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Upper triangular operator matrices and limit points of the essential spectrum

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Abstract:

In this paper, we investigate the limit points set of essential spectrum of upper triangular operator matrices

$$M_c \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$$

We prove that $acc\sigma_e(M_C) \cup W = acc\sigma_e(A) \cup acc\sigma_e(B)$ *where* W *is the union of certain holes in* $acc\sigma e(M_C)$*, which happen to be subsets of* $acc\sigma_e(B) \cap acc\sigma_e(A)$*. Also, several sufficient conditions for* $acc\sigma_e(M_C) = acc\sigma_e(A) \cup acc\sigma_e(B)$ *holds are given.*

Keywords: Fredholm operator; Essential spectra; Limit point; Operator matrices.

MSC (2000): 47A10, 47A11.

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1. Introduction and Preliminaries

Let X, Y be infinite dimensional complex Banach spaces and $\mathcal{B}(X,Y)$ denote the complex algebra of all bounded linear operators from X to Y. For $Y = X$ we write $\mathcal{B}(X,X) = \mathcal{B}(X)$. If $T \in \mathcal{B}(X)$, we denote by T^*, $N(T)$, $R(T)$, $\sigma_{ap}(T)$, $\sigma_{su}(T)$, $\sigma(T)$, respectively the adjoint, the null space, the range, the approximate point spectrum, the surjectivity spectrum and the spectrum of T.

A bounded linear operator is called an upper semi-Fredholm (resp, lower semi-Fredholm) if $\alpha(T) = dimN(T) < \infty$ and $R(T)$ is closed (resp, $\beta(T) = codimR(T) < \infty$). T is semi-Fredholm if it is a lower or upper semi-Fredholm operator. The index of a semi Fredholm operator T is defined by $ind(T) = \alpha(T) - \beta(T)$.

T is a Fredholm operator if is a lower and upper semi-Fredholm operator. The essential spectrum of T is the subset of $\mathbf{C}$ defined by:

$$\sigma_e(T) = \{\lambda \in \mathbf{C} : T - \lambda I \text{ is not a Fredholm operator}\}$$

Let $T \in \mathcal{B}(X,Y)$, T is said to be left Atkinson if T is upper semi-Fredholm and $R(T)$ is complemented in X, and it is said to be right Atkinson if T is lower semi-Fredholm and $N(T)$ is complemented in X (see [1]). The left and right Atkinson spectra are the subsets of $\mathbf{C}$ defined respectively by:

$$\sigma_{le}(T) = \{\lambda \in \mathbf{C} : T - \lambda I \text{ is not a left Atkinson operator}\}$$

$$\sigma_{re}(T) = \{\lambda \in \mathbf{C} : T - \lambda I \text{ is not a right Atkinson operator}\}$$

$\sigma_e(T)$, $\sigma_{re}(T)$ and $\sigma_{le}(T)$ are compact subset and we have

$$\sigma_e(T) = \sigma_{re}(T) \cup \sigma_{le}(T)$$

For $A \in \mathcal{B}(X)$, $B \in \mathcal{B}(Y)$, we denote by $M_C \in \mathcal{B}(X \oplus Y)$ the operator matrix acting on the product of Banach space $X \oplus Y$ [5]:

$$M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$$

It is well know that, in the case of infinite dimensional, the inclusion $\sigma(M_C) \subset \sigma(A) \cup \sigma(B)$, may be strict. This motivates serval authors to study the defect $(\sigma_*(A) \cup \sigma_*(B)) \setminus \sigma_*(M_C)$ where σ_* runs different type of spectra.

If H and K are Hilbert spaces, Du and Pan [5] have studied the description of $\bigcap_{C\in\mathcal{B}(K,H)} \sigma(M_C)$ by showing that

$$\bigcap_{C\in\mathcal{B}(K,H)} \sigma(M_C) = \sigma_{ap}(A) \cup \sigma_{su}(B) \cup \{\lambda \in \mathbf{C} : \alpha(B-\lambda) \neq \beta(A-\lambda)\}$$

Han H.Y. Lee and W. Y. Lee [6] extended the result to the Banach spaces. In [3], D.S. Djordjevic give a description of $\bigcap_{C\in\mathcal{B}(K,H)} \sigma_e(M_C)$, he showed the following theorem.

