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Non-linear new product $A^*B - B^*A$ derivations on $*$-algebras

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Abstract:

Let $\mathcal{A}$ be a prime $*$-algebra with unit I and a nontrivial projection. Then the map $\Phi : \mathcal{A} \to \mathcal{A}$ satisfies in the following condition

$$\Phi(A \diamond B) = \Phi(A) \diamond B + A \diamond \Phi(B)$$

where $A \diamond B = A^*B - B^*A$ for all $A, B \in \mathcal{A}$, is additive. Moreover, if $\Phi(\alpha I)$ is self-adjoint operator for $\alpha \in \{1, i\}$ then Φ is a $*$-derivation.

Keywords: New product derivation: Prime $*$-algebra; Additive map.

MSC (2010): 46J10, 47B48, 46L10.

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1. Introduction

Let $\mathcal{R}$ be a $*$-algebra. For $A, B \in \mathcal{R}$, denoted by $A \bullet B = AB + BA^*$ and $[A, B]_* = AB - BA^*$, which are $*$-Jordan product and $*$-Lie product, respectively. These products are found playing a more and more important role in some research topics, and its study has recently attracted many author's attention (for example, see [3, 9, 11, 15]).

Recall that a map $\Phi : \mathcal{R} \to \mathcal{R}$ is said to be an additive derivation if

$$\Phi(A + B) = \Phi(A) + \Phi(B)$$

and

$$\Phi(AB) = \Phi(A)B + A\Phi(B)$$

for all $A, B \in \mathcal{R}$. A map Φ is additive $*$-derivation if it is an additive derivation and $\Phi(A^*) = \Phi(A)^*$. Derivations are very important maps both in theory and applications, and have been studied intensively ([2, 12, 13, 14]).

Let us define λ-Jordan $*$-product by $A \bullet_\lambda B = AB + \lambda BA^*$. We say that the map Φ with the property of $\Phi(A \bullet_\lambda B) = \Phi(A) \bullet_\lambda B + A \bullet_\lambda \Phi(B)$ is a λ-Jordan $*$-derivation map. It is clear that for $\lambda = -1$ and $\lambda = 1$, the λ-Jordan $*$-derivation map is a $*$-Lie derivation and $*$-Jordan derivation, respectively [1].

A von Neumann algebra $\mathcal{A}$ is a self-adjoint subalgebra of some $B(H)$, the algebra of bounded linear operators acting on a complex Hilbert space, which satisfies the double commutant property: $\mathcal{A}^{''} = \mathcal{A}$ where $\mathcal{A}^{'} = \{T \in B(H), TA = AT, \forall A \in \mathcal{A}\}$ and $\mathcal{A}^{''} = \{\mathcal{A}^{'}\}^{'}$. Denote by $\mathcal{Z}(\mathcal{A}) = \mathcal{A}^{'} \cap \mathcal{A}$ the center of $\mathcal{A}$. A von Neumann algebra $\mathcal{A}$ is called a factor if its center is trivial, that is, $\mathcal{Z}(\mathcal{A}) = \mathbf{C}I$. For $A \in \mathcal{A}$, recall that the central carrier of A, denoted by $\overline{A}$, is the smallest central projection P such that $PA = A$. It is not difficult to see that $\overline{A}$ is the projection onto the closed subspace spanned by $\{BAx : B \in \mathcal{A}, x \in H\}$. If A is self-adjoint, then the core of A, denoted by $\underline{A}$, is $\sup\{S \in \mathcal{Z}(\mathcal{A}) : S = S^*, S \leq A\}$. If $A = P$ is a projection, it is clear that $\underline{P}$ is the largest central projection Q satisfying $Q \leq P$. A projection P is said to be core-free if $\underline{P} = 0$ (see [10]). It is easy to see that $\underline{P} = 0$ if and only if $\overline{I - P} = I$, [6, 7].

Recently, Yu and Zhang in [17] proved that every non-linear $*$-Lie derivation from a factor von Neumann algebra into itself is an additive $*$-derivation. Also, Li, Lu and Fang in [8] have investigated a non-linear λ-Jordan $*$-derivation. They showed that if $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is a von Neumann algebra without central abelian projections and λ is a non-zero scalar, then

$\Phi : \mathcal{A} \longrightarrow \mathcal{B}(\mathcal{H})$ is a non-linear λ-Jordan $*$-derivation if and only if Φ is an additive $*$-derivation.

