



New inequalities for strongly exponentially generalized functions with applications

Artion Kashuri

University Ismail Qemali, Albania

and

Rozana Liko

University Ismail Qemali, Albania

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Abstract

The aim of this paper is to introduce a new class of functions called strongly exponentially generalized $(m, \nu_1, \nu_2, g_1, g_2)$. Some new integral inequalities of trapezium-type for strongly exponentially generalized $(m, \nu_1, \nu_2, g_1, g_2)$ functions with modulus c via Riemann-Liouville fractional integral are established. Also, some new estimates with respect to trapezium-type integral inequalities for strongly exponentially generalized $(m, \nu_1, \nu_2, g_1, g_2)$ functions with modulus c via general fractional integrals are obtained. We show that the strongly exponentially generalized $(m, \nu_1, \nu_2, g_1, g_2)$ functions with modulus c includes several other classes of functions. At the end, some new error estimates for trapezoidal quadrature formula are provided as well. This results may stimulate further research in different areas of pure and applied sciences.

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1. Introduction

The class of convex functions is well known in the literature and is usually defined in the following way:

Definition 1.1. Let I be an interval in \mathbf{R} . A function $\varsigma : I \rightarrow \mathbf{R}$, is said to be convex on I if the inequality

$$(1.1) \quad \varsigma(t\varsigma_1 + (1-t)\varsigma_2) \leq t\varsigma(\varsigma_1) + (1-t)\varsigma(\varsigma_2)$$

holds for all $\varsigma_1, \varsigma_2 \in I$ and $t \in [0, 1]$. Also, we say that ς is concave, if the inequality in (1.1) holds in the reverse direction.

The following inequality, named Hermite-Hadamard inequality, is one of the most famous inequalities in the literature for convex functions.

Theorem 1.2. Let $\varsigma : I \subseteq \mathbf{R} \rightarrow \mathbf{R}$ be a convex function and $\varsigma_1, \varsigma_2 \in I$ with $\varsigma_1 < \varsigma_2$. Then the following inequality holds:

$$(1.2) \quad \varsigma\left(\frac{\varsigma_1 + \varsigma_2}{2}\right) \leq \frac{1}{\varsigma_2 - \varsigma_1} \int_{\varsigma_1}^{\varsigma_2} \varsigma(x) dx \leq \frac{\varsigma(\varsigma_1) + \varsigma(\varsigma_2)}{2}.$$

This inequality (1.2) is also known as trapezium inequality.

The trapezium inequality has remained an area of great interest due to its wide applications in the field of mathematical analysis. Authors of recent decades have studied (1.2) in the premises of newly invented definitions due to motivation of convex function. Interested readers see the references [4]-[12],[15]-[21],[29, 30, 36, 38, 39].

Now, let us recall the following definitions.

Definition 1.3. [16] A function $\varsigma : I \subseteq \mathbf{R} \rightarrow \mathbf{R}$ is called strongly convex with modulus $c \in \mathbf{R}^+$, if

$$\varsigma(tx + (1-t)y) \leq t\varsigma(x) + (1-t)\varsigma(y) - ct(1-t)(x-y)^2$$

holds for every $x, y \in I$ and $t \in [0, 1]$.

Strongly convex functions have been introduced by Polyak, see [16] and references therein. Since strong convexity is a strengthening of the notion of convexity, some properties of strongly convex functions are just stronger versions of known properties of convex functions. Strongly convex functions have been used for proving the convergence of a gradient type algorithm for minimizing a function. They play an important role in optimization theory and mathematical economics.

Definition 1.4. [16, 17] A function $\varsigma : I \subseteq \mathbf{R} \rightarrow \mathbf{R}$ is called strongly m -convex with $m \in [0, 1]$ and modulus $c \in \mathbf{R}^+$, if

$$\varsigma(tx + m(1-t)y) \leq t\varsigma(x) + m(1-t)\varsigma(y) - cmt(1-t)(y-x)^2$$

holds for every $x, y \in I$ and $t \in [0, 1]$.

Definition 1.5. [29] A function: $\varsigma : I \subseteq \mathbf{R} \rightarrow \mathbf{R}$ is said to be m -MT-convex, if ς is positive and $\forall x, y \in I$, and $t \in (0, 1)$, among $m \in (0, 1]$, satisfies the following inequality

$$(1.3) \quad \varsigma(tx + m(1-t)y) \leq \frac{\sqrt{t}}{2\sqrt{1-t}}\varsigma(x) + \frac{m\sqrt{1-t}}{2\sqrt{t}}\varsigma(y).$$

Definition 1.6. [3] A set $O \subseteq \mathbf{R}^n$ is said to be invex respecting the mapping $\tau : O \times O \rightarrow \mathbf{R}^n$, if $x + t\tau(y, x) \in O$ for every $x, y \in O$ and $t \in [0, 1]$.

Definition 1.7. [20] Let $g : [0, 1] \rightarrow \mathbf{R}$ be a nonnegative function and $g \neq 0$. The function ς on the invex set O is said to be g -preinvex with respect to τ , if

$$(1.4) \quad \varsigma(x + t\tau(y, x)) \leq g(1-t)\varsigma(x) + g(t)\varsigma(y)$$

for each $x, y \in O$ and $t \in [0, 1]$ where $\varsigma(\cdot) > 0$.

Definition 1.8. [11] A set $O \subseteq \mathbf{R}^n$ is named as m -invex with respect to the mapping $\tau : O \times O \rightarrow \mathbf{R}^n$ for some fixed $m \in (0, 1]$, if $mx + t\tau(y, mx) \in O$ holds for each $x, y \in O$ and any $t \in [0, 1]$.

Remark 1.9. Taking $m = 1$ in definition 1.8, the mapping $\tau(y, mx)$ reduce to $\tau(y, x)$, and then we get definition 1.6.

Definition 1.10. [32] Let $O \subseteq \mathbf{R}$ be m -invex set respecting the mapping $\tau : O \times O \rightarrow \mathbf{R}$ and $g_1, g_2 : [0, 1] \rightarrow [0, +\infty)$. A function $\varsigma : O \rightarrow \mathbf{R}$ is said to be generalized (m, g_1, g_2) -preinvex, if

$$(1.5) \quad \varsigma(mx + t\tau(y, mx)) \leq mg_1(t)\varsigma(x) + g_2(t)\varsigma(y)$$

is valid for all $x, y \in O$ and $t \in [0, 1]$, for some fixed $m \in (0, 1]$.

Definition 1.11. [18] Let $\varsigma \in L[\iota_1, \iota_2]$. The Riemann-Liouville integrals $J_{\iota_1+}^{\alpha}\varsigma$ and $J_{\iota_2-}^{\alpha}\varsigma$ of order $\alpha > 0$ with $\iota_1 \geq 0$ are defined by

$$J_{\iota_1+}^{\alpha}\varsigma(x) = \frac{1}{\Gamma(\alpha)} \int_{\iota_1}^x (x-t)^{\alpha-1} \varsigma(t) dt, \quad x > \iota_1$$

and

$$J_{\iota_2-}^{\alpha}\varsigma(x) = \frac{1}{\Gamma(\alpha)} \int_x^{\iota_2} (t-x)^{\alpha-1} \varsigma(t) dt, \quad \iota_2 > x,$$

where $\Gamma(\alpha) = \int_0^{+\infty} e^{-u} u^{\alpha-1} du$. Here $J_{\iota_1+}^0\varsigma(x) = J_{\iota_2-}^0\varsigma(x) = \varsigma(x)$.

Note that $\alpha = 1$, the fractional integral reduces to the classical integral.

In [33], Raina R. K. introduced a class of functions defined formally by

$$(1.6) \quad \mathcal{F}_{\rho,\lambda}^{\sigma}(x) = \mathcal{F}_{\rho,\lambda}^{\sigma(0),\sigma(1),\dots}(x) = \sum_{k=0}^{+\infty} \frac{\sigma(k)}{\Gamma(\rho k + \lambda)} x^k,$$

where $\rho, \lambda > 0$, $|x| < R$ and $\sigma = (\sigma(0), \dots, \sigma(k), \dots)$ is a bounded sequence of positive real numbers. Note that, if we take in (1.6) $\rho = 1, \lambda = 0$ and $\sigma(k) = \frac{(\alpha)_k(\beta)_k}{(\gamma)_k}$ for $k = 0, 1, 2, \dots$, where α, β and γ are parameters which can take arbitrary real or complex values (provided that $\gamma \neq 0, -1, -2, \dots$), and the symbol $(a)_k$ denote the quantity

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} = a(a+1)\dots(a+k-1), \quad k = 0, 1, 2, \dots,$$

and restrict its domain to $|x| \leq 1$ (with $x \in \mathbf{C}$), then we have the classical hypergeometric function, that is

$$\mathcal{F}_{\rho,\lambda}^{\sigma}(x) = F(\alpha, \beta; \gamma; x) = \sum_{k=0}^{+\infty} \frac{(\alpha)_k(\beta)_k}{(\gamma)_k k!} x^k.$$

Also, if $\sigma = (1, 1, \dots)$ with $\rho = \alpha$, ($Re(\alpha) > 0$), $\lambda = 1$ and restricting its domain to $z \in \mathbf{C}$ in (1.6) then we have the classical Mittag-Leffler function

$$E_{\alpha}(z) = \sum_{k=0}^{+\infty} \frac{1}{\Gamma(\alpha k + 1)} z^k.$$

Also, Noor in [23], introduced a generalized convex function concept as follows:

Let O be a non empty closed set in a normed space H . We denote by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ the inner product and norm respectively. Let $\phi : O \rightarrow \mathbf{R}$ be a continuous function.