Theorem 1.1 (3). . *For given $(A,B) \in \mathcal{B}(X) \times \mathcal{B}(Y)$ the following holds:*

$$\bigcap_{C\in\mathcal{B}(Y,X)} \sigma_e(M_C) = \sigma_{le}(A) \cup \sigma_{re}(B) \cup W(A,B)$$

Where $W(A,B) = \{\lambda \in \mathbf{C}, N(B-\lambda)$ and $X/\overline{R(A-\lambda)}$are not isomorphic up to a finite dimensional subspace$\}$

In [9], the authors showed the following theorem.

Theorem 1.2 (9). . *Let $(A,B) \in \mathcal{B}(X) \times \mathcal{B}(Y)$ and $C \in \mathcal{B}(Y,X)$. Then*

$$\sigma_e(M_C) \cup W_e = \sigma_e(A) \cup \sigma_e(B)$$

where W_e is the union of certain holes in $\sigma_e(M_C)$, which happen to be subsets of $\sigma_e(A) \cap \sigma_e(B)$.

For a compact subset K of $\mathbf{C}$, let $accK$, $intK$, $isoK$, ∂K and $\eta(K)$ be the set of all points of accumulation of K, the interior of K, the isolated points of K, the boundary of K and the polynomially convex hull of K respectively.

In this paper, we investigate the relationship between $acc\sigma_e(M_C)$ and $acc\sigma_e(A) \cup acc\sigma_e(B)$. We show that the passage from $acc\sigma_e(M_0)$ to $acc\sigma_e(M_C)$ can be described as follows:

$$acc\sigma_e(M_C) \cup W = acc\sigma_e(M_0) = acc\sigma_e(A) \cup acc\sigma_e(B)$$

where W is the union of certain holes in $acc\sigma_e(M_C)$, which happen to be subsets of $acc\sigma_e(A) \cap acc\sigma_e(B)$.

2. Main results

We start this section by proving that the limit point of essential spectrum set of a direct sum is the limit point of essential spectra of its summands.

Proposition 2.1. *Let $(A, B) \in \mathcal{B}(X) \times \mathcal{B}(Y)$ and $C \in \mathcal{B}(Y, X)$. Then*

$$acc\sigma_e(M_0) = acc\sigma_e(A) \cup acc\sigma_e(B)$$

Proof. We have $\lambda \in acc\sigma_e(M_0)$ if and only if $\lambda \in acc(\sigma_e(A) \cup \sigma_e(B)) = acc(\sigma_e(A)) \cup acc(\sigma_e(B))$.

Lemma 2.1. *Let $(A, B) \in \mathcal{B}(X) \times \mathcal{B}(Y)$ and $C \in \mathcal{B}(Y, X)$. Then*

$$acc\sigma_e(M_C) \subseteq acc\sigma_e(M_0) = acc\sigma_e(A) \cup acc\sigma_e(B)$$

Proof. Without loss of generality, let $\lambda = 0 \notin acc\sigma_e(A) \cup acc\sigma_e(B)$, then there exists $\varepsilon > 0$ such that for any λ, $0 < |\lambda| < \varepsilon$, we have $A - \lambda I$ and $B - \lambda I$ are Fredholm. According to [4, Lemma 2.1], we have $M_C - \lambda I$ is Fredholm for any λ, $0 < |\lambda| < \varepsilon$, thus $0 \notin acc(\sigma_e(M_C))$. Therefore $acc\sigma_e(M_C) \subseteq acc\sigma_e(A) \cup acc\sigma_e(B)$.

Definition 2.1. *Let $T \in \mathcal{B}(X)$. We said that T has the property aE at $\lambda \in \mathbf{C}$ if $\lambda \notin acc\sigma_e(T)$.*

The following lemma will be needed in the sequel.

