m_1 + m_2 + \cdots + m_{k-1}] - 2[n_1 + n_2 + \cdots + n_{k-1}] + 4m_kn_k - 2m_k - 2n_k + 2k - 1, \quad 1 \leq k \leq t-1.$$

In view of the above defined labeling pattern, path union of t different copies of $P_{m_i} \times P_{n_i}$, $1 \leq i \leq t$ is odd harmonious .

An odd harmonious labeling of path union of 2 copies $P_5 \times P_4$ and $P_6 \times P_8$ is shown in Figure 2.

Figure 2: An odd harmonious labeling of path union of 2 copies $P_5 \times P_4$ and $P_6 \times P_8$.

Theorem 2.3. The vertex union of t copies of $P_m \times P_n$ is odd harmonious, where $m, n \geq 2$.

Proof. Let G be a vertex union of t copies of the grid graph $P_m \times P_n$, $\forall m, n, t \geq 2$. Let $x_{k,i,1}, x_{k,i,2}, \dots, x_{k,i,n}$ be the vertices of the k^{th} copy of the i^{th} row vertices in G , where $1 \leq k \leq t$, $1 \leq i \leq m$. Now, we join these consecutive copies of the grid graph $P_m \times P_n$ by a vertex. Identifying the vertices $x_{k,m,n}$ with $x_{k+1,1,1}$, $\forall 1 \leq k \leq t-1$ to form a vertex union of t copies of the grid graph $P_m \times P_n$.

Case (i): When $m = n$

Then $|V(G)| = tn^2 - (t - 1)$ and $|E(G)| = 2n(n - 1)t$.

We define a labeling $f : V(G) \rightarrow \{0, 1, 2, \dots, 2[2n(n - 1)t] - 1\}$ as follows:

$$f(x_{k,i,j}) = (i - 1)(2n - 1) + j - 1 + (k - 1)[2n(n - 1)], \quad 1 \leq i, j \leq n, \\ 1 \leq k \leq t.$$

The induced edge labels are

$$f^*(x_{k,i,j}x_{k,i,j+1}) = 2(i - 1)(2n - 1) + 2j - 1 + 2(k - 1)[2n(n - 1)], \quad 1 \leq i \leq n, \\ 1 \leq j \leq n - 1, \quad 1 \leq k \leq t;$$

$$f^*(x_{k,i,j}x_{k,i+1,j}) = 2i(2n - 1) + 2(j - 1) - 2n + 1 + 2(k - 1)[2n(n - 1)], \\ 1 \leq i \leq n - 1, \quad 1 \leq j \leq n, \quad 1 \leq k \leq t.$$

Case(ii): When $m \neq n$

Then $|V(G)| = tmn - (t - 1)$ and $|E(G)| = [(m - 1)(2n - 1) + n - 1]t$.

We define a labeling $f : V(G) \rightarrow \{0, 1, 2, \dots, 2[(m - 1)(2n - 1) + n - 1]t - 1\}$ as follows:

$$f(x_{k,i,j}) = (i - 1)(2n - 1) + j - 1 + (k - 1)[(m - 1)(2n - 1) + n - 1], \\ 1 \leq i \leq m, \quad 1 \leq j \leq n, \quad 1 \leq k \leq t.$$

The induced edge labels are

$$f^*(x_{k,i,j}x_{k,i,j+1}) = 2(i - 1)(2n - 1) + 2j - 1 + 2(k - 1)[(m - 1)(2n - 1) + n - 1], \\ 1 \leq i \leq m, \quad 1 \leq j \leq n - 1, \quad 1 \leq k \leq t;$$

$$f^*(x_{k,i,j}x_{k,i+1,j}) = 2i(2n - 1) + 2(j - 1) - 2n + 1 + 2(k - 1)[(m - 1)(2n - 1) + n - 1], \\ 1 \leq i \leq m - 1, \quad 1 \leq j \leq n, \quad 1 \leq k \leq t.$$

In view of above defined labeling pattern, the vertex union of t copies of $P_m \times P_n$ is odd harmonious, where $m, n \geq 2$.