Definition 1.12. Let $u_1 \in O$. The set O is said to be ϕ -convex set, if there exists a function ϕ such that $u_1 + te^{i\phi}(u_1 - u_2) \in O$, for all $u_1, u_2 \in O$ and $t \in [0, 1]$.

With this definition also is introduced the following.

Definition 1.13. A function ς defined on O is said to be ϕ -convex if and only if there exists a function ϕ such that

$$\varsigma(u_1 + te^{i\phi}(u_1 - u_2)) \leq t\varsigma(u_1) + (1-t)\varsigma(u_2),$$

for all $u_1, u_2 \in O$ and $t \in [0, 1]$.

Furthermore, let us define a function $\phi : [0, \infty) \rightarrow [0, \infty)$ satisfying the following conditions:

$$(1.7) \quad \int_0^1 \frac{\phi(t)}{t} dt < \infty,$$

$$(1.8) \quad \frac{1}{\mathbf{A}} \leq \frac{\phi(s)}{\phi(r)} \leq \mathbf{A} \text{ for } \frac{1}{2} \leq \frac{s}{r} \leq 2$$

$$(1.9) \quad \frac{\phi(r)}{r^2} \leq \mathbf{B} \frac{\phi(s)}{s^2} \text{ for } s \leq r$$

$$(1.10) \quad \left| \frac{\phi(r)}{r^2} - \frac{\phi(s)}{s^2} \right| \leq \mathbf{C} |r - s| \frac{\phi(r)}{r^2} \text{ for } \frac{1}{2} \leq \frac{s}{r} \leq 2$$

where $\mathbf{A}, \mathbf{B}, \mathbf{C} > 0$ are independent of $r, s > 0$. If $\phi(r)r^\alpha$ is increasing for some $\alpha \geq 0$ and $\frac{\phi(r)}{r^\beta}$ is decreasing for some $\beta \geq 0$, then ϕ satisfies (1.7)–(1.10), see reference [37]. Therefore, Sarikaya and Ertuğral [36] defined the following left-sided and right-sided generalized fractional integral operators, respectively, as follows:

$$(1.11) \quad {}_{\imath_1^+} I_\phi \varsigma(x) = \int_{\imath_1}^x \frac{\phi(x-t)}{x-t} \varsigma(t) dt, \quad x > \imath_1,$$

$$(1.12) \quad {}_{\imath_2^-} I_\phi \varsigma(x) = \int_x^{\imath_2} \frac{\phi(t-x)}{t-x} \varsigma(t) dt, \quad x < \imath_2.$$

This fractional integral operators are a new generalization of fractional integrals such as the Riemann-Liouville fractional integral, the k -Riemann-Liouville fractional integral, Katugampola fractional integrals, the conformable fractional integral, Hadamard fractional integrals, etc. To read

more about fractional analysis, see references [13, 14, 22, 35].

An important class of convex functions, which is called exponential convex functions, was introduced and studied by Antczak [2], Dragomir et al [9] and Noor et al [28]. Alirezai and Mathar [1] have investigated their basic properties along with their potential applications in statistics and information theory. Awan et al. [5] and Pecarić and Jakšetić [31] defined another kind of exponential convex functions and have shown that the class of exponential convex functions unifies various unrelated concepts.

Definition 1.14. [2, 9, 27] A function $\varsigma : O \subseteq \mathbf{R} \rightarrow \mathbf{R}$ is said to be exponentially convex function, if

$$(1.13) \quad e^{\varsigma((1-t)\nu_1+t\nu_2)} \leq (1-t)e^{\varsigma(\nu_1)} + te^{\varsigma(\nu_2)}$$

holds for all $\nu_1, \nu_2 \in O$, $t \in [0, 1]$, where ς is positive.

For the applications of exponentially convex functions and strongly exponentially convex functions in different field of statistics, information theory and mathematical sciences, see [1, 2, 5], [24]-[28] and the references therein.

Definition 1.15. [34] A function $\varsigma : O \subseteq \mathbf{R} \rightarrow \mathbf{R}$ is said to be exponentially m -convex function, where $m \in (0, 1]$, if

$$(1.14) \quad e^{\varsigma((1-t)\nu_1+mt\nu_2)} \leq (1-t)e^{\varsigma(\nu_1)} + mte^{\varsigma(\nu_2)}$$

holds for all $\nu_1, \nu_2 \in O$, $t \in [0, 1]$, where ς is positive.

Motivated by the above literatures, the main objective of this article is to establish in Section 2 fractional integral inequalities using a new class of functions called strongly exponentially generalized $(m, \nu_1, \nu_2, g_1, g_2)$ with modulus c . Also, using a new identity pertaining differentiable functions defined on m -invex set as auxiliary result, some new Hermite-Hadamard inequalities for strongly exponentially generalized $(m, \nu_1, \nu_2, g_1, g_2)$ functions with modulus c via Riemann-Liouville fractional integral will obtain. Also, some new estimates with respect to trapezium-type integral inequalities for strongly exponentially generalized $(m, \nu_1, \nu_2, g_1, g_2)$ functions with modulus c via general fractional integrals will given. Various special cases will be discussed. In Section 3, some new error estimates for trapezoidal quadrature formula will be given. At the end, a briefly conclusion is given.

2. Main results

Now, we are in a position to introduce a new class of functions so-called strongly exponentially generalized $(m, \nu_1, \nu_2, g_1, g_2)$ with modulus c as follows:

Definition 2.1. Let $O = [m\iota_1, m\iota_1 + \mathcal{F}_{\rho, \lambda}^\sigma(\iota_2 - m\iota_1)] \subseteq \mathbf{R}$ be m -invex set for some fixed $m \in (0, 1]$, where $\mathcal{F}_{\rho, \lambda}^\sigma(\cdot)$ is defined from (1.6) and $g_1, g_2 : [0, 1] \rightarrow [0, +\infty)$. A function $\varsigma : O \rightarrow (0, +\infty)$ is called strongly exponentially generalized $(m, \nu_1, \nu_2, g_1, g_2)$ with modulus $c > 0$, if

$$e^{\varsigma(m\iota_1 + t\mathcal{F}_{\rho, \lambda}^\sigma(\iota_2 - m\iota_1))} \leq mg_1(t)e^{\nu_1\varsigma(\iota_1)} + g_2(t)e^{\nu_2\varsigma(\iota_2)} - cm(\iota_2 - \iota_1)^2 g_1(t)g_2(t) \quad (2.1)$$

holds for all $\iota_1, \iota_2 \in O$, $t \in [0, 1]$ and $\nu_1, \nu_2 \in \mathbf{R}$.

Remark 2.2. In definition 2.1, taking $c \rightarrow 0^+$ and choose $\nu_1 = \nu_2 = 1$, $g_1(t) = 1 - t$, $g_2(t) = t$, where $\mathcal{F}_{\rho, \lambda}^\sigma(\iota_2 - m\iota_1) = \iota_2 - m\iota_1$, this definition reduce to the definition 1.15.

Remark 2.3. Under the conditions of remark 2.2, taking $m = 1$, we get definition 1.14.

Remark 2.4. Let us discuss some special cases in definition 2.1 as follows:

- (I) Taking $g_1(t) = g(1 - t)$, $g_2(t) = g(t)$, then we get strongly exponentially generalized (m, ν_1, ν_2, g) functions with modulus c .
- (II) Taking $g_1(t) = g_2(t) = t(1 - t)$, then we get strongly exponentially generalized (m, ν_1, ν_2, tgs) functions with modulus c .
- (III) Taking $g_1(t) = \frac{\sqrt{1-t}}{2\sqrt{t}}$, $g_2(t) = \frac{\sqrt{t}}{2\sqrt{1-t}}$, then we get strongly exponentially generalized (m, ν_1, ν_2) -MT functions with modulus c .

In this section, we obtain Hermite-Hadamard type inequalities for strongly exponentially generalized $(m, \nu_1, \nu_2, g_1, g_2)$ function with modulus c via Riemann-Liouville fractional integral.