Lemma 2.2. *If two of M_C, A and B have the property aE at 0, then the third is also has the property aE.*

Proof. *i*) If A and B have the property aE, by lemma 2.1 M_C has the property aE.
ii) If M_C and A have the property aE, then $0 \notin acc(\sigma_e(M_C))$ and $0 \notin acc(\sigma_e(A))$, thus there exists $\varepsilon > 0$ such that $M_C - \lambda I$ and $A - \lambda I$ are Fredholm for every $\lambda, 0 < |\lambda| < \varepsilon$. From [6, Corollary 5], $B - \lambda I$ is Fredholm for every $\lambda, 0 < |\lambda| < \varepsilon$.
iii) If B and M_C have the property aE, the proof is similar to *ii*).

The first main result of this paper is the following theorem.

Theorem 2.1. *Let $(A, B) \in \mathcal{B}(X) \times \mathcal{B}(Y)$ and $C \in \mathcal{B}(Y, X)$. Then*

$$acc\sigma_e(M_C) \cup W = acc\sigma_e(A) \cup acc\sigma_e(B)$$

where W is the union of certain holes in $acc\sigma_e(M_C)$, which happen to be subsets of $acc\sigma_e(B) \cap acc\sigma_e(A)$.

Proof. We first claim that, for every $C \in \mathcal{B}(Y, X)$ we have

$$(acc\sigma_e(A) \cup acc\sigma_e(B)) \setminus acc\sigma_e(A) \cap acc\sigma_e(B) \subseteq acc\sigma_e(M_C) \quad (1)$$

Indeed, let $\lambda \in (acc\sigma_e(A) \cup acc\sigma_e(B)) \setminus acc\sigma_e(A) \cap acc\sigma_e(B)$, then $\lambda \in acc\sigma_e(A) \setminus acc\sigma_e(B)$ or $\lambda \in acc\sigma_e(B) \setminus acc\sigma_e(A)$.

i) If $\lambda \in acc\sigma_e(A) \setminus acc\sigma_e(B)$, then A has not the property aE at λ and B has the property aE at λ. Suppose that $\lambda \notin acc\sigma_e(M_C)$, hence M_C has the property aE at λ, by lemma 2.2 A has the property aE at λ, contradiction. So $\lambda \in acc\sigma_e(M_C)$.

ii) If $\lambda \in acc\sigma_e(B) \setminus acc\sigma_e(A)$, by the same argument of *i)* we have $\lambda \in acc\sigma_e(M_C)$.

Next, we claim that for every $C \in \mathcal{B}(Y, X)$ we have

$$\eta(acc\sigma_e(M_C)) = \eta(acc\sigma_e(A) \cup acc\sigma_e(B)) \quad (2)$$

Since $acc\sigma_e(M_C) \subseteq acc\sigma_e(A) \cup acc\sigma_e(B)$, we need to prove $\partial(acc\sigma_e(A) \cup acc\sigma_e(B)) \subseteq \partial acc\sigma_e(M_C)$. But since $int(acc\sigma_e(M_C)) \subseteq int(acc\sigma_e(A) \cup acc\sigma_e(B))$, by the maximum modules theorem, it suffices to show that $\partial(acc\sigma_e(A) \cup acc\sigma_e(B)) \subseteq acc\sigma_e(M_C)$. Without loss of generality, suppose $0 \in \partial(acc\sigma_e(A) \cup acc\sigma_e(B))$. There are two cases to consider.

Case 1: If $0 \in acc(\partial(acc\sigma_e(A) \cup acc\sigma_e(B)))$, then there exists $(\lambda_n) \subseteq \partial(acc\sigma_e(A) \cup acc\sigma_e(B))$ such that $\lim\limits_{n\to\infty} \lambda_n = 0$, since

$$\partial(acc\sigma_e(A)) \subseteq \partial(\sigma_e(A)) \subseteq \sigma_{le}(A) \subseteq \sigma_e(M_C)$$

and

$$\partial(acc\sigma_e(B)) \subseteq \partial(\sigma_e(B)) \subseteq \sigma_{re}(B) \subseteq \sigma_e(M_C)$$

we have, $\lambda_n \in \sigma_e(M_C)$, $n = 1, 2, ...$, hence $0 \in acc(\sigma_e(M_C))$.

