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Maps preserving the square zero of η-Lie product of operators

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Abstract:

Let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators on an infinite dimensional Hilbert space $\mathcal{H}$. In this paper, we identify the form of the unital surjective additive map $\varphi : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ which preserves the square zero of η-Lie product of operators for some scalar η with $\eta \neq 0, 1, -1$.

Keywords: Preserver problem; Square zero operator; η-Lie product.

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1. Introduction

Let $\mathcal{A}$ be a Banach algebra, $A, B \in \mathcal{A}$ and η be a scalar. The Lie product, η-Lie product and triple Jordan products are defined as $[A, B] = AB - BA$, $[A, B]_\eta = AB + \eta BA$ and $A * B = ABA$, respectively. In last decade, Many mathematician research on the preserving problems. Specially, maps preserving a certain property of products were often considered, see $[1-4]$, [6], [8] and $[10-12]$. We point to some of them close to our purpose.

Authors in [10], considered the maps that strongly preserve the η-Lie product, that is $\phi(A)\phi(P) + \eta\phi(P)\phi(A) = AP + \eta PA$, for every A, some idempotent P and some scalar η. Author in [12], identified the forms of bijective maps preserving Lie products from a factor von Neumann algebra into another factor von Neumann algebra.

Let $\mathcal{B}(\mathcal{X})$ be the algebra of all bounded linear operators on a Banach space $\mathcal{X}$. In [4], authors characterized the form of unital surjective maps on $\mathcal{B}(\mathcal{X})$ preserving the nonzero idempotency of product of operators in both directions. Also in [11], authors characterized the form of linear surjective maps on $\mathcal{B}(\mathcal{X})$ preserving the nonzero idempotency of either products of operators or triple Jordan products of operators.

We say an operator $A \in \mathcal{B}(\mathcal{X})$ is a square zero operator, when $A^2 = 0$. Let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators on an infinite dimensional Hilbert space $\mathcal{H}$. In this paper, we identify the form of surjective additive map $\phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ such that $\phi(I) = I$ and preserves the square zero of η-Lie product of operators for some scalar η with $\eta \neq 0, 1, -1$. The complete form of our main result is as following:

Main Theorem. Let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators on an infinite dimensional Hilbert space $\mathcal{H}$. Let $\phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be an unital surjective additive map which satisfies

$$[A, B]_\eta^2 = 0 \Leftrightarrow [\phi(A), \phi(B)]_\eta^2 = 0,$$

for every $A, B \in \mathcal{B}(\mathcal{H})$ and for some scalar η with $\eta \neq 0, 1, -1$. Then there exists either a bounded invertible linear or a conjugate linear operator $T : \mathcal{H} \to \mathcal{H}$ such that

$$\phi(A) = TAT^{-1} \quad \text{or} \quad \phi(A) = TA^*T^{-1}$$

for every $A \in \mathcal{B}(\mathcal{H})$.

2. Proofs

First we recall some notations. We denote by $\mathcal{I}(\mathcal{H})$ the set of all idempotent operators in $\mathcal{B}(\mathcal{H})$. For every nonzero $x, y \in \mathcal{H}$, the symbol $x \otimes y$ stands for the rank one linear operator on $\mathcal{H}$ defined by $(x \otimes y)z =< z, y > x$ for any $z \in \mathcal{H}$. Note that every rank one operator in $\mathcal{B}(\mathcal{H})$ can be written in this way.

The rank one operator $x \otimes y$ is idempotent if and only if $< x, y >= 1$. Let $P, Q \in \mathcal{B}(\mathcal{H})$ be idempotent operators. We say that P and Q are orthogonal if and only if $PQ = QP = 0$.