Theorem 2.5. Let $O = [m\iota_1, m\iota_1 + \mathcal{F}_{\rho,\lambda}^\sigma(\iota_2 - m\iota_1)] \subseteq \mathbf{R}$ be m -invex set for some fixed $m \in (0, 1]$, where $\iota_1 < \iota_2$ and $\mathcal{F}_{\rho,\lambda}^\sigma(\iota_2 - m\iota_1) > 0$. Suppose $g_1, g_2 : [0, 1] \rightarrow [0, +\infty)$ be continuous functions. Let $\varsigma, \vartheta : O \rightarrow (0, +\infty)$ be strongly exponentially generalized $(m, \nu_1, \nu_2, g_1, g_2)$ functions with modulus $c > 0$. If $\varsigma, \vartheta \in L(O)$, then for $\nu_1, \nu_2 \in \mathbf{R}$ and $\alpha > 0$, the following inequality holds:

$$\begin{aligned} & \frac{\Gamma(\alpha)}{\left(F_{\rho,\lambda}^\sigma(\iota_2 - m\iota_1)\right)^\alpha} \left\{ J_{(mi_1+F_{\rho,\lambda}^\sigma(\iota_2-mi_1))}^\alpha - e^{s(mi_1)} + J_{(mi_1+F_{\rho,\lambda}^\sigma(\iota_2-mi_2))}^\alpha - e^{\vartheta(mi_1)} \right\} \\ (2.2) \quad & \leq m \left(e^{\nu_1\varsigma(\iota_1)} + e^{\nu_1\vartheta(\iota_1)} \right) H_{g_1}(\alpha) + \left(e^{\nu_2\varsigma(\iota_2)} + e^{\nu_2\vartheta(\iota_2)} \right) H_{g_2}(\alpha) \\ & \quad - 2cm(\iota_2 - \iota_1)^2 \Psi_{g_1, g_2}(\alpha), \end{aligned}$$

where

$$\begin{aligned} \Psi_{g_1, g_2}(\alpha) &:= \int_0^1 t^{\alpha-1} g_1(t) g_2(t) dt, \quad H_{g_i}(\alpha) := \int_0^1 t^{\alpha-1} g_i(t) dt, \quad \forall i = 1, 2. \\ (2.3) \quad & \end{aligned}$$

Proof. From strongly exponentially generalized $(m, \nu_1, \nu_2, g_1, g_2)$ with modulus c of ς and ϑ for all $t \in [0, 1]$, we have

$$e^{\varsigma(m\iota_1+t\mathcal{F}_{\rho,\lambda}^\sigma(\iota_2-m\iota_1))} \leq mg_1(t)e^{\nu_1\varsigma(\iota_1)} + g_2(t)e^{\nu_2\varsigma(\iota_2)} - cm(\iota_2 - \iota_1)^2 g_1(t)g_2(t)$$

and

$$e^{\vartheta(m\iota_1+t\mathcal{F}_{\rho,\lambda}^\sigma(\iota_2-m\iota_1))} \leq mg_1(t)e^{\nu_1\vartheta(\iota_1)} + g_2(t)e^{\nu_2\vartheta(\iota_2)} - cm(\iota_2 - \iota_1)^2 g_1(t)g_2(t).$$

Adding both sides of the above inequalities, we get

$$\begin{aligned} (2.4) \quad & e^{\varsigma(m\iota_1+t\mathcal{F}_{\rho,\lambda}^\sigma(\iota_2-m\iota_1))} + e^{\vartheta(m\iota_1+t\mathcal{F}_{\rho,\lambda}^\sigma(\iota_2-m\iota_1))} \leq m \left(e^{\nu_1\varsigma(\iota_1)} + e^{\nu_1\vartheta(\iota_1)} \right) g_1(t) \\ & + \left(e^{\nu_2\varsigma(\iota_2)} + e^{\nu_2\vartheta(\iota_2)} \right) g_2(t) - 2cm(\iota_2 - \iota_1)^2 g_1(t)g_2(t). \end{aligned}$$

Multiplying both sides of inequality (2.4) with $t^{\alpha-1}$ and integrating over $[0, 1]$, we obtain

$$\begin{aligned} & \int_0^1 t^{\alpha-1} \left[e^{\varsigma(m\iota_1 + t\mathcal{F}_{\rho,\lambda}^\sigma(\iota_2 - m\iota_1))} + e^{\vartheta(m\iota_1 + t\mathcal{F}_{\rho,\lambda}^\sigma(\iota_2 - m\iota_1))} \right] dt \\ & \leq m \left(e^{\nu_1 \varsigma(\iota_1)} + e^{\nu_1 \vartheta(\iota_1)} \right) \int_0^1 t^{\alpha-1} g_1(t) dt + \left(e^{\nu_2 \varsigma(\iota_2)} + e^{\nu_2 \vartheta(\iota_2)} \right) \int_0^1 t^{\alpha-1} g_2(t) dt \\ & \quad - 2cm(\iota_2 - \iota_1)^2 \int_0^1 t^{\alpha-1} g_1(t) g_2(t) dt. \end{aligned}$$

Using definition 1.11, we get the required result. \square

Corollary 2.6. In Theorem 2.5, taking $c \rightarrow 0^+$, we get

$$\frac{\Gamma(\alpha)}{\left(\mathcal{F}_{\rho,\lambda}^\sigma(\iota_2 - m\iota_1)\right)^\alpha} \left\{ J_{(m\iota_1 + \mathcal{F}_{\rho,\lambda}^\sigma(\iota_2 - m\iota_1))^-}^\alpha e^{\varsigma(m\iota_1)} + J_{(m\iota_1 + \mathcal{F}_{\rho,\lambda}^\sigma(\iota_2 - m\iota_1))^-}^\alpha e^{\vartheta(m\iota_1)} \right\}$$

$$(2.5) \quad \leq m \left(e^{\nu_1 \varsigma(\iota_1)} + e^{\nu_1 \vartheta(\iota_1)} \right) H_{g_1}(\alpha) + \left(e^{\nu_2 \varsigma(\iota_2)} + e^{\nu_2 \vartheta(\iota_2)} \right) H_{g_2}(\alpha).$$

Corollary 2.7. In Theorem 2.5, if we choose $m = 1$ and $\mathcal{F}_{\rho,\lambda}^\sigma(\iota_2 - m\iota_1) = \iota_2 - m\iota_1$, we have

$$\begin{aligned} & \frac{\Gamma(\alpha)}{(\iota_2 - \iota_1)^\alpha} \left\{ J_{\iota_2^-}^\alpha e^{\varsigma(\iota_1)} + J_{\iota_2^-}^\alpha e^{\vartheta(\iota_1)} \right\} \\ & \leq \left(e^{\nu_1 \varsigma(\iota_1)} + e^{\nu_1 \vartheta(\iota_1)} \right) H_{g_1}(\alpha) + \left(e^{\nu_2 \varsigma(\iota_2)} + e^{\nu_2 \vartheta(\iota_2)} \right) H_{g_2}(\alpha) - 2c(\iota_2 - \iota_1)^2 \Psi_{g_1, g_2}(\alpha). \end{aligned} \quad (2.6)$$

Corollary 2.8. In Theorem 2.5, if we choose $\alpha = 1$, we obtain

$$\begin{aligned} & \frac{1}{\mathcal{F}_{\rho,\lambda}^\sigma(\iota_2 - m\iota_1)} \int_{m\iota_1}^{m\iota_1 + \mathcal{F}_{\rho,\lambda}^\sigma(\iota_2 - m\iota_1)} \left[e^{\varsigma(t)} + e^{\vartheta(t)} \right] dt \\ & \leq m \left(e^{\nu_1 \varsigma(\iota_1)} + e^{\nu_1 \vartheta(\iota_1)} \right) H_{g_1} + \left(e^{\nu_2 \varsigma(\iota_2)} + e^{\nu_2 \vartheta(\iota_2)} \right) H_{g_2} - 2cm(\iota_2 - \iota_1)^2 F_{g_1, g_2}, \end{aligned} \quad (2.7)$$

where

$$(2.8) \quad F_{g_1, g_2} := \int_0^1 g_1(t) g_2(t) dt, \quad H_{g_i} := \int_0^1 g_i(t) dt, \quad \forall i = 1, 2.$$

Theorem 2.9. Let $O = [m\iota_1, m\iota_1 + \mathcal{F}_{\rho,\lambda}^\sigma(\iota_2 - m\iota_1)] \subseteq \mathbf{R}$ be m -invex set for some fixed $m \in (0, 1]$, where $\iota_1 < \iota_2$ and $\mathcal{F}_{\rho,\lambda}^\sigma(\iota_2 - m\iota_1) > 0$. Suppose $g_1, g_2 : [0, 1] \rightarrow [0, +\infty)$ be continuous functions. Let $\varsigma, \vartheta : O \rightarrow (0, +\infty)$ be strongly exponentially generalized $(m, \nu_1, \nu_2, g_1, g_2)$ functions with modulus $c > 0$. If $\varsigma, \vartheta \in L(O)$, then for $\nu_1, \nu_2 \in \mathbf{R}$ and $\alpha > 0$, the following inequality holds:

$$\frac{\Gamma(\alpha)}{\left(\mathcal{F}_{\rho,\lambda}^\sigma(\iota_2 - m\iota_1)\right)^\alpha} \left\{ J_{(m\iota_1)^+}^\alpha e^{\varsigma(m\iota_1 + \mathcal{F}_{\rho,\lambda}^\sigma(\iota_2 - m\iota_1))} + J_{(m\iota_1 + \mathcal{F}_{\rho,\lambda}^\sigma(\iota_2 - m\iota_1))^+}^\alpha e^{\vartheta(m\iota_1)} \right\}$$

$$(2.9) \leq m \left(e^{\nu_1 \varsigma(\iota_1)} C_{g_1}(\alpha) + e^{\nu_1 \vartheta(\iota_1)} H_{g_1}(\alpha) \right) + e^{\nu_2 \varsigma(\iota_2)} C_{g_2}(\alpha) + e^{\nu_2 \vartheta(\iota_2)} H_{g_2}(\alpha) - cm(\iota_2 - \iota_1)^2 D_{g_1, g_2}(\alpha),$$

where

$$(2.10) \quad D_{g_1, g_2}(\alpha) := \int_0^1 \left[t^\alpha + (1-t)^{\alpha-1} \right] g_1(t) g_2(t) dt,$$

$$(2.11) \quad C_{g_i}(\alpha) := \int_0^1 (1-t)^{\alpha-1} g_i(t) dt, \quad \forall i = 1, 2$$

and $H_{g_1}(\alpha), H_{g_2}(\alpha)$ are defined as in Theorem 2.5.

