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Interpolation and approximation from sublattices of $C_0(X; R)$

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Abstract:

In this paper, we give a proof of a result concerning simultaneous interpolation and approximation from sublattices of the space of real continuous functions vanishing at infinity.

Keywords: Bonsall; Lattices; Interpolation; Approximation.

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1. Introduction

Throughout this paper, we shall assume that X is a locally compact Hausdorff space.

A continuous real function f on X is said to *vanish at infinity* if for every $\varepsilon > 0$ the set $\{x \in X : \ |f(x)| \geq \varepsilon\}$ is compact. Let $C_0(X;\mathcal{R})$ be the vector space of all continuous real functions on X vanishing at infinity and equipped with the supremum norm.

Recall that for each $x \in X$,

$$\begin{aligned}(f \wedge g)(x) &= inf(f(x), g(x))\\ (f \vee g)(x) &= sup(f(x), g(x)).\end{aligned}$$

A subset B of $C_0(X;\mathcal{R})$ is called a lattice (or a sublattice) if $f \wedge g$ and $f \vee g$ belong to B whenever $f \in B$ and $g \in B$.

We denote the closure of a subset T of a topological space by $\overline{T}$.

In the paper [1], Boel, Carlsen and Hansen present a proof of a result of simultaneous interpolation and approximation from certain subalgebras of $C_0(X;\mathcal{C})$, where $\mathcal{C}$ is the complex field, through Stone-Weierstrass Theorem. Motivated by their paper, we give a proof of a theorem of this type for sublattices of $C_0(X;\mathcal{R})$ by using Bonsall Theorem [2].

2. Main result

In order to show the main theorem we list some results. First of all, we present a version of Bonsall Theorem [2] concerning characterization of the uniform closure of sublattices of $C_0(X;\mathcal{R})$.

Theorem 1. Let $B \subset C_0(X;\mathcal{R})$ be a sublattice and let $f \in C_0(X;\mathcal{R})$. Then $f \in \overline{B}$ if, and only if, for any pair of points $x, y \in X$ and any $\varepsilon > 0$, there is $\psi \in B$ such that

$$f(x) < \psi(x) + \varepsilon,$$

$$f(y) > \psi(y) - \varepsilon.$$

Proof. It is clear that the condition is necessary. Conversely, let $\varepsilon > 0$ be given. Let $t \in X$ be given. For any $x \in X$ there is $\psi_x \in B$ such that $f(x) < \psi_x(x) + \varepsilon$ and $f(t) > \psi_x(t) - \varepsilon$. Let

$$K_x = \{v \in X : f(v) \geq \psi_x(v) + \varepsilon\}.$$

The closed set K_x is a subset of the compact set $\{s \in X : |f(s) - \psi_x(s)| \geq \varepsilon\}$. Hence K_x is compact and does not contain x. Therefore $\cap_{x\in X} K_x = \emptyset$. By the finite intersection property, there are $x_1, ..., x_m \in X$ such that $K_{x_1} \cap \cdots \cap K_{x_m} = \emptyset$. Let $\phi_t = \psi_{x_1} \vee \cdots \vee \psi_{x_m}$. Note that $\phi_t \in B$ and $f(t) > \psi_{x_i}(t) - \varepsilon$ for all $i = 1, ..., m$. Hence

$$f(t) > \phi_t(t) - \varepsilon. \tag{2.1}$$

On the other hand, given $x \in X$ there is some $i \in \{1, ..., m\}$ such that $x \notin K_{x_i}$.

Therefore

$$f(x) < \psi_{x_i}(x) + \varepsilon \leq \phi_t(x) + \varepsilon. \tag{2.2}$$

The closed set $I_t = \{v \in X : f(v) \leq \phi_t(v) - \varepsilon\}$ is a subset of the compact set $\{s \in X : |f(s) - \phi_t(s)| \geq \varepsilon\}$. Then I_t is compact and by (2.1) I_t does not contain t. Hence $\cap_{t\in X} I_t = \emptyset$. By the finite intersection property, there are $t_1, ..., t_n \in X$ such that $I_{t_1} \cap \cdots \cap I_{t_n} = \emptyset$. Let $\psi = \phi_{t_1} \wedge \cdots \wedge \phi_{t_n}$. Then $\psi \in B$. If $x \in X$, there is some $j \in \{1, ..., n\}$ such that $x \notin I_{t_j}$. Therefore

$$\psi(x) - \varepsilon \leq \phi_{t_j}(x) - \varepsilon < f(x). \tag{2.3}$$

It follows by (2.2) that

$$f(x) < \phi_{t_j}(x) + \varepsilon$$

for each $1 \leq j \leq n$.

Hence

$$f(x) < \psi(x) + \varepsilon. \tag{2.4}$$

It follows from (2.3) and (2.4) that

$$|f(x) - \psi(x)| < \varepsilon$$

for all $x \in X$.

Therefore f belongs to the uniform closure of B.

Lemma 2. Let X be a locally compact Hausdorff space, $t_1, \ldots, t_m$ distinct points in X, and $y_1, \ldots, y_m$ distinct real numbers. If L is a dense linear subspace of $C_0(X;\mathcal{R})$, then there exists a function $h \in L$ such that $h(t_j) = y_j$, $j = 1, \ldots, m$.