On the other hand, many mathematician devoted themselves to study the $*$-Jordan product $A \bullet B = AB + BA^*$. In [18], F. Zhang proved that every non-linear $*$-Jordan derivation map $\Phi : \mathcal{A} \to \mathcal{A}$ on a factor von neumann algebra with $I_{\mathcal{A}}$ the identity of it is an additive $*$-derivation.

In [16], we showed that $*$-Jordan derivation map on every factor von Neumann algebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is additive $*$-derivation.

Very recently the authors of [5] discussed some bijective maps preserving the new product A^*B+B^*A between von Neumann algebras with no central abelian projections. In other words, Φ holds in the following condition

$$\Phi(A^*B + B^*A) = \Phi(A)^*\Phi(B) + \Phi(B)^*\Phi(A).$$

They showed that such a map is sum of a linear $*$-isomorphism and a conjugate linear $*$-isomorphism.

Motivated by the above results, in this paper, we prove that if $\mathcal{A}$ is a prime $*$-algebra then $\Phi : \mathcal{A} \to \mathcal{A}$ which holds in the following condition

$$\Phi(A \diamond B) = \Phi(A) \diamond B + A \diamond \Phi(B)$$

where $A \diamond B = A^*B - B^*A$ for all $A, B \in \mathcal{A}$, is additive $*$-derivation.

We say that $\mathcal{A}$ is prime, that is, for $A, B \in \mathcal{A}$ if $A\mathcal{A}B = \{0\}$, then $A = 0$ or $B = 0$. For example, every simple or prime generally primitive C^*-algebras are prime (e.g., $\mathcal{B}(\mathcal{H})$, $\mathcal{K}(\mathcal{H})$ for every Hilbert space) [4].

2. Main Results

Our main theorem is as follows:

Theorem 2.1. *Let $\mathcal{A}$ be a prime $*$-algebra with unit I and a nontrivial projection. Then the map $\Phi : \mathcal{A} \to \mathcal{A}$ satisfies in the following condition*

$$\Phi(A \diamond B) = \Phi(A) \diamond B + A \diamond \Phi(B) \tag{2.1}$$

where $A \diamond B = A^*B - B^*A$ for all $A, B \in \mathcal{A}$. is additive.

Proof. Let P_1 be a nontrivial projection in $\mathcal{A}$ and $P_2 = I_{\mathcal{A}} - P_1$. Denote $\mathcal{A}_{ij} = P_i\mathcal{A}P_j,\ i, j = 1, 2$, then $\mathcal{A} = \sum_{i,j=1}^{2} \mathcal{A}_{ij}$. For every $A \in \mathcal{A}$ we may write $A = A_{11} + A_{12} + A_{21} + A_{22}$. In all that follow, when we write A_{ij}, it indicates that $A_{ij} \in \mathcal{A}_{ij}$. For showing additivity of Φ on $\mathcal{A}$, we use above partition of $\mathcal{A}$ and give some claims that prove Φ is additive on each $\mathcal{A}_{ij},\ i,j=1,2$.

We prove the above theorem by several claims.

Claim 1. $\Phi(0)=0$.

This claim is easy to prove.

Claim 2. $\Phi(iA)=i\Phi(A)+A^*K$, *where,* $K=\Phi(iI)-i\Phi(I)$.

Consider

$$\Phi(-iA\diamond I)=\Phi(A\diamond iI).$$

So, we have

$$\begin{aligned}&\Phi(-iA)^*-\Phi(-iA)+iA^*\Phi(I)+i\Phi(I)^*A=i\Phi(A)^*+i\Phi(A)\\ (2.2)\qquad &+A^*\Phi(iI)-\Phi(iI)^*A.\end{aligned}$$

Consider

$$\Phi(-iA\diamond iI)=\Phi(I\diamond A)$$

So, we have

$$\imath\Phi(-iA)^*+i\Phi(-iA)+iA^*\Phi(iI)+i\Phi(iI)^*A=\Phi(I)^*A-A^*\Phi(I)+\Phi(A)-\Phi(A)^*.$$