Proof. From strongly exponentially generalized $(m, \nu_1, \nu_2, g_1, g_2)$ with modulus c of ς and ϑ for all $t \in [0, 1]$, we have

$$e^{\varsigma(m\iota_1 + t\mathcal{F}_{\rho,\lambda}^\sigma(\iota_2 - m\iota_1))} \leq mg_1(t)e^{\nu_1 \varsigma(\iota_1)} + g_2(t)e^{\nu_2 \varsigma(\iota_2)} - cm(\iota_2 - \iota_1)^2 g_1(t)g_2(t)$$

and

$$e^{\vartheta(m\iota_1 + t\mathcal{F}_{\rho,\lambda}^\sigma(\iota_2 - m\iota_1))} \leq mg_1(t)e^{\nu_1 \vartheta(\iota_1)} + g_2(t)e^{\nu_2 \vartheta(\iota_2)} - cm(\iota_2 - \iota_1)^2 g_1(t)g_2(t).$$

Multiplying first above inequality with $(1-t)^{\alpha-1}$, the second with $t^{\alpha-1}$ and adding both sides, we get

$$(1-t)^{\alpha-1} e^{\varsigma(m\iota_1 + t\mathcal{F}_{\rho,\lambda}^\sigma(\iota_2 - m\iota_1))} + t^{\alpha-1} e^{\vartheta(m\iota_1 + t\mathcal{F}_{\rho,\lambda}^\sigma(\iota_2 - m\iota_1))}$$

$$(2.12) \quad \leq (1-t)^{\alpha-1} \left[mg_1(t)e^{\nu_1 \varsigma(\iota_1)} + g_2(t)e^{\nu_2 \varsigma(\iota_2)} \right] + t^{\alpha-1} \left[mg_1(t)e^{\nu_1 \vartheta(\iota_1)} + g_2(t)e^{\nu_2 \vartheta(\iota_2)} \right] - cm(\iota_2 - \iota_1)^2 \int_0^1 \left[t^\alpha + (1-t)^{\alpha-1} \right] g_1(t) g_2(t) dt.$$

Integrating over $[0, 1]$ both sides of inequality (2.12) and using definition 1.11, we get the required result. \square

Corollary 2.10. In Theorem 2.9, taking $c \rightarrow 0^+$, we get

$$\frac{\Gamma(\alpha)}{(\mathcal{F}_{\rho,\lambda}^\sigma(\iota_2 - m\iota_1))^\alpha} \left\{ J_{(m\iota_1)^+}^\alpha e^{\varsigma(m\iota_1 + \mathcal{F}_{\rho,\lambda}^\sigma(\iota_2 - m\iota_1))} + J_{(m\iota_1 + \mathcal{F}_{\rho,\lambda}^\sigma(\iota_2 - m\iota_1))^-}^\alpha e^{\vartheta(m\iota_1)} \right\}$$

$$(2.13) \leq m \left(e^{\nu_1 \varsigma(\iota_1)} C_{g_1}(\alpha) + e^{\nu_1 \vartheta(\iota_1)} H_{g_1}(\alpha) \right) + e^{\nu_2 \varsigma(\iota_2)} C_{g_2}(\alpha) + e^{\nu_2 \vartheta(\iota_2)} H_{g_2}(\alpha).$$

Corollary 2.11. In Theorem 2.9, if we choose $m = 1$ and $\mathcal{F}_{\rho,\lambda}^\sigma(\iota_2 - m\iota_1) = \iota_2 - m\iota_1$, we have

$$\frac{\Gamma(\alpha)}{(\iota_2 - \iota_1)^\alpha} \left\{ J_{\iota_1^+}^\alpha e^{\varsigma(\iota_2)} + J_{\iota_2^-}^\alpha e^{\vartheta(\iota_1)} \right\}$$

$$(2.14) \leq e^{\nu_1 \varsigma(\iota_1)} C_{g_1}(\alpha) + e^{\nu_1 \vartheta(\iota_1)} H_{g_1}(\alpha) + e^{\nu_2 \varsigma(\iota_2)} C_{g_2}(\alpha) + e^{\nu_2 \vartheta(\iota_2)} H_{g_2}(\alpha) \\ - c(\iota_2 - \iota_1)^2 D_{g_1, g_2}(\alpha).$$

Corollary 2.12. In Theorem 2.9, if we choose $\alpha = 1$, we obtain Corollary 2.8.

Remark 2.13. Under the conditions of Theorems 2.5 and 2.9, using remark 2.4, we can get several new integral inequalities. The details are left to the interested reader.

For establishing some new results regarding generalizations of Hermite-Hadamard type integral inequalities associated with strongly exponentially generalized $(m, \nu_1, \nu_2, g_1, g_2)$ with modulus c via general fractional integrals, we need the following lemma.

Lemma 2.14. Let $O = [m\iota_1, m\iota_1 + \mathcal{F}_{\rho,\lambda}^\sigma(\iota_2 - m\iota_1)] \subseteq \mathbf{R}$ be m -invex set for some fixed $m \in (0, 1]$, where $\iota_1 < \iota_2$ and $\mathcal{F}_{\rho,\lambda}^\sigma(\iota_2 - m\iota_1) > 0$. If $\varsigma : O \rightarrow \mathbf{R}$ is a differentiable function on O° such that $\varsigma' \in L(O)$, then the following identity for generalized fractional integrals holds:

$$\frac{e^{\varsigma(m\iota_1)} + e^{\varsigma(m\iota_1 + \mathcal{F}_{\rho,\lambda}^\sigma(\iota_2 - m\iota_1))}}{2} - \frac{1}{2\Pi_m(1)\mathcal{F}_{\rho,\lambda}^\sigma(\iota_2 - m\iota_1)} \\ \times \left\{ {}_{(m\iota_1)^+} I_\phi e^{\varsigma(m\iota_1 + \mathcal{F}_{\rho,\lambda}^\sigma(\iota_2 - m\iota_1))} + {}_{(m\iota_1 + \mathcal{F}_{\rho,\lambda}^\sigma(\iota_2 - m\iota_1))^-} I_\phi e^{\varsigma(m\iota_1)} \right\}$$

$$(2.15) \quad = \frac{\mathcal{F}_{\rho,\lambda}^{\sigma}(\nu_2 - m\nu_1)}{2\Pi_m(1)} \\ \times \int_0^1 [\Pi_m(t) - \Pi_m(1-t)] e^{\varsigma(m\nu_1 + t\mathcal{F}_{\rho,\lambda}^{\sigma}(\nu_2 - m\nu_1))} \varsigma'(m\nu_1 + t\mathcal{F}_{\rho,\lambda}^{\sigma}(\nu_2 - m\nu_1)) dt,$$

where

$$(2.16) \quad \Pi_m(t) := \int_0^t \frac{\phi(\mathcal{F}_{\rho,\lambda}^{\sigma}(\nu_2 - m\nu_1)u)}{u} du < \infty.$$

We denote

$$(2.17) \quad \Xi_{\varsigma,\Pi_m}(\nu_1, \nu_2) := \frac{\mathcal{F}_{\rho,\lambda}^{\sigma}(\nu_2 - m\nu_1)}{2\Pi_m(1)} \\ \times \int_0^1 [\Pi_m(t) - \Pi_m(1-t)] e^{\varsigma(m\nu_1 + t\mathcal{F}_{\rho,\lambda}^{\sigma}(\nu_2 - m\nu_1))} \varsigma'(m\nu_1 + t\mathcal{F}_{\rho,\lambda}^{\sigma}(\nu_2 - m\nu_1)) dt.$$