An odd harmonious labeling of vertex union of 3 copies of $P_3 \times P_3$ ($m = n$), is shown in Figure 3.

Figure 3: An odd harmonious labeling of vertex union of 3 copies of $P_3 \times P_3$.

An odd harmonious labeling of vertex union of 2 copies of $P_3 \times P_4$ ($m \neq n$) is shown in Figure 4.

Figure 4: An odd harmonious labeling of vertex union of 2 copies of $P_3 \times P_4$.

Theorem 2.4. *The vertex union of t different copies of the grid graph $P_{m_i} \times P_{n_i}$, $1 \leq i \leq t$ is odd harmonious.*

Proof. Let G be a vertex union of t different copies of the grid graph $P_{m_i} \times P_{n_i}$, $1 \leq i \leq t$. Let $x_{k,i,1}, x_{k,i,2}, \dots, x_{k,i,n_k}$ be the vertices of the k^{th} copy of the i^{th} row vertices in G , where $1 \leq i \leq m_k$, $1 \leq k \leq t$. Now, we join these consecutive copies of the grid graph $P_{m_i} \times P_{n_i}$, $1 \leq i \leq t$ by a vertex. Identifying the vertices x_{k,m_k,n_k} with $x_{k+1,1,1}$, $\forall 1 \leq k \leq t-1$ to form a vertex union of t copies of the grid graph $P_{m_i} \times P_{n_i}$, $1 \leq i \leq t$. Then $|V(G)| = (n_1m_1 + n_2m_2 + \dots + n_tm_t) - (t-1)$ and

$$|E(G)| = 2(m_1n_1 + m_2n_2 + \dots + m_tn_t) - (m_1 + m_2 + \dots + m_t) - (n_1 + n_2 + \dots + n_t).$$

We define a labeling $f : V(G) \rightarrow \{0, 1, 2, \dots, 2|E(G)| - 1\}$ as follows:

$$f(x_{k,1,j}) = 2[m_1n_1 + m_2n_2 + \dots + m_{k-1}n_{k-1}] - [m_1 + \dots + m_{k-1}] - [n_1 + \dots + n_{k-1}] + (j-1), \quad 1 \leq j \leq n_k, \quad 1 \leq k \leq t;$$

$$f(x_{k,i,j}) = (2n_k - 1)(i-1) + j - 1 + 2[m_1n_1 + m_2n_2 + \dots + m_{k-1}n_{k-1}] - [m_1 + \dots + m_{k-1}] - [n_1 + \dots + n_{k-1}], \quad 1 \leq j \leq n_k, \quad 2 \leq i \leq m_k, \quad 1 \leq k \leq t.$$

The induced edge labels are

$$f^*(x_{k,1,j}x_{k,1,j+1}) = 4[m_1n_1 + \dots + m_{k-1}n_{k-1}] - 2[m_1 + \dots + m_{k-1}] - 2[n_1 + \dots + n_{k-1}] + 2j - 1, \quad 1 \leq j \leq n_k, \quad 1 \leq k \leq t;$$

$$f^*(x_{k,1,j}x_{k,2,j}) = (2n_k - 1) + 2(j-1) + 4[m_1n_1 + \dots + m_{k-1}n_{k-1}] - 2[m_1 + \dots + m_{k-1}] - 2[n_1 + \dots + n_{k-1}], \quad 1 \leq j \leq n_k, \quad 1 \leq k \leq t;$$

$$f^*(x_{k,i,j}x_{k,i,j+1}) = 2(2n_k - 1)(i-1) + 2j - 1 + 4[m_1n_1 + \dots + m_{k-1}n_{k-1}] - 2[m_1 + \dots + m_{k-1}] - 2[n_1 + \dots + n_{k-1}], \quad 2 \leq i \leq m_k, \quad 1 \leq j \leq n_k - 1, \quad 1 \leq k \leq t;$$

$$f^*(x_{k,i,j}x_{k,i+1,j}) = (2n_k - 1)(2i-1) + 2(j-1) + 4[m_1n_1 + \dots + m_{k-1}n_{k-1}] -$$

$$2[m_1 + \cdots + m_{k-1}] - 2[n_1 + \cdots + n_{k-1}], 2 \leq i \leq m_k - 1, 1 \leq j \leq n_k, 1 \leq k \leq t.$$

In the view of above defined labeling pattern, vertex union of t different copies of the grid graph $P_{m_i} \times P_{n_i}$, $1 \leq i \leq t$ is odd harmonious.