Proposition 2.1. *Let $A \in \mathcal{B}(\mathcal{H})$, $x, y \in \mathcal{H}$ such that $< x, y >= 1$ and let η be a scalar such that $\eta \neq 0, 1, -1$. Then $[A, x \otimes y]_\eta^2 = 0$ if and only if only one of the following statements occurs: (i) $Ax < Ax, y >= -\eta x < A^2x, y >$ and $Ax = -\eta x < Ax, y >$. (ii) $A^*y = 0$.*

Proof. Assume first that $Ax < Ax, y >= -\eta x < A^2x, y >$ and $Ax = -\eta x < Ax, y >$. Hence

$$\begin{aligned}
&[A, x \otimes y]_\eta^2 = (Ax \otimes y + \eta x \otimes A^*y)^2 \\
&=< Ax, y > Ax \otimes y + \eta Ax \otimes A^*y \\
&+\eta^2 < Ax, y > x \otimes A^*y + \eta < A^2x, y > x \otimes y \\
&= -\eta x < A^2x, y > \otimes y - \eta^2 x < Ax, y > \otimes yA \\
&+\eta^2 < Ax, y > x \otimes A^*y + \eta < A^2x, y > x \otimes y = 0.
\end{aligned}$$

Now if $A^*y = 0$, then

$$\begin{aligned}
&[A, x \otimes y]_\eta^2 = (Ax \otimes y + \eta x \otimes A^*y)^2 \\
&= (Ax \otimes y)^2 =< Ax, y > Ax \otimes y \\
&=< x, A^*y > Ax \otimes y = 0.
\end{aligned}$$

Conversely, Assume that $[A, x \otimes y]_\eta^2 = 0$. It is clear that

$$B^2 = 0 \Leftrightarrow B(Bx) = 0, \quad \forall x \in \mathcal{X} \Leftrightarrow \text{Im B} \subseteq \ker B.$$

This together with assumption implies

$$[A, x \otimes y]_\eta^2 = 0 \Leftrightarrow \text{Im}[A, x \otimes y]_\eta \subseteq \ \ker[A, x \otimes y]_\eta.$$

Let $A^*y \neq 0$. If A^*y and y are linearly independent, then In the following lemmas, assume that $\phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is an unital surjective additive map which satisfies

$$[A,B]_\eta^2 = 0 \Leftrightarrow [\phi(A),\phi(B)]_\eta^2 = 0,$$

for every $A, B \in \mathcal{B}(\mathcal{H})$ and for some scalar η with $\eta \neq 0, 1, -1$.

Lemma 2.2. *ϕ preserves the square zero operators in both directions.*

Proof. Let $A \in \mathcal{B}(\mathcal{H})$. By assumptions we have

$$\begin{aligned} A^2 = 0 \quad &\Leftrightarrow \quad (1+\eta)^2 A^2 = [A, I]_\eta^2 = 0 \\ &\Leftrightarrow \quad [\phi(A), I]_\eta^2 = 0 \\ &\Leftrightarrow \quad (1+\eta)^2 \phi(A)^2 = 0 \\ &\Leftrightarrow \quad \phi(A)^2 = 0. \end{aligned}$$

The following theorem is a straightforward consequence of Theorem 2.1 in [7].

Theorem 2.3. *Let $\mathcal{H}$ be an infinite dimensional Hilbert space and $\phi : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H})$ be a surjective additive map satisfying $\phi(I) = I$. Assume that ϕ preserves the square zero operators in both directions. Then ϕ is injective and preserves the idempotent operators in both directions.*

Lemma 2.4. *ϕ is injective and preserves the idempotent operators in both directions.*

Proof. It is clear by assumptions and Theorem 2.3.

Lemma 2.5. *There exists either a bounded invertible linear or a conjugate linear operator $T : \mathcal{H} \to \mathcal{H}$ such that*

$$\phi(P) = TPT^{-1}$$

or

$$\phi(P) = TP^*T^{-1}$$

for every $P \in \mathcal{I}(\mathcal{H})$.

Proof. Since ϕ is additive and by Lemma 2.4 preserves idempotent operators in both directions, then ϕ preserves the orthogonality of idempotent operators in both directions. Thus we can obtain the form of ϕ on idempotents by Lemma 3.1 in [5].

Remark 2.6. *Let T be the same operator defined in Lemma 2.5. It is clear that $\Psi = T^{-1}\phi T : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ satisfies the assumptions on ϕ. Therefore, without loss of generality we can assume that $\phi(P) = P$ or $\phi(P) = P^*$ for every $P \in \mathcal{I}(\mathcal{H})$.*

Now we are in a position to prove our main result.

