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Some bounds for relative autocommutativity degree

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Abstract:

We consider the probability that a randomly chosen element of a subgroup of a finite group G is fixed by an automorphism of G. We obtain several bounds for this probability and characterize some finite groups with respect to this probability.

Keywords: Autocommutativity degree; Automorphism group; Autoisoclinism.

MSC (2010): 20P05, 20D60, 20F28.

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1. Introduction

Let G be a finite group and $Aut(G)$ be its automorphism group. The relative autocommutativity degree $\Pr(K, Aut(G))$ of a subgroup K of G is the probability that a randomly chosen element of K is fixed by an automorphism of G. In other words

$$\Pr(K, Aut(G)) = \frac{|\{(a,\nu) \in K \times Aut(G) : \nu(a) = a\}|}{|K||Aut(G)|}. \tag{1.1}$$

The notion of $\Pr(K, Aut(G))$ was introduced in [6] and studied in [6, 10]. A generalization of $\Pr(K, Aut(G))$ can also be found in [2, 11]. Note that $\Pr(G, Aut(G))$ is the probability that an automorphism of G fixes an element of it. The ratio $\Pr(G, Aut(G))$ is also known as the autocommutativity degree of G. It is worth mentioning that autocommutativity degree of G was initially studied by Sherman [12] in 1975.

In this paper, we obtain several bounds for $\Pr(K, Aut(G))$. We remark that some of these bounds are better than some existing bounds. We also characterize some finite groups with respect to $\Pr(K, Aut(G))$. We shall conclude this paper showing that the bounds for $\Pr(K, Aut(G))$ are also applicable for $\Pr(K_1, Aut(G_1))$ if (K_1, G_1) and (K, G) are autoisoclinic.

For any element $a \in G$ and $\nu \in Aut(G)$ we write $[a, \nu] := a^{-1}\nu(a)$, the autocommutator of a and ν. We also write $S(K, Aut(G)) := \{[a, \nu] : a \in K \text{ and } \nu \in Aut(G)\}$, $L(K, Aut(G)) := \{a \in K : \nu(a) = a \text{ for all } \nu \in Aut(G)\}$ and $[K, Aut(G)] := \langle S(K, Aut(G))\rangle$. Note that $L(K, Aut(G))$ is a normal subgroup of K contained in $K \cap Z(G)$ and $L(K, Aut(G)) = \bigcap\limits_{\nu \in Aut(G)} C_K(\nu)$, where $Z(G)$ is the center of G and $C_K(\nu) := \{a \in K : \nu(a) = a\}$ is a subgroup of K. If $K = G$ then $L(K, Aut(G)) = L(G)$, the absolute centre of G (see [5]). It is also not difficult to see that K is abelian if $\frac{K}{L(K, Aut(G))}$ is cyclic. Let $C_{Aut(G)}(a) := \{\nu \in Aut(G) : \nu(a) = a\}$ for $a \in K$ and $C_{Aut(G)}(K) := \{\nu \in Aut(G) : \nu(a) = a \text{ for all } a \in K\}$. Then $C_{Aut(G)}(a)$ is a subgroup of $Aut(G)$ and $C_{Aut(G)}(K) = \bigcap\limits_{a \in K} C_{Aut(G)}(a)$.

It is easy to see that

$$\begin{aligned}\{(a,\nu) \in K \times Aut(G) : \nu(a) = a\} &= \bigsqcup_{a \in K} (\{a\} \times C_{Aut(G)}(a)) \\ &= \bigsqcup_{\nu \in Aut(G)} (C_K(\nu) \times \{\nu\}),\end{aligned}$$

where $\sqcup$ stands for union of disjoint sets. Hence

$$|K||Aut(G)|\Pr(K, Aut(G)) = \sum_{a\in K} |C_{Aut(G)}(a)| = \sum_{\nu\in Aut(G)} |C_K(\nu)|. \tag{1.2}$$

Also, for $\nu \in Aut(G)$ and $a \in G$, $(\nu, a) \mapsto \nu(a)$ is an action of $Aut(G)$ on G. The orbit of $a \in G$ is given by $orb(a) := \{\nu(a) : \nu \in Aut(G)\}$ and $|orb(a)| = |Aut(G)|/|C_{Aut(G)}(a)|$.