Proof. From (2.17), we have

$$\begin{aligned} \Xi_{\varsigma,\Pi_m}(\nu_1, \nu_2) &= \frac{\mathcal{F}_{\rho,\lambda}^{\sigma}(\nu_2 - m\nu_1)}{2\Pi_m(1)} \\ &\times \left[\int_0^1 \Pi_m(t) e^{\varsigma(m\nu_1 + t\mathcal{F}_{\rho,\lambda}^{\sigma}(\nu_2 - m\nu_1))} \varsigma'(m\nu_1 + t\mathcal{F}_{\rho,\lambda}^{\sigma}(\nu_2 - m\nu_1)) dt \right. \\ &\quad \left. - \int_0^1 \Pi_m(1-t) e^{\varsigma(m\nu_1 + t\mathcal{F}_{\rho,\lambda}^{\sigma}(\nu_2 - m\nu_1))} \varsigma'(m\nu_1 + t\mathcal{F}_{\rho,\lambda}^{\sigma}(\nu_2 - m\nu_1)) dt \right] \\ (2.18) \quad &= \frac{\mathcal{F}_{\rho,\lambda}^{\sigma}(\nu_2 - m\nu_1)}{2} \times \left[\Xi_{\varsigma,\Pi_m}^{(1)}(\nu_1, \nu_2) - \Xi_{\varsigma,\Pi_m}^{(2)}(\nu_1, \nu_2) \right], \end{aligned}$$

where

$$(2.19) \quad \Xi_{\varsigma,\Pi_m}^{(1)}(\nu_1, \nu_2) := \int_0^1 \Pi_m(t) e^{\varsigma(m\nu_1 + t\mathcal{F}_{\rho,\lambda}^{\sigma}(\nu_2 - m\nu_1))} \varsigma'(m\nu_1 + t\mathcal{F}_{\rho,\lambda}^{\sigma}(\nu_2 - m\nu_1)) dt$$

and

$$\Xi_{\varsigma, \Pi_m}^{(2)}(\nu_1, \nu_2) := \int_0^1 \Pi_m(1-t) e^{\varsigma(m\nu_1 + t\mathcal{F}_{\rho, \lambda}^\sigma(\nu_2 - m\nu_1))} \varsigma'(m\nu_1 + t\mathcal{F}_{\rho, \lambda}^\sigma(\nu_2 - m\nu_1)) dt. \quad (2.20)$$

Now, integrating by parts (2.19), changing the variable $u = m\nu_1 + t\mathcal{F}_{\rho, \lambda}^\sigma(\nu_2 - m\nu_1)$ and using definition 1.11, we get

$$\begin{aligned} \Xi_{\varsigma, \Pi_m}^{(1)}(\nu_1, \nu_2) &:= \frac{\Pi_m(t) e^{\varsigma(m\nu_1 + t\mathcal{F}_{\rho, \lambda}^\sigma(\nu_2 - m\nu_1))}}{\mathcal{F}_{\rho, \lambda}^\sigma(\nu_2 - m\nu_1)} \Big|_0^1 \\ &- \frac{1}{\mathcal{F}_{\rho, \lambda}^\sigma(\nu_2 - m\nu_1)} \int_0^1 \frac{\phi(\mathcal{F}_{\rho, \lambda}^\sigma(\nu_2 - m\nu_1)t)}{t} e^{\varsigma(m\nu_1 + t\mathcal{F}_{\rho, \lambda}^\sigma(\nu_2 - m\nu_1))} dt \\ &= \frac{\Pi_m(1) e^{\varsigma(m\nu_1 + \mathcal{F}_{\rho, \lambda}^\sigma(\nu_2 - m\nu_1))}}{\mathcal{F}_{\rho, \lambda}^\sigma(\nu_2 - m\nu_1)} - \frac{1}{\mathcal{F}_{\rho, \lambda}^{\sigma, 2}(\nu_2 - m\nu_1)} e^{\varsigma(m\nu_1 + \mathcal{F}_{\rho, \lambda}^\sigma(\nu_2 - m\nu_1)) - I_\phi} e^{\varsigma(m\nu_1)}. \end{aligned} \quad (2.21)$$

Similarly, using (2.20), we obtain

$$\Xi_{\varsigma, \Pi_m}^{(2)}(\nu_1, \nu_2) = -\frac{\Pi_m(1) e^{\varsigma(m\nu_1)}}{\mathcal{F}_{\rho, \lambda}^\sigma(\nu_2 - m\nu_1)} + \frac{1}{\mathcal{F}_{\rho, \lambda}^{\sigma, 2}(\nu_2 - m\nu_1)} e^{\varsigma(m\nu_1 + \mathcal{F}_{\rho, \lambda}^\sigma(\nu_2 - m\nu_1)) + I_\phi}. \quad (2.22)$$

Substituting (2.21) and (2.22) in (2.18), we get (2.15). The proof of Lemma 2.14 is completed. \square

Remark 2.15. In Lemma 2.14, if we choose $\phi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$ for $\alpha > 0$, we have

$$\begin{aligned} &\frac{e^{\varsigma(m\nu_1)} + e^{\varsigma(m\nu_1 + \mathcal{F}_{\rho, \lambda}^\sigma(\nu_2 - m\nu_1))}}{2} \\ &- \frac{\Gamma(\alpha + 1)}{2\mathcal{F}_{\rho, \lambda}^{\sigma, \alpha}(\nu_2 - m\nu_1)} \times \left\{ J_{(m\nu_1)^+}^\alpha e^{\varsigma(m\nu_1 + \mathcal{F}_{\rho, \lambda}^\sigma(\nu_2 - m\nu_1))} + J_{(m\nu_1 + \mathcal{F}_{\rho, \lambda}^\sigma(\nu_2 - m\nu_1))^-}^\alpha e^{\varsigma(m\nu_1)} \right\} \\ &= \frac{\mathcal{F}_{\rho, \lambda}^\sigma(\nu_2 - m\nu_1)}{2} \int_0^1 [t^\alpha - (1-t)^\alpha] e^{\varsigma(m\nu_1 + t\mathcal{F}_{\rho, \lambda}^\sigma(\nu_2 - m\nu_1))} \varsigma'(m\nu_1 + t\mathcal{F}_{\rho, \lambda}^\sigma(\nu_2 - m\nu_1)) dt. \end{aligned} \quad (2.23)$$

Using Lemma 2.14, we now state the following theorem.

Theorem 2.16. Let $O = [m\iota_1, m\iota_1 + \mathcal{F}_{\rho,\lambda}^\sigma(\iota_2 - m\iota_1)] \subseteq \mathbf{R}$ be m -invex set for some fixed $m \in (0, 1]$, where $\iota_1 < \iota_2$ and $\mathcal{F}_{\rho,\lambda}^\sigma(\iota_2 - m\iota_1) > 0$. Suppose $g_1, g_2 : [0, 1] \rightarrow [0, +\infty)$ be continuous functions. Let $\varsigma : O \rightarrow (0, +\infty)$ be a differentiable strongly exponentially generalized $(m, \nu_1, \nu_2, g_1, g_2)$ function with modulus $c > 0$ on O° such that $\varsigma' \in L(O)$ and $\nu_1, \nu_2 \in \mathbf{R}$. If $|\varsigma'|^q$ is generalized (m, g_1, g_2) -preinvex function, then for $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, the following inequality holds:

$$(2.24) \quad \begin{aligned} |\Xi_{\varsigma, \Pi_m}(\iota_1, \iota_2)| &\leq \frac{\mathcal{F}_{\rho,\lambda}^\sigma(\iota_2 - m\iota_1)}{2\Pi_m(1)} \sqrt[p]{B_{\Pi_m}(p)} \\ &\times \left[m^2 e^{q\nu_1 \varsigma(\iota_1)} |\varsigma'(\iota_1)|^q G_{g_1} + m \Delta_\varsigma(q; \nu_1, \nu_2, \iota_1, \iota_2) F_{g_1, g_2} + e^{q\nu_2 \varsigma(\iota_2)} |\varsigma'(\iota_2)|^q G_{g_2} \right. \\ &\left. - cm(\iota_2 - \iota_1)^2 \left(m |\varsigma'(\iota_1)|^q M_{g_1, g_2} + |\varsigma'(\iota_2)|^q N_{g_1, g_2} \right) \right]^{\frac{1}{q}}, \end{aligned}$$

where

$$(2.25) \quad B_{\Pi_m}(p) := \int_0^1 |\Pi_m(t) - \Pi_m(1-t)|^p dt, \quad G_{g_i} := \int_0^1 [g_i(t)]^2 dt, \quad \forall i = 1, 2,$$

$$(2.26) \quad M_{g_1, g_2} := \int_0^1 [g_1(t)]^2 g_2(t) dt, \quad N_{g_1, g_2} := \int_0^1 g_1(t) [g_2(t)]^2 dt,$$

$$(2.27) \quad \Delta_\varsigma(q; \nu_1, \nu_2, \iota_1, \iota_2) := e^{q\nu_1 \varsigma(\iota_1)} |\varsigma'(\iota_2)|^q + e^{q\nu_2 \varsigma(\iota_2)} |\varsigma'(\iota_1)|^q$$

and F_{g_1, g_2} is defined from (2.8).