Case 2: If $0 \in iso(\partial(acc\sigma_e(A) \cup acc\sigma_e(B)))$, since $acc\sigma_e(A) \cup acc\sigma_e(B)$ is closed, then $iso(\partial(acc\sigma_e(A) \cup acc\sigma_e(B))) = iso(acc\sigma_e(A) \cup acc\sigma_e(B))$.
$0 \in iso(acc\sigma_e(A) \cup acc\sigma_e(B))$, thus there exists $\varepsilon > 0$ such that $\lambda \notin acc(acc\sigma_e(A) \cup acc\sigma_e(B))$ for every λ, $0 < |\lambda| < \varepsilon$. Since $0 \in acc\sigma_e(A) \cup acc\sigma_e(B) = acc(\sigma_e(A) \cup \sigma_e(B))$, there exists $(\mu_n) \subseteq \sigma_e(A) \cup \sigma_e(B)$ such that $\lim\limits_{n\to\infty} \mu_n = 0$, $\mu_n \neq 0$ for all n, thus there exists certain positive integer N such that $0 < |\mu_n| < \varepsilon$ for any $n \geq N$. Let $\lambda_n = \mu_{N+1+n}$, then $\lambda_n \in iso(\sigma_e(A) \cup \sigma_e(B))$, $n = 1, 2, ..$ and $\lim\limits_{n\to\infty} \lambda_n = 0$. Since $\sigma_e(A)$ and $\sigma_e(B)$ are closed, then

$$\begin{aligned} iso(\sigma_e(A) \cup \sigma_e(B)) &\subseteq iso(\sigma_e(A)) \cup iso(\sigma_e(B)) \\ &\subseteq \partial\sigma_e(A) \cup \partial\sigma_e(B) \\ &\subseteq \sigma_{le}(A) \cup \sigma_{re}(B) \subseteq \sigma_e(M_C) \end{aligned}$$

Then, $\lambda_n \in iso(\sigma_e(A) \cup \sigma_e(B)) \subseteq \sigma_e(M_C)$, $n = 1, 2, ...$ Since $\lim\limits_{n\to\infty} \lambda_n = 0$, so $0 \in acc\sigma_e(M_C)$.

Therefore $\partial(acc\sigma_e(A) \cup acc\sigma_e(B)) \subseteq acc\sigma_e(M_C)$. This proves (2).

$acc\sigma_e(M_C) \subseteq acc\sigma_e(A) \cup acc\sigma_e(B)$ and (2) says that the passage from $acc\sigma_e(M_C)$ to $acc\sigma_e(A) \cup acc\sigma_e(B)$ is the filling in certain of the holes in $acc\sigma_e(M_C)$. But since $(acc\sigma_e(A) \cup acc\sigma_e(B)) \setminus acc\sigma_e(M_C)$ is contained in $acc\sigma_e(A) \cap acc\sigma_e(B)$, it follows that the filling in certain of the holes in $acc\sigma_e(M_C)$ should occur in $acc\sigma_e(A) \cap acc\sigma_e(B)$.

Corollary 2.1. *Let $(A, B) \in \mathcal{B}(X) \times \mathcal{B}(Y)$. If $acc\sigma_e(A) \cap acc\sigma_e(B)$ has no interior points, then for every $C \in \mathcal{B}(Y, X)$ we have*

$$acc\sigma_e(M_C) = acc\sigma_e(A) \cup acc\sigma_e(B)$$

Second main result is the following theorem.

Theorem 2.2. *Let $(A, B) \in \mathcal{B}(X) \times \mathcal{B}(Y)$ and $C \in \mathcal{B}(Y, X)$. Then the following assertions are equivalent*

1. $\sigma_e(M_C) = \sigma_e(A) \cup \sigma_e(B)$,
2. $acc\sigma_e(M_C) = acc\sigma_e(A) \cup acc\sigma_e(B)$.