Hence, (1.2) gives the following generalization of [1, Proposition 2]

$$\Pr(K, Aut(G)) = \frac{1}{|K|}\sum_{a\in K}\frac{1}{|orb(a)|}. \tag{1.3}$$

Note that $\Pr(K, Aut(G)) = 1$ if and only if $K = L(K, Aut(G))$. Therefore, throughout the paper we consider $K \neq L(K, Aut(G))$.

2. Some upper bounds

We begin with the following upper bound for $\Pr(K, Aut(G))$.

Theorem 2.1. *If K is a subgroup of G then*

$$\Pr(K, Aut(G)) \leq \frac{1}{2}\left(1 + \frac{1}{|K : L(K, Aut(G))|}\right)$$

with equality if and only if $|orb(a)| = 2$ for all $a \in K \setminus L(K, Aut(G))$.

Proof. By (1.3), we get

$$\Pr(K, Aut(G)) = \frac{1}{|K|}\left(|L(K, Aut(G))| + \sum_{a\in K\setminus L(K,Aut(G))}\frac{1}{|orb(a)|}\right). \tag{2.1}$$

Since $|orb(a)| \geq 2$ for all $a \in K \setminus L(K, Aut(G))$, the result follows from (2.1).

Corollary 2.2. *If K is a non-abelian subgroup of G, then* $\Pr(K, Aut(G)) \leq \frac{5}{8}$. *Further,* $\Pr(K, Aut(G)) = \frac{5}{8}$ *if and only if* $|orb(a)| = 2$ *for all* $a \in K \setminus L(K, Aut(G))$ *and* $\frac{K}{L(K,Aut(G))} \cong \mathbf{Z}_2 \times \mathbf{Z}_2$.

Proof. The inequality follows from Theorem 2.1 noting that $\frac{|K|}{|L(K,Aut(G))|} \geq 4$ if K is non-abelian.
Note that $\Pr(K, Aut(G)) = \frac{5}{8}$ if and only if $\frac{|K|}{|L(K,Aut(G))|} = 4$ and equality holds in Theorem 2.1. Hence, the result follows.

Theorem 2.3. *If K is a subgroup of G and p the smallest prime dividing $|Aut(G)|$, then*

$$\Pr(K, Aut(G)) \leq \frac{(p-1)|L(K, Aut(G))| + |K|}{p|K|} - \frac{|X_K|(|Aut(G)| - p)}{p|K||Aut(G)|}$$

where $X_K = \{a \in K : C_{Aut(G)}(a) = \{I\}\}$ *and I is the identity of $Aut(G)$.*

Proof. Note that $X_K \cap L(K, Aut(G)) = \emptyset$. Therefore

$$\sum_{a \in K} |C_{Aut(G)}(a)| = |X_K| + |Aut(G)||L(K, Aut(G))| + \sum_{a \in K \setminus (X_K \cup L(K,Aut(G)))} |C_{Aut(G)}(a)|.$$

For $a \in K \setminus (X_K \cup L(K, Aut(G)))$ we have $C_{Aut(G)}(a) < Aut(G)$ which implies $|C_{Aut(G)}(a)| \leq \frac{|Aut(G)|}{p}$. Therefore

$$\sum_{a \in K} |C_{Aut(G)}(a)| \leq |X_K| + |Aut(G)||L(K, Aut(G))| + \frac{|Aut(G)|(|K| - |X_K| - |L(K,Aut(G))|)}{p}. \tag{2.2}$$

Hence, the result follows from (1.2) and (2.2).

We would like to mention here that Theorem 2.3 gives better upper bound than the upper bound given by [6, Theorem 2.3 (i)]. We also have the following improvement of [6, Corollary 2.2].

Corollary 2.4. *Let K be a subgroup of G. Then*

$$\Pr(K, Aut(G)) \leq \frac{p + q - 1}{pq}$$

where p and q are the smallest prime divisors of $|Aut(G)|$ and $|K|$ respectively. Further, if $q \geq p$ then $\Pr(K, Aut(G)) \leq \frac{2p-1}{p^2} \leq \frac{3}{4}$.