An odd harmonious labeling of vertex union of 3 different copies of $P_3 \times P_3$, $P_3 \times P_4$, $P_4 \times P_5$ is shown in Figure 5.

Figure 5: An odd harmonious labeling of vertex union of 3 different copies $P_3 \times P_3$, $P_3 \times P_4$, $P_4 \times P_5$.

Theorem 2.5. The t -super subdivision of grid graph $P_m \times P_n$ is odd harmonious, where $m, n \geq 2$.

Proof. Let $x_{i,j}$ ($1 \leq i \leq n, 1 \leq j \leq m$) be the vertices of the grid graph $P_m \times P_n$. We know that the number of vertices in $P_m \times P_n$ is $p = mn$ and the number of edges $q = 2mn - (m + n)$. Let G be a graph obtained by t -super subdivision of $P_m \times P_n$. Then we see that the number of vertices in G is $P = |V(G)| = p + tq$ and the number of edges in G is $Q = |E(G)| = 2tq$.

Let $u_{i,j,k}$ ($1 \leq i \leq n - 1, 1 \leq j \leq m, 1 \leq k \leq t$) be vertices for the vertical edges in G and $v_{i,j,k}$ ($1 \leq i \leq n, 1 \leq j \leq m - 1, 1 \leq k \leq t$) be

vertices for horizontal edges in G .

We define the labeling $f : V(G) \rightarrow \{0, 1, 2, \dots, 4tq - 1\}$ as follows:

$$f(x_{i,j}) = 2t(i-1)(2m-1) + 2t(j-1), \text{ if } i \text{ is odd, } 1 \leq j \leq m;$$

$$f(x_{i,j}) = 2t(i-1)(2m-1) + 2t(m-j), \text{ if } i \text{ is even, } 1 \leq j \leq m;$$

$$f(v_{1,j,k}) = 2t(j-1) + 2k - 1, 1 \leq j \leq m-1, 1 \leq k \leq t;$$

$$f(v_{i,j,k}) = 2t(i-2)(2m-1) + 2t(m-1) + 2t-1 + 2k + 4t(j-1), \\ 1 \leq j \leq m-1, i = 3, 5, \dots, 1 \leq k \leq t;$$

$$f(v_{i,j,k}) = 2t(i-2)(2m-1) + 2t(m-1) + 2t-1 + 2k + 4t(m-j-1), \\ 1 \leq j \leq m-1, i = 2, 4, \dots, 1 \leq k \leq t;$$

$$f(u_{i,j,k}) = 2t(i-1)(2m-1) + 2t(m-1) + 2k-1 + 4t(m-j), 1 \leq j \leq m, \\ i = 3, 5, \dots, 1 \leq k \leq t;$$

$$f(u_{i,j,k}) = 2t(i-1)(2m-1) + 2t(m-1) + 2k-1 + 4t(j-1), 1 \leq j \leq m, \\ i = 2, 4, \dots, 1 \leq k \leq t.$$

The induced edge labels are

$$f^*(x_{1,j}v_{1,s,k}) = 2t(j-1) + 2t(s-1) + 2k-1, 1 \leq j \leq m, 1 \leq s \leq m-1;$$

$$f^*(x_{1,j}u_{1,s,k}) = 2t(j-1) + 2t(m-1) + 2k-1 + 4t(m-s), 1 \leq j, s \leq m;$$

$$f^*(x_{i,j}u_{l,s,k}) = 2t(i-1)(2m-1) + 2t(m-j) + 2t(l-1)(2m-1) + 2t(m-1) + 2k-1 + 4t(m-s), \\ i = 2, 4, \dots, 1 \leq j, s \leq m, l = 1, 3, 5, \dots, 1 \leq k \leq t;$$