Proof. From Lemma 2.14, strongly exponentially generalized $(m, \nu_1, \nu_2, g_1, g_2)$ with modulus c of ς , generalized (m, g_1, g_2) -preinvexity of $|\varsigma'|^q$, Hölder inequality and properties of the modulus, we have

$$\begin{aligned} |\Xi_{\varsigma, \Pi_m}(\iota_1, \iota_2)| &\leq \frac{\mathcal{F}_{\rho,\lambda}^\sigma(\iota_2 - m\iota_1)}{2\Pi_m(1)} \\ &\times \int_0^1 |\Pi_m(t) - \Pi_m(1-t)| \left| e^{\varsigma(m\iota_1 + t\mathcal{F}_{\rho,\lambda}^\sigma(\iota_2 - m\iota_1))} \varsigma'(m\iota_1 + t\mathcal{F}_{\rho,\lambda}^\sigma(\iota_2 - m\iota_1)) \right| dt \\ &\leq \frac{\mathcal{F}_{\rho,\lambda}^\sigma(\iota_2 - m\iota_1)}{2\Pi_m(1)} \sqrt[p]{B_{\Pi_m}(p)} \left(\int_0^1 e^{q\varsigma(m\iota_1 + t\mathcal{F}_{\rho,\lambda}^\sigma(\iota_2 - m\iota_1))} |\varsigma'(m\iota_1 + t\mathcal{F}_{\rho,\lambda}^\sigma(\iota_2 - m\iota_1))|^q dt \right)^{\frac{1}{q}} \end{aligned}$$

$$\begin{aligned}
 & \leq \frac{\mathcal{F}_{\rho,\lambda}^\sigma(\nu_2 - m\nu_1)}{2\Pi_m(1)} \sqrt[p]{B_{\Pi_m}(p)} \\
 & \times \left\{ \int_0^1 \left[mg_1(t) e^{q\nu_1\zeta(\nu_1)} + g_2(t) e^{q\nu_2\zeta(\nu_2)} - cm(\nu_2 - \nu_1)^2 g_1(t)g_2(t) \right] \right. \\
 & \quad \left. \times \left[mg_1(t) |\zeta'(\nu_1)|^q + g_2(t) |\zeta'(\nu_2)|^q \right] dt \right\}^{\frac{1}{q}} \\
 & = \frac{\mathcal{F}_{\rho,\lambda}^\sigma(\nu_2 - m\nu_1)}{2\Pi_m(1)} \sqrt[p]{B_{\Pi_m}(p)} \\
 & \times \left[m^2 e^{q\nu_1\zeta(\nu_1)} |\zeta'(\nu_1)|^q G_{g_1} + m \Delta_f(q; \nu_1, \nu_2, \nu_1, \nu_2) F_{g_1, g_2} + e^{q\nu_2\zeta(\nu_2)} |\zeta'(\nu_2)|^q G_{g_2} \right. \\
 & \quad \left. - cm(\nu_2 - \nu_1)^2 \left(m |\zeta'(\nu_1)|^q M_{g_1, g_2} + |\zeta'(\nu_2)|^q N_{g_1, g_2} \right) \right]^{\frac{1}{q}}.
 \end{aligned}$$

The proof of Theorem 2.16 is completed. \square

We point out some special cases of Theorem 2.16.

Corollary 2.17. In Theorem 2.16, taking $c \rightarrow 0^+$, we get

$$\begin{aligned}
 (2.28) \quad & \left| \Xi_{f, \Pi_m}(\nu_1, \nu_2) \right| \leq \frac{\mathcal{F}_{\rho,\lambda}^\sigma(\nu_2 - m\nu_1)}{2\Pi_m(1)} \sqrt[p]{B_{\Pi_m}(p)} \\
 & \times \sqrt[q]{m^2 e^{q\nu_1\zeta(\nu_1)} |\zeta'(\nu_1)|^q G_{g_1} + m \Delta_\zeta(q; \nu_1, \nu_2, \nu_1, \nu_2) F_{g_1, g_2} + e^{q\nu_2\zeta(\nu_2)} |\zeta'(\nu_2)|^q G_{g_2}}.
 \end{aligned}$$

Corollary 2.18. In Theorem 2.16, if we choose $\phi(t) = t$, we have

$$\begin{aligned}
 (2.29) \quad & \left| \frac{e^{\zeta(m\nu_1)} + e^{\zeta(m\nu_1 + \mathcal{F}_{\rho,\lambda}^\sigma(\nu_2 - m\nu_1))}}{2} - \frac{1}{\mathcal{F}_{\rho,\lambda}^\sigma(\nu_2 - m\nu_1)} \int_{m\nu_1}^{m\nu_1 + \mathcal{F}_{\rho,\lambda}^\sigma(\nu_2 - m\nu_1)} e^{\zeta(t)} dt \right| \\
 & \leq \frac{\mathcal{F}_{\rho,\lambda}^\sigma(\nu_2 - m\nu_1)}{2 \sqrt[p]{p+1}} \\
 & \times \left[m^2 e^{q\nu_1\zeta(\nu_1)} |\zeta'(\nu_1)|^q G_{g_1} + m \Delta_\zeta(q; \nu_1, \nu_2, \nu_1, \nu_2) F_{g_1, g_2} + e^{q\nu_2\zeta(\nu_2)} |\zeta'(\nu_2)|^q G_{g_2} \right. \\
 & \quad \left. - cm(\nu_2 - \nu_1)^2 \left(m |\zeta'(\nu_1)|^q M_{g_1, g_2} + |\zeta'(\nu_2)|^q N_{g_1, g_2} \right) \right]^{\frac{1}{q}}.
 \end{aligned}$$

Corollary 2.19. In Theorem 2.16, if we choose $\phi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$ for $\alpha > 0$ and $\mathcal{F}_{\rho,\lambda}^\sigma(\iota_2 - m\iota_1) = \iota_2 - m\iota_1$, we obtain

$$(2.30) \quad \begin{aligned} & \left| \frac{e^{\varsigma(m\iota_1)} + e^{\varsigma(\iota_2)}}{2} - \frac{\Gamma(\alpha+1)}{2(\iota_2 - m\iota_1)^\alpha} \times \left\{ J_{(m\iota_1)^+}^\alpha e^{\varsigma(\iota_2)} + J_{\iota_2^-}^\alpha e^{\varsigma(m\iota_1)} \right\} \right| \\ & \leq \frac{(\iota_2 - m\iota_1)}{2} \sqrt[p]{B(p, \alpha)} \end{aligned}$$

$$\begin{aligned} & \times \left[m^2 e^{q\nu_1 \varsigma(\iota_1)} \left| \varsigma'(\iota_1) \right|^q G_{g_1} + m \Delta_\varsigma(q; \nu_1, \nu_2, \iota_1, \iota_2) F_{g_1, g_2} + e^{q\nu_2 \varsigma(\iota_2)} \left| \varsigma'(\iota_2) \right|^q G_{g_2} \right. \\ & \left. - cm(\iota_2 - \iota_1)^2 \left(m \left| \varsigma'(\iota_1) \right|^q M_{g_1, g_2} + \left| \varsigma'(\iota_2) \right|^q N_{g_1, g_2} \right) \right]^{\frac{1}{q}}, \end{aligned}$$

where

$$B(p, \alpha) := \int_0^1 \left| t^\alpha - (1-t)^\alpha \right|^p dt.$$

Theorem 2.20. Let $O = [m\iota_1, m\iota_1 + \mathcal{F}_{\rho,\lambda}^\sigma(\iota_2 - m\iota_1)] \subseteq \mathbf{R}$ be m -invex set for some fixed $m \in (0, 1]$, where $\iota_1 < \iota_2$ and $\mathcal{F}_{\rho,\lambda}^\sigma(\iota_2 - m\iota_1) > 0$. Suppose $g_1, g_2 : [0, 1] \rightarrow [0, +\infty)$ be continuous functions. Let $\varsigma : O \rightarrow (0, +\infty)$ be a differentiable strongly exponentially generalized $(m, \nu_1, \nu_2, g_1, g_2)$ function with modulus $c > 0$ on O° such that $\varsigma' \in L(O)$ and $\nu_1, \nu_2 \in \mathbf{R}$. If $|\varsigma'|^q$ is generalized (m, g_1, g_2) -preinvex function, then for $q \geq 1$, the following inequality holds:

$$(2.31) \quad \begin{aligned} & \left| \Xi_{\varsigma, \Pi_m}(\iota_1, \iota_2) \right| \leq \frac{\mathcal{F}_{\rho,\lambda}^\sigma(\iota_2 - m\iota_1)}{2\Pi_m(1)} \left[B_{\Pi_m}(1) \right]^{1-\frac{1}{q}} \\ & \times \left[m^2 e^{q\nu_1 \varsigma(\iota_1)} \left| \varsigma'(\iota_1) \right|^q P_{\Pi_m, g_1} + m \Delta_\varsigma(q; \nu_1, \nu_2, \iota_1, \iota_2) S_{\Pi_m, g_1, g_2} + e^{q\nu_2 \varsigma(\iota_2)} \left| \varsigma'(\iota_2) \right|^q P_{\Pi_m, g_2} \right. \\ & \left. - cm(\iota_2 - \iota_1)^2 \left(m \left| \varsigma'(\iota_1) \right|^q U_{\Pi_m, g_1, g_2} + \left| \varsigma'(\iota_2) \right|^q T_{\Pi_m, g_1, g_2} \right) \right]^{\frac{1}{q}}, \end{aligned}$$

where

$$(2.32) \quad S_{\Pi_m, g_1, g_2} := \int_0^1 \left| \Pi_m(t) - \Pi_m(1-t) \right| g_1(t) g_2(t) dt,$$

$$(2.33) \quad P_{\Pi_m, g_i} := \int_0^1 |\Pi_m(t) - \Pi_m(1-t)| [g_i(t)]^2 dt, \quad \forall i = 1, 2,$$

$$(2.34) \quad U_{\Pi_m, g_1, g_2} := \int_0^1 |\Pi_m(t) - \Pi_m(1-t)| [g_1(t)]^2 g_2(t) dt,$$

$$(2.35) \quad T_{\Pi_m, g_1, g_2} := \int_0^1 |\Pi_m(t) - \Pi_m(1-t)| g_1(t) [g_2(t)]^2 dt$$

and $\Delta_\varsigma(q; \nu_1, \nu_2, \iota_1, \iota_2)$, $B_{\Pi_m}(1)$ are defined as in Theorem 2.16.