Proof. We have $|K : L(K, Aut(G))| \geq q$ since $K \neq L(K, Aut(G))$. Therefore, by Theorem 2.3, we get

$$\Pr(K, Aut(G)) \leq \frac{1}{p}\left(1 + \frac{p-1}{|K : L(K, Aut(G))|}\right) \leq \frac{p+q-1}{pq}.$$

Corollary 2.5. *If K is a non-abelian subgroup of G then*

$$\Pr(K, Aut(G)) \leq \frac{q^2+p-1}{pq^2}$$

where p and q denote respectively the smallest prime divisors of $|Aut(G)|$ and $|K|$. Further, if $q \geq p$ then $\Pr(K, Aut(G)) \leq \frac{p^2+p-1}{p^3} \leq \frac{5}{8}$.

Proof. The fact that K is a non-abelian subgroup of G implies $|K : L(K, Aut(G))| \geq q^2$. Hence

$$\Pr(K, Aut(G)) \leq \frac{1}{p}\left(1 + \frac{p-1}{|K : L(K, Aut(G))|}\right) \leq \frac{q^2+p-1}{pq^2}$$

by Theorem 2.3.

Now we obtain some characterizations of a subgroup K of G if equality holds in Corollaries 2.4 and 2.5.

Theorem 2.6. *If K is a subgroup of G and $\Pr(K, Aut(G)) = \frac{p+q-1}{pq}$, where p, q are the smallest prime divisors of $|Aut(G)|$ and $|K|$, respectively, then*

$$\frac{K}{L(K, Aut(G))} \cong \mathbf{Z_q}.$$

Proof. If p and q denote respectively the smallest prime divisors of $|Aut(G)|$ and $|K|$ then, by Theorem 2.3, we get

$$\frac{p+q-1}{pq} \leq \frac{1}{p}\left(1 + \frac{p-1}{|K : L(K, Aut(G))|}\right)$$

which gives $|K : L(K, Aut(G))| \leq q$. Hence, $\frac{K}{L(K,Aut(G))} \cong \mathbf{Z_q}$.
It is worth mentioning here that Theorem 2.6 generalizes [6, Theorem 2.4].

Theorem 2.7. *If K is a subgroup of G and* $\Pr(K, Aut(G)) = \frac{q^2+p-1}{pq^2}$*, where p, q are the smallest prime divisors of $|Aut(G)|$ and $|K|$, respectively, then*

$$\frac{K}{L(K, Aut(G))} \cong \mathbf{Z_q} \times \mathbf{Z_q}.$$

Further, if $|K|$ and $|Aut(G)|$ are even and $\Pr(K, Aut(G)) = \frac{5}{8}$*, then* $\frac{K}{L(K,Aut(G))} \cong \mathbf{Z_2} \times \mathbf{Z_2}$.

Proof. If p and q denote respectively the smallest prime divisors of $|Aut(G)|$ and $|K|$ then, by Theorem 2.3, we get

$$\frac{q^2+p-1}{pq^2} \leq \frac{1}{p}\left(1 + \frac{p-1}{|K : L(K, Aut(G))|}\right).$$

This gives $|K : L(K, Aut(G))| \leq q^2$. Since K is non-abelian, $|K : L(K, Aut(G))| \neq 1, q$. Hence, $\frac{K}{L(K,Aut(G))} \cong \mathbf{Z_q} \times \mathbf{Z_q}$.
The following result gives partial converses of Theorems 2.6 and 2.7, respectively.

Proposition 2.8. *Let K be a subgroup of G. Let p, q be the smallest primes dividing $|Aut(G)|$, $|K|$, respectively, and $|Aut(G) : C_{Aut(G)}(a)| = p$ for all $a \in K \setminus L(K, Aut(G))$.*

(a) *If* $\frac{K}{L(K,Aut(G))} \cong \mathbf{Z_q}$*, then* $\Pr(K, Aut(G)) = \frac{p+q-1}{pq}$.

(b) *If* $\frac{K}{L(K,Aut(G))} \cong \mathbf{Z_q} \times \mathbf{Z_q}$*, then* $\Pr(K, Aut(G)) = \frac{q^2+p-1}{pq^2}$.