$$f^*(x_{i,j}u_{l,s,k}) = 2t(i-1)(2m-1) + 2t(m-j) + 2t(l-1)(2m-1) + 2t(m-1) + 2k-1 + 4t(s-1), \\ i = 2, 4, \dots, 1 \leq j, s \leq m, l = 2, 4, 6, \dots, 1 \leq k \leq t;$$

$$f^*(x_{i,j}u_{l,s,k}) = 2t(i-1)(2m-1) + 2t(j-1) + 2t(l-1)(2m-1) + 2t(m-1) + 2k-1 + 4t(m-s), \\ i = 1, 3, \dots, 1 \leq j, s \leq m, l = 1, 3, 5, \dots, 1 \leq k \leq t;$$

$$f^*(x_{i,j}u_{l,s,k}) = 2t(i-1)(2m-1) + 2t(j-1) + 2t(l-1)(2m-1) + 2t(m-1) + 2k-1 + 4t(s-1), \\ i = 1, 3, \dots, 1 \leq j, s \leq m, l = 2, 4, \dots, 1 \leq k \leq t;$$

$$f^*(x_{i,j}v_{l,s,k}) = 2t(i-1)(2m-1) + 2t(j-1) + 2t(l-2)(2m-1) + 2t(m-1) + 2t-1 + 2k + 4t(s-1), \quad i = 1, 3, \dots, 1 \leq j \leq m, 1 \leq s \leq m-1, \\ l = 3, 5, \dots, 1 \leq k \leq t;$$

$$f^*(x_{i,j}v_{l,s,k}) = 2t(i-1)(2m-1) + 2t(j-1) + 2t(l-2)(2m-1) + 2t(m-1) + 2t-1 + 2k + 4t(m-s-1), \quad i = 3, 5, \dots, 1 \leq j \leq m, 1 \leq s \leq m-1, \\ l = 2, 4, \dots, 1 \leq k \leq t;$$

$$f^*(x_{i,j}v_{l,s,k}) = 2t(i-1)(2m-1) + 2t(m-j) + 2t(l-2)(2m-1) + 2t(m-1) + 2t-1 + 2k + 4t(s-1), \quad i = 2, 4, \dots, 1 \leq j \leq m, 1 \leq s \leq m-1, \\ l = 3, 5, \dots, 1 \leq k \leq t;$$

$$f^*(x_{i,j}v_{l,s,k}) = 2t(i-1)(2m-1) + 2t(m-j) + 2t(l-2)(2m-1) + 2t(m-1) + 2t-1 + 2k + 4t(m-s-1), \quad i = 2, 4, \dots, 1 \leq j \leq m, 1 \leq s \leq m-1, \\ l = 2, 4, \dots, 1 \leq k \leq t.$$

In the view of above defined labeling pattern, t -super subdivision of $P_m \times P_n$, $m, n \geq 2$ is odd harmonious.

An odd harmonious labeling of 2-super subdivision of $P_3 \times P_3$ is shown in Figure 6.

Figure 6: An odd harmonious labeling of 2-super subdivision of $P_3 \times P_3$.

Theorem 2.6. One point union of path of graph $P_n^t(t.n.P_m \times P_m)$, $n \geq 1$, $m \geq 2$ is odd harmonious if t is odd.

Proof. Let $G = P_n^t(t.n.P_m \times P_m)$ be a graph obtained by replacing each vertices of P_n^t except the central vertex by the graph $P_m \times P_m$. That means G is the graph obtained by replacing each vertices of $K_{1,t}$ except the apex vertex by the path union of n copies of the graph $P_m \times P_m$. Let u_0 be the central vertex for the graph G with t branches. Let $x_{s,k,i,j}$ ($\forall 1 \leq k \leq n, 1 \leq i, j \leq m$) be the vertices of k^{th} copy of path union of n copies of $P_m \times P_m$ lies in the s^{th} branch of the graph G , $\forall s = 1, 2, \dots, t$.