Proof. First we show that $W_e \subseteq W$.
Indeed, if $\lambda \in W_e$, from theorem 1.2, we have $\lambda \in (\sigma_e(A) \cup \sigma_e(B)) \setminus \sigma_e(M_C)$, then $\lambda \notin \sigma_e(M_C)$, hence $\lambda \notin acc\sigma_e(M_C)$. It suffice to show that

$$\lambda \in acc\sigma_e(A) \cup acc\sigma_e(B) = acc(\sigma_e(A) \cup \sigma_e(B))$$

Suppose that $\lambda \notin acc(\sigma_e(A) \cup \sigma_e(B))$, since $\lambda \in \sigma_e(A) \cup \sigma_e(B)$, then

$$\begin{aligned} \lambda \in iso(\sigma_e(A) \cup \sigma_e(B)) &\subseteq iso(\sigma_e(A)) \cup iso(\sigma_e(B)) \\ &\subseteq \partial\sigma_e(A) \cup \partial\sigma_e(B) \\ &\subseteq \sigma_{le}(A) \cup \sigma_{re}(B) \subseteq \sigma_e(M_C) \end{aligned}$$

Hence $\lambda \in \sigma_e(M_C)$, contradiction. Therefore

$$\lambda \in (acc\sigma_e(A) \cup acc\sigma_e(B)) \setminus acc\sigma_e(M_C)$$

By theorem 2.1, we have $\lambda \in W$. So $W_e \subseteq W$.

Furthermore, $W_e \subseteq W$ implies that

$$acc\sigma_e(M_C) = acc\sigma_e(A) \cup acc\sigma_e(B) \Longrightarrow \sigma_e(M_C) = \sigma_e(A) \cup \sigma_e(B)$$

Conversely, if $\sigma_e(M_C) = \sigma_e(A) \cup \sigma_e(B)$ let $\lambda \notin acc\sigma_e(M_C)$, without loss of generality, we assume that $0 \notin acc\sigma_e(M_C)$, then there exists $\varepsilon > 0$ such that $M_C - \lambda$ is Fredholm for all λ, $0 < |\lambda| < \varepsilon$, hence $\lambda \notin \sigma_e(M_C) = \sigma_e(A) \cup \sigma_e(B)$. Thus both $A - \lambda$ and $B - \lambda$ are Fredholm for every λ, $0 < |\lambda| < \varepsilon$. Therefore $0 \notin acc(\sigma_e(A)) \cup acc(\sigma_e(B))$. Since $acc\sigma_e(M_C) \subseteq acc\sigma_e(A) \cup acc\sigma_e(B)$ always holds, then $acc\sigma_e(M_C) = acc\sigma_e(A) \cup acc\sigma_e(B)$.

It is immediate to check the following result.

Corollary 2.2. *Let $(A, B) \in \mathcal{B}(X) \times \mathcal{B}(Y)$. If $acc\sigma_e(A) \cap acc\sigma_e(B)$ has no interior points, then for every $C \in \mathcal{B}(Y, X)$, we have we have*

$$\sigma_e(M_C) = \sigma_e(A) \cup \sigma_e(B) \qquad (**)$$

In particular, if either $A \in \mathcal{B}(X)$ or $B \in \mathcal{B}(Y)$ is a Riesz, then $(**)$ holds.

Now, For $(A, B) \in \mathcal{B}(X) \times \mathcal{B}(Y)$, let L_A (resp R_B) be the left (resp. right) multiplication operator given by $L_A(X) = AX$; (resp. $R_B(X) = XB$), and let $\delta_{A,B}(X) = AX - XB = L_A(X) - R_B(X)$ be the usual generalized derivation associated with A and B. We denote by $N^\infty(A) = \bigcup_{n \geq 1} N(A^n)$ the generalized kernel of A [1].

Corollary 2.3. *Let $A \in \mathcal{B}(X)$ and $B \in \mathcal{B}(Y)$. If C is in the closure of the set*

$$R(\delta_{A,B}) + N(\delta_{A,B}) + \bigcup_{\lambda \in \mathbf{C}} N^{\infty}(L_{A-\lambda}) + \bigcup_{\lambda \in \mathbf{C}} N^{\infty}(R_{B-\lambda})$$

then :

$$acc\sigma_e(M_C) = acc\sigma_e(A) \cup acc\sigma_e(B)$$

Proof. If C is in the closure of the set

$$R(\delta_{A,B}) + N(\delta_{A,B}) + \bigcup_{\lambda \in \mathbf{C}} N^{\infty}(L_{A-\lambda}) + \bigcup_{\lambda \in \mathbf{C}} N^{\infty}(R_{B-\lambda})$$

then $\sigma_e(M_C) = \sigma_e(A) \cup \sigma_e(B)$, by [2]. From theorem 2.2, we obtain the result.

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