Proof. Since $|Aut(G) : C_{Aut(G)}(a)| = p$ for all $a \in K \setminus L(K, Aut(G))$, we have $|C_{Aut(G)}(a)| = \frac{|Aut(G)|}{p}$ for all $a \in K \setminus L(K, Aut(G))$. Therefore, by (1.2), we get

$$\begin{aligned}\Pr(K, Aut(G)) &= \tfrac{|L(K,Aut(G))|}{|K|} + \tfrac{1}{|K||Aut(G)|} \sum_{a \in K \setminus L(K,Aut(G))} |C_{Aut(G)}(a)| \\ &= \tfrac{|L(K,Aut(G))|}{|K|} + \tfrac{|K|-|L(K,Aut(G))|}{p|K|}.\end{aligned}$$

Thus

$$\Pr(K, Aut(G)) = \frac{1}{p}\left(1 + \frac{p-1}{|K : L(K, Aut(G))|}\right). \tag{2.3}$$

Hence, the results follow from (2.3).

For any subgroup K of G, let $m_K = \min\{|orb(a)| : a \in K \setminus L(K, Aut(G))\}$. The following theorem gives an upper bound for $\Pr(K, Aut(G))$ involving m_K.

Theorem 2.9. *If K is a subgroup of G, then*

$$\Pr(K, Aut(G)) \leq \frac{1}{m_K}\left(1 + \frac{m_K - 1}{|K : L(K, Aut(G))|}\right)$$

with equality if and only if $m_K = |orb(a)|$ for all $a \in K \setminus L(K, Aut(G))$.

Proof. Since $|orb(a)| \geq m_K$ for all $a \in K \setminus L(K, Aut(G))$, we have

$$\sum_{a \in K \setminus L(K, Aut(G))} \frac{1}{|orb(a)|} \leq \frac{|K| - |L(K, Aut(G))|}{m_K}.$$

Hence, the result follows from (2.1).

For any two integers $r \geq s$, we have

$$\frac{1}{s}\left(1 + \frac{s-1}{|K : L(K, Aut(G))|}\right) \geq \frac{1}{r}\left(1 + \frac{r-1}{|K : L(K, Aut(G))|}\right). \tag{2.4}$$

Therefore, if p is the smallest prime dividing $|Aut(G)|$ then $2 \leq p \leq m_K$ and hence, by (2.4), we have

$$\begin{aligned} \tfrac{1}{m_K}\left(1 + \tfrac{m_K - 1}{|K:L(K,Aut(G))|}\right) &\leq \tfrac{1}{p}\left(1 + \tfrac{p-1}{|K:L(K,Aut(G))|}\right) \\ &\leq \tfrac{1}{2}\left(1 + \tfrac{1}{|K:L(K,Aut(G))|}\right). \end{aligned}$$

This shows that Theorem 2.9 gives better upper bound than the upper bounds obtained in [6, Theorem 2.3 (i)] and Theorem 2.1.

Note that if we replace $Aut(G)$ by the inner automorphism group $Inn(G)$ of G, then from (2.1), we get $\Pr(K, Inn(G)) = \Pr(K, G)$ where

$$\Pr(K, G) = \frac{|\{(u, v) \in K \times G : uv = vu\}|}{|K||G|}.$$

Various properties of the ratio $\Pr(K, G)$ are studied in [3] and [9]. We conclude this section showing that $\Pr(K, Aut(G))$ is bounded by $\Pr(K, G)$.

Proposition 2.10. *If K is a subgroup of G then*

$$\Pr(K, Aut(G)) \leq \Pr(K, G).$$

Proof. From [9, Lemma 1], we get

$$\operatorname{Pr}(K,G) = \frac{1}{|K|}\sum_{a\in K}\frac{1}{|Cl_G(a)|} \tag{2.5}$$

where $Cl_G(a) = \{\nu(a) : \nu \in Inn(G)\}$. Since $Cl_G(a) \subseteq orb(a)$ for all $a \in K$, the result follows from (1.3) and (2.5).

3. Some lower bounds

We begin this section with the following bound.

Theorem 3.1. *If K a subgroup of G, then*

$$\operatorname{Pr}(K, Aut(G)) \geq \frac{|L(K, Aut(G))|}{|K|} + \frac{p(|K| - |X_K| - |L(K, Aut(G))|) + |X_K|}{|K||Aut(G)|}$$

where p is the smallest prime dividing $|Aut(G)|$,
$X_K = \{a \in K : C_{Aut(G)}(a) = \{I\}\}$ and I is the identity of $Aut(G)$.