Join the vertices of $u_{s,1,1,1}$ with u_0 by an edge to form the one point union for path of grid graph G . Also join the vertices $u_{s,k,m,m}$ to $u_{s,k+1,1,1}$ for $k = 1, 2, \dots, n-1$, $s = 1, 2, \dots, t$ by an edge. This graph G with $|V(G)| = tnm^2 + 1$ and $|E(G)| = tn[2m(m-1) + 1]$.

We define the labeling $f : V(G) \rightarrow \{0, 1, 2, \dots, 2tn[2m(m-1) + 1] - 1\}$ as follows:

$$f(u_0) = 0;$$

$$f(x_{s,k,i,j}) = 2 + 4(t-s) + t(j-1) + t(2m-1)(i-1) + 2tm(m-1) + t(k-2)[2m(m-1) + 1], \text{ if both } i \text{ and } j \text{ are odd, } k = 2, 4, \dots, 1 \leq s \leq t;$$

$$f(x_{s,k,i,j}) = 2s - 1 + t(j-2) + t(2m-1)(i-1) + t[2m(m-1) + 2] + t(k-2)[2m(m-1) + 1], \text{ if } i \text{ is odd and } j \text{ is even, } k = 2, 4, \dots, 1 \leq s \leq t;$$

$$f(x_{s,k,i,j}) = 2mt + 1 + t(j-1) + t(2m-1)(i-2) + 2(s-1) + 2tm(m-1) + t(k-2)[2m(m-1) + 1], \text{ if } i \text{ is even and } j \text{ is odd, } k = 2, 4, \dots, 1 \leq s \leq t;$$

$$f(x_{s,k,i,j}) = 2[1 + t(m-1)] + t(j-2) + 4(t-s) + t(2m-1)(i-2) + t[2m(m-1) + 2] + t(k-2)[2m(m-1) + 1], \text{ if both } i \text{ and } j \text{ are even, } k = 2, 4, \dots, 1 \leq s \leq t;$$

$$f(x_{s,k,i,j}) = 2s - 1 + t(j-1) + t(2m-1)(i-1) + t(k-1)[2m(m-1) + 1], \text{ if both } i \text{ and } j \text{ are odd, } k = 1, 3, \dots, 1 \leq s \leq t;$$

$$f(x_{s,k,i,j}) = 2 + 4(t-s) + t(j-2) + t(2m-1)(i-1) + t(k-1)[2m(m-1) + 1], \text{ if } i \text{ is odd and } j \text{ is even, } k = 1, 3, \dots, 1 \leq s \leq t;$$

$$f(x_{s,k,i,j}) = 2[1 + t(m-1)] + t(j-1) + 4(t-s) + t(2m-1)(i-2) + t(k-1)[2m(m-1) + 1], \text{ if } i \text{ is even and } j \text{ is odd, } k = 1, 3, \dots, 1 \leq s \leq t;$$

$$f(x_{s,k,i,j}) = 2mt + 1 + t(j-2) + t(2m-1)(i-2) + 2(s-1) + t(k-1)[2m(m-1) + 1], \text{ if both } i \text{ and } j \text{ are even, } k = 1, 3, \dots, 1 \leq s \leq t.$$

The induced edge labels are

$$f^*(x_{s,k,i,j}x_{s,k,i,j+1}) = 1 - 2s + 4t + 2t(j-1) + 2t(2m-1)(i-1) + 2t(k-1)[2m(m-1) + 1], \text{ if } k \text{ and } i \text{ are odd, } 1 \leq j \leq m-1, 1 \leq s \leq t;$$

$$f^*(x_{s,k,i,j}x_{s,k,i,j+1}) = 1 - 2s + 2t(2m + 1) + 2t(j - 1) + 2t(2m - 1)(i - 2) + 2t(k - 1)[2m(m - 1) + 1], \text{ if } k \text{ is odd and } i \text{ is even, } 1 \leq j \leq m - 1, 1 \leq s \leq t;$$