Proof. From Lemma 2.14, strongly exponentially generalized $(m, \nu_1, \nu_2, g_1, g_2)$ with modulus c of ς , generalized (m, g_1, g_2) -preinvexity of $|\varsigma'|^q$, the well-known power mean inequality and properties of the modulus, we have

$$\begin{aligned} & |\Xi_{\varsigma, \Pi_m}(\iota_1, \iota_2)| \leq \frac{\mathcal{F}_{\rho, \lambda}^\sigma(\iota_2 - m\iota_1)}{2} \\ & \times \int_0^1 |\Pi_m(t) - \Pi_m(1-t)| \left| e^{\varsigma(m\iota_1 + t\mathcal{F}_{\rho, \lambda}^\sigma(\iota_2 - m\iota_1))} \varsigma'(m\iota_1 + t\mathcal{F}_{\rho, \lambda}^\sigma(\iota_2 - m\iota_1)) \right| dt \\ & \leq \frac{\mathcal{F}_{\rho, \lambda}^\sigma(\iota_2 - m\iota_1)}{2\Pi_m(1)} \left[B_{\Pi_m}(1) \right]^{1-\frac{1}{q}} \\ & \times \left(\int_0^1 |\Pi_m(t) - \Pi_m(1-t)| \left| e^{q\varsigma(m\iota_1 + t\mathcal{F}_{\rho, \lambda}^\sigma(\iota_2 - m\iota_1))} \right| \left| \varsigma'(m\iota_1 + t\mathcal{F}_{\rho, \lambda}^\sigma(\iota_2 - m\iota_1)) \right|^q dt \right)^{\frac{1}{q}} \\ & \leq \frac{\mathcal{F}_{\rho, \lambda}^\sigma(\iota_2 - m\iota_1)}{2\Pi_m(1)} \left[B_{\Pi_m}(1) \right]^{1-\frac{1}{q}} \\ & \times \left\{ \int_0^1 |\Pi_m(t) - \Pi_m(1-t)| \left[mg_1(t)e^{q\nu_1\varsigma(\iota_1)} + g_2(t)e^{q\nu_2\varsigma(\iota_2)} - cm(\iota_2 - \iota_1)^2 g_1(t)g_2(t) \right] \right. \\ & \quad \left. \times \left[mg_1(t) \left| \varsigma'(\iota_1) \right|^q + g_2(t) \left| \varsigma'(\iota_2) \right|^q \right] dt \right\}^{\frac{1}{q}} \\ & = \frac{\mathcal{F}_{\rho, \lambda}^\sigma(\iota_2 - m\iota_1)}{2\Pi_m(1)} \left[B_{\Pi_m}(1) \right]^{1-\frac{1}{q}} \\ & \times \left[m^2 e^{q\nu_1\varsigma(\iota_1)} \left| \varsigma'(\iota_1) \right|^q P_{\Pi_m, g_1} + m\Delta_\varsigma(q; \nu_1, \nu_2, \iota_1, \iota_2) S_{\Pi_m, g_1, g_2} + e^{q\nu_2\varsigma(\iota_2)} \left| \varsigma'(\iota_2) \right|^q P_{\Pi_m, g_2} \right. \end{aligned}$$

$$-cm(\nu_2 - \nu_1)^2 \left(m \left| \zeta'(\nu_1) \right|^q U_{\Pi_m, g_1, g_2} + \left| \zeta'(\nu_2) \right|^q T_{\Pi_m, g_1, g_2} \right)^{\frac{1}{q}}.$$

The proof of Theorem 2.20 is completed. \square

We point out some special cases of Theorem 2.20.

Corollary 2.21. *In Theorem 2.20, taking $c \rightarrow 0^+$, we get*

$$(2.36) \quad \left| \Xi_{\varsigma, \Pi_m}(\nu_1, \nu_2) \right| \leq \frac{\mathcal{F}_{\rho, \lambda}^{\sigma}(\nu_2 - m\nu_1)}{2\Pi_m(1)} \left[B_{\Pi_m}(1) \right]^{1-\frac{1}{q}}$$

$$\times \sqrt{m^2 e^{q\nu_1 \varsigma(\nu_1)} \left| \zeta'(\nu_1) \right|^q P_{\Pi_m, g_1} + m \Delta_{\varsigma}(q; \nu_1, \nu_2, \nu_1, \nu_2) S_{\Pi_m, g_1, g_2} + e^{q\nu_2 \varsigma(\nu_2)} \left| \zeta'(\nu_2) \right|^q P_{\Pi_m, g_2}}.$$

Corollary 2.22. *In Theorem 2.20, if we choose $\phi(t) = t$, we have*

$$(2.37) \quad \begin{aligned} & \left| \frac{e^{\varsigma(m\nu_1)} + e^{\varsigma(m\nu_1 + \mathcal{F}_{\rho, \lambda}^{\sigma}(\nu_2 - m\nu_1))}}{2} - \frac{1}{\mathcal{F}_{\rho, \lambda}^{\sigma}(\nu_2 - m\nu_1)} \int_{m\nu_1}^{m\nu_1 + \mathcal{F}_{\rho, \lambda}^{\sigma}(\nu_2 - m\nu_1)} e^{\varsigma(t)} dt \right| \\ & \leq 2^{\frac{1-2q}{q}} \mathcal{F}_{\rho, \lambda}^{\sigma}(\nu_2 - m\nu_1) \end{aligned}$$

$$\begin{aligned} & \times \left\{ m^2 e^{q\nu_1 \varsigma(\nu_1)} \left| \zeta'(\nu_1) \right|^q \Omega_{g_1} + m \Delta_{\varsigma}(q; \nu_1, \nu_2, \nu_1, \nu_2) \Theta_{g_1, g_2} + e^{q\nu_2 \varsigma(\nu_2)} \left| \zeta'(\nu_2) \right|^q \Omega_{g_2} \right. \\ & \left. - cm(\nu_2 - \nu_1)^2 \left(m \left| \zeta'(\nu_1) \right|^q Q_{g_1, g_2} + \left| \zeta'(\nu_2) \right|^q R_{g_1, g_2} \right) \right\}^{\frac{1}{q}}, \end{aligned}$$

where

$$(2.38) \quad \Theta_{g_1, g_2} := \int_0^1 |2t - 1| g_1(t) g_2(t) dt, \quad \Omega_{g_i} := \int_0^1 |2t - 1| [g_i(t)]^2 dt, \quad \forall i = 1, 2,$$

$$(2.39) \quad Q_{g_1, g_2} := \int_0^1 |2t - 1| [g_1(t)]^2 g_2(t) dt, \quad R_{g_1, g_2} := \int_0^1 |2t - 1| g_1(t) [g_2(t)]^2 dt.$$

Corollary 2.23. In Theorem 2.20, taking $c \rightarrow 0^+$, where $\phi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$ for $\alpha > 0$ and $\mathcal{F}_{\rho,\lambda}^\sigma(\iota_2 - m\iota_1) = \iota_2 - m\iota_1$, we obtain

$$(2.40) \quad \begin{aligned} & \left| \frac{e^{\varsigma(m\iota_1)} + e^{\varsigma(\iota_2)}}{2} - \frac{\Gamma(\alpha+1)}{2(\iota_2 - m\iota_1)^\alpha} \times \left\{ J_{(m\iota_1)^+}^\alpha e^{\varsigma(\iota_2)} + J_{\iota_2^-}^\alpha e^{\varsigma(m\iota_1)} \right\} \right| \\ & \leq \frac{(\iota_2 - m\iota_1)}{2} \left[B_{\Pi_m}(1) \right]^{1-\frac{1}{q}} \\ & \times \sqrt[q]{m^2 e^{q\nu_1 \varsigma(\iota_1)} |\varsigma'(\iota_1)|^q P_{\Pi_m, g_1} + m \Delta_\varsigma(q; \nu_1, \nu_2, \iota_1, \iota_2) S_{\Pi_m, g_1, g_2} + e^{q\nu_2 \varsigma(\iota_2)} |\varsigma'(\iota_2)|^q P_{\Pi_m, g_2}}. \end{aligned}$$

Remark 2.24. Under the conditions of Theorems 2.16 and 2.20, using remark 2.4, for the appropriate choices of function $\phi(t) = t, \frac{t^\frac{\alpha}{k}}{k\Gamma_k(\alpha)}; \phi(t) = \frac{t}{\alpha} \exp \left[\left(-\frac{1-\alpha}{\alpha} \right) t \right]$ where $\alpha \in (0, 1)$, etc., we can get several new integral inequalities. Also, under assumptions of Theorems 2.16 and 2.20, taking $K = \|\varsigma'\|_\infty := \sup_{x \in O} |\varsigma'(x)|$, we can deduce many other inequalities.