Proof. Note that $X_K \cap L(K, Aut(G)) = \emptyset$. Therefore

$$\begin{aligned}\sum_{a\in K}|C_{Aut(G)}(a)| = \quad & |X_K| + |Aut(G)||L(K, Aut(G))| \\ & + \sum_{a\in K\setminus(X_K\cup L(K,Aut(G)))}|C_{Aut(G)}(a)|.\end{aligned}$$

If $a \in K \setminus (X_K \cup L(K, Aut(G)))$ then $\{I\} < C_{Aut(G)}(a)$ which implies $|C_{Aut(G)}(a)| \geq p$. Therefore

$$\begin{aligned}\sum_{a\in K}|C_{Aut(G)}(a)| \geq \quad & |X_K| + |Aut(G)||L(K, Aut(G))| \\ & + p(|K| - |X_K| - |L(K, Aut(G))|).\end{aligned} \tag{3.1}$$

Hence, the result follows from (1.2) and (3.1).

Now we obtain two lower bounds analogous to the lower bounds obtained in [9, Theorem A] and [8, Theorem 1].

Theorem 3.2. *If K is a subgroup of G, then*

$$\operatorname{Pr}(K, Aut(G)) \geq \frac{1}{|S(K, Aut(G))|}\left(1 + \frac{|S(K, Aut(G))| - 1}{|K : L(K, Aut(G))|}\right)$$

with equality if and only if $orb(a) = aS(K, Aut(G))$ for all $a \in K\setminus L(K, Aut(G))$.

Proof. For all $a \in K \setminus L(K, Aut(G))$ and $\nu \in Aut(G)$ we get $\nu(a) = a[a, \nu] \in aS(K, Aut(G))$. It follows that $orb(a) \subseteq aS(K, Aut(G))$ and hence

$$|orb(a)| \leq |S(K, Aut(G))|$$

for all $a \in K \setminus L(K, Aut(G))$. By (1.3), we have

$$\begin{aligned}\Pr(K, Aut(G)) &= \tfrac{1}{|K|}\left(\sum_{a\in L(K,Aut(G))} \tfrac{1}{|orb(a)|} + \sum_{a\in K\setminus L(K,Aut(G))} \tfrac{1}{|orb(a)|}\right)\\ &\geq \tfrac{|L(K,Aut(G))|}{|K|} + \tfrac{1}{|K|}\sum_{a\in K\setminus L(K,Aut(G))} \tfrac{1}{|S(K,Aut(G))|}.\end{aligned}$$

Hence, the result follows.

The following corollary is a generalization of [1, Equation (3)].

Corollary 3.3. *If $K$is a subgroup of G, then*

$$\Pr(K, Aut(G)) \geq \frac{1}{|[K, Aut(G)]|}\left(1 + \frac{|[K, Aut(G)]| - 1}{|K : L(K, Aut(G))|}\right).$$

Proof. The result follows from Theorem 3.2 and (2.4) noting that

$$|[K, Aut(G)]| \geq |S(K,\ Aut(G))|.$$

It is clear from the above proof that Theorem 3.2 gives better lower bound than Corollary 3.3.

Also

$$\tfrac{1}{|[K,Aut(G)]|}\left(1 + \tfrac{|[K,Aut(G)]|-1}{|K:L(K,Aut(G))|}\right) \geq \tfrac{|L(K,Aut(G))|}{|K|} + \tfrac{p(|K|-|L(K,Aut(G))|)}{|K||Aut(G)|}.$$

Hence, the lower bound given by Corollary 3.3 is better than that in [6, Theorem 2.3 (i)].

The following result is a generalization of [1, Proposition 3] which gives several equivalent conditions for equality in Corollary 3.3.

Proposition 3.4. *If K is a subgroup of G then the following statements are equivalent.*

(a) $\Pr(K, Aut(G)) = \tfrac{1}{|[K,Aut(G)]|}\left(1 + \tfrac{|[K,Aut(G)]|-1}{|K:L(K,Aut(G))|}\right)$.

(b) $|orb(a)| = |[K, Aut(G)]|$ *for all* $a \in K \setminus L(K, Aut(G))$.

(c) $orb(a) = a[K, Aut(G)]$ *for all* $a \in K\setminus L(K, Aut(G))$, *and so* $[K, Aut(G)] \subseteq L(K, Aut(G))$.

(d) $C_{Aut(G)}(a) \lhd Aut(G)$ and $\frac{Aut(G)}{C_{Aut(G)}(a)} \cong [K, Aut(G)]$ *for all* $a \in K \setminus L(K, Aut(G))$.