$$f^*(x_{s,k,i,j}x_{s,k,i,j+1}) = 1 - 2s + 6t + 2t(j - 1) + 2t(2m - 1)(i - 1) + 4tm(m - 1) + 2t(k - 2)[2m(m - 1) + 1], \text{ if } k \text{ is even and } i \text{ is odd, } 1 \leq j \leq m - 1, 1 \leq s \leq t;$$

$$f^*(x_{s,k,i,j}x_{s,k,i,j+1}) = 1 - 2s + 4t + 4tm^2 + 2t(j - 1) + 2t(2m - 1)(i - 2) + 2t(k - 2)[2m(m - 1) + 1], \text{ if both } k \text{ and } i \text{ are even, } 1 \leq j \leq m - 1, 1 \leq s \leq t;$$

$$f^*(x_{s,k,i,j}x_{s,k,i+1,j}) = 1 - 2s + 2t(m + 1) + 2t(j - 1) + 2t(2m - 1)(i - 1) + 2t(k - 1)[2m(m - 1) + 1], \text{ if } k \text{ and } j \text{ are odd, } 1 \leq i \leq m - 1, 1 \leq s \leq t;$$

$$f^*(x_{s,k,i,j}x_{s,k,i+1,j}) = 1 - 2s + 2t(m + 2) + 2t(j - 2) + 2t(2m - 1)(i - 1) + 2t(k - 1)[2m(m - 1) + 1], \text{ if } k \text{ is odd and } j \text{ is even, } 1 \leq i \leq m - 1, 1 \leq s \leq t;$$

$$f^*(x_{s,k,i,j}x_{s,k,i+1,j}) = 1 - 2s + 2t(m + 2) + 2t(j - 1) + 2t(2m - 1)(i - 1) + 4tm(m - 1) + 2t(k - 2)[2m(m - 1) + 1], \text{ if } k \text{ is even and } j \text{ is odd, } 1 \leq i \leq m - 1, 1 \leq s \leq t;$$

$$f^*(x_{s,k,i,j}x_{s,k,i+1,j}) = 1 - 2s + 6t + 2tm + 2t(j - 2) + 4tm(m - 1) + 2t(2m - 1)(i - 1) + 2t(k - 2)[2m(m - 1) + 1], \text{ if } k \text{ and } j \text{ are even, } 1 \leq i \leq m - 1, 1 \leq s \leq t;$$

$$f^*(x_{s,k,m,m}x_{s,k+1,1,1}) = 1 - 2s + 4t + 4tm(m - 1) + 2t(k - 1)[2m(m - 1) + 1], \text{ if } m \text{ is odd, } 1 \leq k \leq m - 1, 1 \leq s \leq t;$$

$$f^*(x_{s,k,m,m}x_{s,k+1,1,1}) = 1 - 2s + 2t + tm(2m + 1) + t(m - 2)(2m - 1) + 2t(k - 1)[2m(m - 1) + 1], \text{ if } m \text{ is even, } 1 \leq k \leq m - 1, 1 \leq s \leq t.$$

In the view of above defined labeling pattern, one point union of path of grid graph $P_n^t(t.n.P_m \times P_m)$, $n \geq 1$, $m \geq 2$ is odd harmonious if t is odd.

An odd harmonious labeling of one point union of path of grid graph $P_2^3(3.2.P_4 \times P_4)$ is shown in Figure 7.

Figure 7: An odd harmonious labeling of $P_2^3(3.2.P_4 \times P_4)$.

3. Concluding Remarks

In this work, we establish that path union of t copies of $P_m \times P_n$, path union of t different copies of $P_{m_i} \times P_{n_i}$ where $1 \leq i \leq t$, vertex union of t copies of $P_m \times P_n$, vertex union of t different copies of $P_{m_i} \times P_{n_i}$ where $1 \leq i \leq t$, one point union of path of $P_n^t(t.n.P_m \times P_m)$, t super subdivision of grid graph $P_m \times P_n$ admit an odd harmonious labeling. Further, we proceed with to prove more families of graphs are odd harmonious.

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