3. Applications

In this section, we provide some new error estimates for trapezoidal quadrature formula. Let P be the partition of the points $\iota_1 = x_0 < x_1 < \dots < x_k = \iota_2$ of the interval $[\iota_1, \iota_2]$. Let consider the following quadrature formula:

$$\int_{\iota_1}^{\iota_2} e^{\varsigma(x)} dx = T(\varsigma, P) + E(\varsigma, P),$$

where

$$T(\varsigma, P) = \sum_{i=0}^{k-1} \left[\frac{e^{\varsigma(x_i)} + e^{\varsigma(x_{i+1})}}{2} \right] (x_{i+1} - x_i)$$

is the trapezoidal version and $E(\varsigma, P)$ is denote the associated approximation error.

Proposition 3.1. Let $\varsigma : [\iota_1, \iota_2] \rightarrow (0, +\infty)$ be a differentiable strongly exponentially generalized (ν_1, ν_2, g_1, g_2) -convex function with modulus $c > 0$ on (ι_1, ι_2) , where $\iota_1 < \iota_2$ and $\nu_1, \nu_2 \in \mathbf{R}$. Suppose $g_1, g_2 : [0, 1] \rightarrow [0, +\infty)$

be continuous functions. If $|\zeta'|^q$ is generalized (g_1, g_2) -convex on $[\nu_1, \nu_2]$ for $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then

$$(3.1) \quad |E(\zeta, P)| \leq \frac{1}{2\sqrt[p+1]{p+1}} \times \sum_{i=0}^{k-1} (x_{i+1} - x_i)^2$$

$$\times \left[e^{q\nu_1\zeta(x_i)} |\zeta'(x_i)|^q G_{g_1} + \Delta_\zeta(q; \nu_1, \nu_2, x_i, x_{i+1}) F_{g_1, g_2} + e^{q\nu_2\zeta(x_{i+1})} |\zeta'(x_{i+1})|^q G_{g_2} \right. \\ \left. - c(x_{i+1} - x_i)^2 \left(|\zeta'(x_i)|^q M_{g_1, g_2} + |\zeta'(x_{i+1})|^q N_{g_1, g_2} \right) \right]^{\frac{1}{q}},$$

where

$$(3.2) \quad \Delta_\zeta(q; \nu_1, \nu_2, x_i, x_{i+1}) := e^{q\nu_1\zeta(x_i)} |\zeta'(x_{i+1})|^q + e^{q\nu_2\zeta(x_{i+1})} |\zeta'(x_i)|^q$$

and F_{g_1, g_2} , G_{g_1} , G_{g_2} , M_{g_1, g_2} , N_{g_1, g_2} are defined as in Theorem 2.16.

Proof. Applying Theorem 2.16 for $m = 1$, $\mathcal{F}_{\rho, \lambda}^\sigma(\nu_2 - m\nu_1) = \nu_2 - m\nu_1$ and $\phi(t) = t$ on the subintervals $[x_i, x_{i+1}]$ ($i = 0, \dots, k-1$) of the partition P , we have

$$(3.3) \quad \left| \frac{e^{\zeta(x_i)} + e^{\zeta(x_{i+1})}}{2} - \frac{1}{x_{i+1} - x_i} \int_{x_i}^{x_{i+1}} e^{\zeta(x)} dx \right| \leq \frac{(x_{i+1} - x_i)}{2\sqrt[p+1]{p+1}}$$

$$\times \left[e^{q\nu_1\zeta(x_i)} |\zeta'(x_i)|^q G_{g_1} + \Delta_\zeta(q; \nu_1, \nu_2, x_i, x_{i+1}) F_{g_1, g_2} + e^{q\nu_2\zeta(x_{i+1})} |\zeta'(x_{i+1})|^q G_{g_2} \right. \\ \left. - c(x_{i+1} - x_i)^2 \left(|\zeta'(x_i)|^q M_{g_1, g_2} + |\zeta'(x_{i+1})|^q N_{g_1, g_2} \right) \right]^{\frac{1}{q}}.$$

Hence from (3.3), we get

$$|E(\zeta, P)| = \left| \int_a^b e^{\zeta(x)} dx - T(\zeta, P) \right| \\ \leq \left| \sum_{i=0}^{k-1} \left\{ \int_{x_i}^{x_{i+1}} e^{\zeta(x)} dx - \frac{e^{\zeta(x_i)} + e^{\zeta(x_{i+1})}}{2} (x_{i+1} - x_i) \right\} \right|$$

$$\begin{aligned}
 &\leq \sum_{i=0}^{k-1} \left| \left\{ \int_{x_i}^{x_{i+1}} e^{\varsigma(x)} dx - \frac{e^{\varsigma(x_i)} + e^{\varsigma(x_{i+1})}}{2} (x_{i+1} - x_i) \right\} \right| \\
 &\leq \frac{1}{2\sqrt[p+1]{p+1}} \times \sum_{i=0}^{k-1} (x_{i+1} - x_i)^2 \\
 &\times \left[e^{q\nu_1\varsigma(x_i)} |\varsigma'(x_i)|^q G_{g_1} + \Delta_\varsigma(q; \nu_1, \nu_2, x_i, x_{i+1}) F_{g_1, g_2} + e^{q\nu_2\varsigma(x_{i+1})} |\varsigma'(x_{i+1})|^q G_{g_2} \right. \\
 &\quad \left. - c(x_{i+1} - x_i)^2 \left(|\varsigma'(x_i)|^q M_{g_1, g_2} + |\varsigma'(x_{i+1})|^q N_{g_1, g_2} \right) \right]^{\frac{1}{q}}.
 \end{aligned}$$

The proof of Proposition 3.1 is completed. \square

Proposition 3.2. Let $\varsigma : [\iota_1, \iota_2] \rightarrow (0, +\infty)$ be a differentiable strongly exponentially generalized (ν_1, ν_2, g_1, g_2) -convex function with modulus $c > 0$ on (ι_1, ι_2) , where $\iota_1 < \iota_2$ and $\nu_1, \nu_2 \in \mathbf{R}$. Suppose $g_1, g_2 : [0, 1] \rightarrow [0, +\infty)$ be continuous functions. If $|\varsigma'|^q$ is generalized (g_1, g_2) -convex on $[\iota_1, \iota_2]$ for $q \geq 1$, then

$$\begin{aligned}
 (3.4) \quad |E(\varsigma, P)| &\leq 2^{\frac{1-2q}{q}} \times \sum_{i=0}^{k-1} (x_{i+1} - x_i)^2 \\
 &\times \left[e^{q\nu_1\varsigma(x_i)} |\varsigma'(x_i)|^q \Omega_{g_1} + \Delta_\varsigma(q; \nu_1, \nu_2, x_i, x_{i+1}) \Theta_{g_1, g_2} + e^{q\nu_2\varsigma(x_{i+1})} |\varsigma'(x_{i+1})|^q \Omega_{g_2} \right. \\
 &\quad \left. - c(x_{i+1} - x_i)^2 \left(|\varsigma'(x_i)|^q Q_{g_1, g_2} + |\varsigma'(x_{i+1})|^q R_{g_1, g_2} \right) \right]^{\frac{1}{q}},
 \end{aligned}$$

where $\Delta_\varsigma(q; \nu_1, \nu_2, x_i, x_{i+1})$ is defined from (3.2), Θ_{g_1, g_2} , Ω_{g_1} , Ω_{g_2} ; Q_{g_1, g_2} , R_{g_1, g_2} are defined respectively from (2.38) and (2.39).

Proof. The proof is analogous as to that of Proposition 3.1, taking $m = 1$, $\mathcal{F}_{\rho, \lambda}^\sigma(\iota_2 - m\iota_1) = \iota_2 - m\iota_1$ and $\phi(t) = t$ in Theorem 2.20. \square

Remark 3.3. Under the conditions of Theorems 2.16 and 2.20, using remark 2.4, for the appropriate choices of function $\phi(t) = t$, $\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$; $\phi(t)$

$= \frac{t}{\alpha} \exp \left[\left(-\frac{1-\alpha}{\alpha} \right) t \right]$ where $\alpha \in (0, 1)$, etc., we can deduce several new bounds for the trapezoidal quadrature formula using above ideas and techniques. Also, under assumptions of Theorems 2.16 and 2.20, taking $K = \|\varsigma'\|_\infty := \sup_{x \in O} |\varsigma'(x)|$, we can obtain many other new bounds.

4. Conclusion

The so-called strongly exponentially generalized $(m, \nu_1, \nu_2, g_1, g_2)$ functions can be applied to obtain several results in convex analysis, related optimization theory and may stimulate further research in different areas of pure and applied sciences.

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Artion Kashuri

Department of Mathematics,
Faculty of Technical Science,
University Ismail Qemali,
Vlora,
Albania
e-mail: artionkashuri@gmail.com
Corresponding author

and

Rozana Liko

Department of Mathematics,
Faculty of Technical Science,
University Ismail Qemali,
Vlora,
Albania
e-mail: rozanaliko86@gmail.com