(e) $[K, Aut(G)] = \{a^{-1}\nu(a) : \nu \in Aut(G)\}$ *for all* $a \in K \setminus L(K, Aut(G))$.

Proof. First note that for all $a \in K$

$$orb(a) \subseteq a[K, Aut(G)]. \tag{3.2}$$

Suppose that (a) holds. Then, by (1.3), we have

$$\sum_{a \in K \setminus L(K, Aut(G))} \left(\frac{1}{|orb(a)|} - \frac{1}{|[K, Aut(G)]|} \right) = 0.$$

Now using (3.2), we get (b). Also, if (b) holds then from (1.3), we have (a). Thus (a) and (b) are equivalent.

Suppose that (b) holds. Then for all $a \in K \setminus L(K, Aut(G))$ we have $|orb(a)| = |a[K, Aut(G)]|$. Hence, using (3.2) we get (c). If $[K, Aut(G)] \nsubseteq L(K, Aut(G))$ then there exist $z \in [K, Aut(G)] \setminus L(K, Aut(G))$. Therefore $orb(z) = z[K, Aut(G)] = [K, Aut(G)]$, a contradiction. Hence $[K, Aut(G)] \subseteq L(K, Aut(G))$. It can be seen that the mapping $f : Aut(G) \to [K, Aut(G)]$ given by $\nu \mapsto a^{-1}\nu(a)$, where a is a fixed element of $K \setminus L(K, Aut(G))$, is a surjective homomorphism with kernel $C_{Aut(G)}(a)$. Therefore (d) follows.

Since $|Aut(G)|/|C_{Aut(G)}(a)| = |orb(a)|$ for all $a \in K \setminus L(K, Aut(G))$ we have (b).

Thus (b), (c), and (d) are equivalent.

Also $orb(a) = a[K, Aut(G)]$ if and only if $a^{-1}orb(a) = [K, Aut(G)]$ for all $a \in K \setminus L(K, Aut(G))$, which gives the equivalence of (c) and (e). This completes the proof.

Let $M_K = \max\{|orb(a)| : a \in K \setminus L(K, Aut(G))\}$. The following theorem gives a lower bound for $\Pr(K, Aut(G))$ involving M_K.

Theorem 3.5. *If K is a subgroup of G then*

$$\Pr(K, Aut(G)) \geq \frac{1}{M_K} \left(1 + \frac{M_K - 1}{|K : L(K, Aut(G))|} \right)$$

with equality if and only if $M_K = |orb(a)|$ *for all* $a \in K \setminus L(K, Aut(G))$.

Proof. Since $|orb(a)| \leq M_K$ for all $a \in K \setminus L(K, Aut(G))$, we have

$$\sum_{a \in K \setminus L(K,Aut(G))} \frac{1}{|orb(a)|} \geq \frac{|K| - |L(K, Aut(G))|}{M_K}.$$

Hence, the result follows from (2.1).
For any $a \in K \setminus L(K, Aut(G))$ we have $orb(a) \subseteq aS(K, Aut(G))$ where $aS(K, Aut(G)) = \{ak : k \in S(K, Aut(G))\}$. Therefore $|S(K, Aut(G))| \geq M_K$ and hence, by (2.4), we have

$$\frac{1}{M_K}\left(1 + \frac{M_K - 1}{|K : L(K, Aut(G))|}\right) \geq \frac{1}{|S(K, Aut(G))|}\left(1 + \frac{|S(K, Aut(G))| - 1}{|K : L(K, Aut(G))|}\right).$$

This shows that Theorem 3.5 gives better lower bound than Theorem 3.2.

4. Autoisoclinism between pairs of groups

Hall [4], in the year 1940, introduced isoclinism between two groups. After many years, autoisoclinism between two groups was introduced by Moghaddam et al. [7] in 2013. Let G_1 and G_2 be two groups. Suppose there exist isomorphisms $\phi : \frac{G_1}{L(G_1)} \to \frac{G_2}{L(G_2)}$, $\gamma : Aut(G_1) \to Aut(G_2)$ and $\beta : [G_1, Aut(G_1)] \to [G_2, Aut(G_2)]$ such that the diagram