Equivalently, we obtain

$$\begin{aligned}&-\Phi(-iA)^*-\Phi(-iA)-A^*\Phi(iI)-\Phi(iI)^*A=i\Phi(I)^*A-iA^*\Phi(I)\\ (2.3)\quad &+i\Phi(A)-i\Phi(A)^*.\end{aligned}$$

By adding equations (2.2) and (2.3) we have

$$-\Phi(-iA)-i\Phi(A)=-iA^*\Phi(I)+A^*\Phi(iI).$$

Substituting iA instead of A in the above equation implies

$$\Phi(iA)=i\Phi(A)+A^*(\Phi(iI)-i\Phi(I))$$

that

$$K=\Phi(iI)-i\Phi(I).$$

So

$$\Phi(iA)=i\Phi(A)+A^*K$$

Claim 3. $\Phi(-A) = -\Phi(A)$

By considering $\Phi(iA) = i\Phi(A) + A^*K$ and applying iA instead of A we have

$$\begin{aligned}\Phi(-A) &= i\Phi(iA) - iA^*K\\ \Phi(-A) &= i(i\Phi(A) + A^*K) - iA^*K\\ \Phi(-A) &= -\Phi(A) + iA^*K - iA^*K\\ \Phi(-A) &= -\Phi(A).\end{aligned} \tag{2.4}$$

Claim 4. *For each $A_{11} \in \mathcal{A}_{11}$, $A_{12} \in \mathcal{A}_{12}$ we have*

$$\Phi(A_{11} + A_{12}) = \Phi(A_{11}) + \Phi(A_{12}).$$

Let $T = \Phi(A_{11} + A_{12}) - \Phi(A_{11}) - \Phi(A_{12})$, we should prove that $T = 0$. For $X_{21} \in \mathcal{A}_{21}$ we can write that

$$\begin{aligned}&\Phi(A_{11} + A_{12}) \diamond X_{21} + (A_{11} + A_{12}) \diamond \Phi(X_{21}) = \Phi((A_{11} + A_{12}) \diamond X_{21})\\ &= \Phi(A_{11} \diamond X_{21}) + \Phi(A_{12} \diamond X_{21}) = \Phi(A_{11}) \diamond X_{21} + A_{11} \diamond \Phi(X_{21})\\ &+\Phi(A_{12}) \diamond X_{21} + A_{12} \diamond \Phi(X_{21})\\ &= (\Phi(A_{11}) + \Phi(A_{12})) \diamond X_{21} + (A_{11} + A_{12}) \diamond \Phi(X_{21}).\end{aligned}$$

So, we obtain

$$T \diamond X_{21} = 0.$$

Since $T = T_{11} + T_{12} + T_{21} + T_{22}$ we have

$$T_{21}^* X_{21} + T_{22}^* X_{21} - X_{21}^* T_{21} - X_{21}^* T_{22} = 0.$$

From the above equation and primeness of $\mathcal{A}$ we have $T_{22} = 0$ and

$$T_{21}^* X_{21} - X_{21}^* T_{21} = 0. \tag{2.5}$$

On the other hand, similarly by applying iX_{21} instead of X_{21} in above, we obtain

$$iT_{21}^* X_{21} + iT_{22}^* X_{21} + iX_{21}^* T_{21} + iX_{21}^* T_{22} = 0.$$

Since $T_{22} = 0$ we obtain from the above equation that

$$-T_{21}^* X_{21} - X_{21}^* T_{21} = 0. \tag{2.6}$$

From (2.5) and (2.6) we have

$$X_{21}^* T_{21} = 0.$$

Since $\mathcal{A}$ is prime, then we get $T_{21} = 0$.