Proof of Main Theorem. Let $A \in \mathcal{B}(\mathcal{H})$ such that $\ker A \neq 0$. Let $x \in \ker A$ be nonzero. Hence there exists a nonzero vector $y \in \mathcal{H}$ such that $< x, y >= 1$. Let the first case of Lemma 2.5 occurs. So by Remark 2.6, $\phi(x \otimes y) = x \otimes y$. By $Ax = 0$ and Proposition 2.1 we infer that $[A, x \otimes y]_\eta^2 = 0$ and by assumption

$$[\phi(A), \phi(x \otimes y)]_\eta^2 = [\phi(A), x \otimes y]_\eta^2 = 0.$$

Using again Proposition 2.1 implies

$$\phi(A)x < \phi(A)x, y >= -\eta x < \phi(A)^2 x, y > \tag{1}$$

and

$$\phi(A)x = -\eta x < \phi(A)x, y > \tag{2}$$

or $\phi(A)^* y = 0$. We assert that $\phi(A)x = 0$. We assume on the contrary that $\phi(A)x \neq 0$. Let us first assume that (1) and (2) occur.

Thus

$$-\eta x < \phi(A)x, y >^2 = -\eta x < \phi(A)^2 x, y >$$

and since $\eta \neq 0$, $< \phi(A)x, y >^2 = < \phi(A)^2 x, y >$. It easily follows that x, $\phi(A)x$ and $\phi(A)^2 x$ are linearly dependent, because otherwise, there exists a vector y such that $< x, y >= 1$ and $< \phi(A)x, y >^2 \neq < \phi(A)^2 x, y >$.

If x and $\phi(A)x$ are linearly dependent, then $\phi(A)x = \alpha x$ for some nonzero scalar α. From (2) we obtain $\alpha x < x, y >= -\eta \alpha x < x, y >$ which implies that $\eta = -1$, that is a contradiction. If x and $\phi(A)x$ are linearly independent, then we conclude that $\phi(A)^2 x \in \text{span}\{\phi(A)x, x\}$ and so $\phi(A)^2 x = \alpha \phi(A)x + \beta x$ for some scalars α, β. It implies that $< \phi(A)^2 x, y >= \alpha < \phi(A)x, y > + \beta$. According to (1)

$$\phi(A)x < \phi(A)x, y >= -\eta x(\alpha < \phi(A)x, y > + \beta).$$

Since x and $\phi(A)x$ are linearly independent, $<\phi(A)x, y>=0$ which by (2) implies, $<x, y>=0$, a contradiction.

Now let $\phi(A)^*y=0$. Since we assume that $\phi(A)x \neq 0$, there exists a vector y such that $<x, y>=1$ and $<\phi(A)x, y>\neq 0$. This implies $<x, \phi(A)^*y>\neq 0$. It is a contradiction, because $\phi(A)^*y=0$. The proof of assertion is completed and so $\ker A \subseteq \ker \phi(A)$, when $\ker A \neq 0$. This implies that if $\ker A \neq 0$, then $\ker \phi(A) \neq 0$ and this with a similar discussion as above yields that $\ker \phi(A) \subseteq \ker A$. Therefore, $\ker \phi(A) = \ker A$ for every operator A such that $\ker A \neq 0$. Moreover, this implies that $\ker A \neq 0$ if and only if $\ker \phi(A) \neq 0$ which yields that $\ker A = 0$ if and only if $\ker \phi(A) = 0$. Hence $\ker \phi(A) = \ker A$ for every operator A and so $F(\phi(A)) = F(A)$ and since ϕ is additive, $F(\phi(A)+\phi(B)) = F(A+B)$ for every $A, B \in \mathcal{B}(\mathcal{H})$. The form of such ϕ has been given in [9], Theorem 3.5. By this theorem, $\phi(A) = UA + R$ such that $U = I - 2\phi(0)$ and $R = \phi(0)$. Since ϕ is additive, $\phi(A) = A$. With a similar discussion, we obtain $\phi(A) = A^*$, when the second case in Lemma 2.5 occurs. These together with Remark 2.6 complete the proof.

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