$$\begin{array}{ccc}
\frac{G_1}{L(G_1)} \times Aut\,(G_1) & \xrightarrow{\phi \times \gamma} & \frac{G_2}{L(G_2)} \times Aut\,(G_2) \\
\Big\downarrow a_{(G_1,Aut(G_1))} & & \Big\downarrow a_{(G_2,Aut(G_2))} \\
[G_1, Aut\,(G_1)] & \xrightarrow{\beta} & [G_2, Aut\,(G_2)]
\end{array}$$

commutes, where the maps $a_{(G_i,Aut(G_i))} : \frac{G_i}{L(G_i)} \times Aut(G_i) \to [G_i, Aut(G_i)]$ for $i = 1, 2$ are given by

$$a_{(G_i,Aut(G_i))}(x_i L(G_i), \nu_i) = [x_i, \nu_i].$$

Then the groups G_1 and G_2 are called autoisoclinic and the triple (ϕ, γ, β) is an autoisoclinism between them. A generalization of this notion of autoisoclinism between two groups is given below.

Definition 4.1. *Let K_1 and K_2 be two subgroups of the groups G_1 and G_2 respectively. A pair of groups (K_1, G_1) is said to be autoisoclinic to another pair of groups (K_2, G_2) if there exist isomorphisms $\phi : \frac{K_1}{L(K_1,Aut(G_1))} \to$*

$\frac{K_2}{L(K_2,Aut(G_2))}$, $\gamma : Aut(G_1) \to Aut(G_2)$ and $\beta : [K_1, Aut(G_1)] \to [K_2, Aut(G_2)]$ *such that the diagram*

$$\begin{array}{ccc} \frac{K_1}{L(K_1,Aut(G_1))} \times Aut\,(G_1) & \xrightarrow{\phi \times \gamma} & \frac{K_2}{L(K_2,Aut(G_2))} \times Aut\,(G_2) \\ \Big\downarrow a_{(K_1,Aut(G_1))} & & \Big\downarrow a_{(K_2,Aut(G_2))} \\ [K_1, Aut\,(G_1)] & \xrightarrow{\beta} & [K_2, Aut\,(G_2)] \end{array}$$

commutes, where the maps $a_{(K_i,Aut(G_i))} : \frac{K_i}{L(K_i,Aut(G_i))} \times Aut(G_i) \to [K_i, Aut(G_i)]$ *for* $i = 1, 2$ *are given by*

$$a_{(K_i,Aut(G_i))}(x_i L(K_i, Aut(G_i)), \nu_i) = [x_i, \nu_i].$$

Such a triple (ϕ, γ, β) *is said to be an autoisoclinism between the pairs* (K_1, G_1) *and* (K_2, G_2).

Theorem 4.2. *Let* G_1 *and* G_2 *be two finite groups with subgroups* K_1 *and* K_2*, respectively. If the pairs* (K_1, G_1) *and* (K_2, G_2) *are autoisoclinic, then*

$$\Pr(K_1, Aut(G_1)) = \Pr(K_2, Aut(G_2)).$$

Proof. Consider the sets $\mathcal{S} = \{(x_1 L(K_1, Aut(G_1)), \nu_1) \in \frac{K_1}{L(K_1,Aut(G_1))} \times Aut(G_1) : \nu_1(x_1) = x_1\}$ and $\mathcal{T} = \{(x_2 L(K_2, Aut(G_2)), \nu_2) \in \frac{K_2}{L(K_2,Aut(G_2))} \times Aut(G_2) : \nu_2(x_2) = x_2\}$. Since (K_1, G_1) is autoisoclinic to (K_2, G_2) we have $|\mathcal{S}| = |\mathcal{T}|$. Again, it is clear that

(4.1) $|\{(x_1, \nu_1) \in K_1 \times Aut(G_1) : \nu_1(x_1) = x_1\}| = |L(K_1, Aut(G_1))||\mathcal{S}|$

and

(4.2) $|\{(x_2, \nu_2) \in K_2 \times Aut(G_2) : \nu_2(x_2) = x_2\}| = |L(K_2, Aut(G_2))||\mathcal{T}|.$

Hence, the result follows from (1.1), (4.1), and (4.2).

Note that Theorem 4.2 is a generalization of [10, Lemma 2.5]. We conclude the paper by noting that the bounds obtained in Section 2 and Section 3 for $\Pr(K, Aut(G))$ are also applicable for $\Pr(K_1, Aut(G_1))$ if (K_1, G_1) is autoisoclinic to (K, G).

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