Proof. Let L be a dense linear subspace of $C_0(X;\mathcal{R})$. Let $S = \{x_1, \ldots, x_m\}$ be a subset of X. Consider the following linear mapping

$$\begin{array}{rcl} T : C_0(X;\mathcal{R}) & \to & \mathcal{R}^m \\ f & \mapsto & (f(x_1), \ldots, f(x_m)). \end{array}$$

Notice that T is linear and continuous. It follows from the Tietze Extension Theorem ([4] p. 389) that T is surjective. Moreover, $T(L)$ is closed because it is a linear subspace of $\mathcal{R}^m$. Then by density of L and continuity of T, it follows that

$$\mathcal{R}^m = T(C_0(X;\mathcal{R})) = T(\overline{L}) \subset \overline{T(L)} = T(L).$$

Therefore, there exists $h \in L$ such that

$$(h(x_1), \ldots, h(x_m)) = (y_1, \ldots, y_m).$$

Lemma 3. Let X be a locally compact Hausdorff space, L a dense linear sublattice of $C_0(X;\mathcal{R})$, and $x_1, \ldots, x_n$ distinct points in X. Consider the locally compact Hausdorff space

$$\widehat{X} = X \setminus \{x_1, \ldots, x_n\}$$

and the sublattice

$$M = \{f|_{\widehat{X}} : f \in L, f(x_1) = \ldots = f(x_n) = 0\}.$$

Then M is dense in $C_0(\widehat{X};\mathcal{R})$.

Proof. Take an arbitrary $\psi \in C_0(\widehat{X};\mathcal{R})$, $\varepsilon > 0$ and let x, y be any distinct points in $\widehat{X}$. Consider the following subset

$$S = \{x, y, x_1, \ldots, x_n\}$$

of X. By Lemma 2 there exists $h \in L$ such that $h(x) = \psi(x)$, $h(y) = \psi(y)$ and $h(x_j) = 0$ for $j = 1, \ldots, n$.

Notice that $h|_{\widehat{X}} \in M$. Moreover,

$$(\psi(x) - h|_{\widehat{X}}(x)) = 0 < \varepsilon$$

$$(\psi(y) - h|_{\widehat{X}}(y)) = 0 > -\varepsilon.$$

Then, it follows from Theorem 1 that $\psi \in \overline{M}$.

□

Here is the main result. It can be showed through Deutsch Theorem [3]. We give a proof by using Bonsall Theorem.

Theorem 4. Let X be a locally compact Hausdorff space. Suppose that $f \in C_0(X;\mathcal{R})$, $\varepsilon > 0$, and $x_1, ..., x_n$ are distinct points in X. If L is a dense vector sublattice of $C_0(X;\mathcal{R})$, then there exists a function $g \in L$ such that $g(x_j) = f(x_j)$ for $j = 1, ..., n$ and $|g(x) - f(x)| < \varepsilon$ for all $x \in X$.

Proof. It follows from Lemma 2 that there exists $h \in L$, such that $h(x_j) = f(x_j)$, $j = 1, ..., n$. Then, by Lemma 3 there exists $\phi \in L$ such that $\phi(x_j) = 0$ for $j = 1, ..., n$ and $|\phi(x) - (f(x) - h(x))| < \varepsilon$ for all $x \in \widehat{X}$. Notice that $\phi(x_j) - (f(x_j) - h(x_j)) = 0$ for $j = 1, ..., n$. Taking $g = \phi + h$, the result follows.

Example 5. Consider for each $t \in (0, 1)$, the function

$$h_t(x) = \begin{cases} 1 & \text{if } x = t \\ (2x - t)/t & \text{if } x \in [t/2, t] \\ (2x - t - 1)/(t - 1) & \text{if } x \in [t, (t + 1)/2] \\ 0 & \text{otherwise .} \end{cases}$$

Let

$$h_o(x) = \begin{cases} 1 & \text{if } x = 0 \\ -2x + 1 & \text{if } x \in [0, 1/2] \\ 0 & \text{otherwise .} \end{cases}$$

Let

$$h_1(x) = \begin{cases} 1 & \text{if } x = 1 \\ 2x - 1 & \text{if } x \in [1/2, 1] \\ 0 & \text{otherwise .} \end{cases}$$

Let L be the vector sublattice of $C([0, 1];\mathcal{R})$ generated by the set $\{h_t : t \in [0, 1]\}$. Then L is uniformly dense in $C[0, 1]$ by [6, Theorem 4.2]. Since $[0, 1]$ is a compact Hausdorff space, it follows that $C([0, 1];\mathcal{R}) = C_0([0, 1];\mathcal{R})$. By using Theorem 4 we conclude that L has the simultaneous approximation and interpolation property.

Example 6. Let $L \subset C_0([0,\infty);\mathcal{R})$ be the vector sublattice of all real Lipschitz functions on $[0,\infty)$ that vanish at infinity. It follows from [5, Theorem A, p. 166] that L is dense in $C_0([0,\infty);\mathcal{R})$. Then, by Theorem 4, L has the the simultaneous approximation and interpolation property.

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