It suffices to show that $T_{12} = T_{11} = 0$. For this purpose for $X_{12} \in \mathcal{A}_{12}$ we write

$$\begin{aligned}
&\Phi(((A_{11} + A_{12}) \diamond X_{12}) \diamond P_1) = \Phi((A_{11} + A_{12}) \diamond X_{12}) \diamond P_1 + ((A_{11} + A_{12}) \diamond X_{12}) \diamond \Phi(P_1)\\
&= (\Phi(A_{11} + A_{12}) \diamond X_{12} + (A_{11} + A_{12}) \diamond \Phi(X_{12})) \diamond P_1 + (A_{11} + A_{12}) \diamond X_{12} \diamond \Phi(P_1)\\
&= \Phi(A_{11} + A_{12}) \diamond X_{12} \diamond P_1 + A_{11} \diamond \Phi(X_{12}) \diamond P_1 + A_{12} \diamond \Phi(X_{12}) \diamond P_1\\
&+ A_{11} \diamond X_{12} \diamond \Phi(P_1) + A_{12} \diamond X_{12} \diamond \Phi(P_1).
\end{aligned}$$

So, we showed that

$$\begin{aligned}
&\Phi(((A_{11} + A_{12}) \diamond X_{12}) \diamond P_1) = \Phi(A_{11} + A_{12}) \diamond X_{12} \diamond P_1 + A_{11} \diamond \Phi(X_{12}) \diamond P_1\\
&+ A_{12} \diamond \Phi(X_{12}) \diamond P_1 + A_{11} \diamond X_{12} \diamond \Phi(P_1) + A_{12} \diamond X_{12} \diamond \Phi(P_1).
\end{aligned} \tag{2.7}$$

Since $A_{12} \diamond X_{12} \diamond P_1 = 0$ we have

$$\begin{aligned}
&\Phi(((A_{11} + A_{12}) \diamond X_{12}) \diamond P_1) = \Phi((A_{11} \diamond X_{12}) \diamond P_1) + \Phi((A_{12} \diamond X_{12}) \diamond P_1)\\
&= \Phi(A_{11} \diamond X_{12}) \diamond P_1 + (A_{11} \diamond X_{12}) \diamond \Phi(P_1) + \Phi(A_{12} \diamond X_{12}) \diamond P_1 + (A_{12} \diamond X_{12}) \diamond \Phi(P_1)\\
&= (\Phi(A_{11}) \diamond X_{12} + A_{11} \diamond \Phi(X_{12})) \diamond P_1 + (A_{11} \diamond X_{12}) \diamond \Phi(P_1)\\
&+ (\Phi(A_{12}) \diamond X_{12} + A_{12} \diamond \Phi(X_{12})) \diamond P_1 + (A_{12} \diamond X_{12}) \diamond \Phi(P_1)\\
&= \Phi(A_{11}) \diamond X_{12} \diamond P_1 + A_{11} \diamond \Phi(X_{12}) \diamond P_1 + A_{11} \diamond X_{12} \diamond \Phi(P_1)\\
&+ \Phi(A_{12}) \diamond X_{12} \diamond P_1 + A_{12} \diamond \Phi(X_{12}) \diamond P_1 + A_{12} \diamond X_{12} \diamond \Phi(P_1).
\end{aligned}$$

So,

$$\begin{aligned}
&\Phi(((A_{11} + A_{12}) \diamond X_{12}) \diamond P_1) = \Phi(A_{11}) \diamond X_{12} \diamond P_1 + A_{11} \diamond \Phi(X_{12}) \diamond P_1\\
&+ A_{11} \diamond X_{12} \diamond \Phi(P_1) + \Phi(A_{12}) \diamond X_{12} \diamond P_1\\
&+ A_{12} \diamond \Phi(X_{12}) \diamond P_1 + A_{12} \diamond X_{12} \diamond \Phi(P_1).
\end{aligned} \tag{2.8}$$

From (2) and (2.7) we have

$$\Phi(A_{11} + A_{12}) \diamond X_{12} \diamond P_1 = \Phi(A_{11}) \diamond X_{12} \diamond P_1 + \Phi(A_{12}) \diamond X_{12} \diamond P_1.$$

It follows that $T \diamond X_{12} \diamond P_1 = 0$, so $T_{11}^* X_{12} - X_{12}^* T_{11} = 0$. We have $T_{11}^* X_{12} = 0$ or $T_{11} X P_2 = 0$ for all $X \in \mathcal{A}$, then we have $T_{11} = 0$. Similarly, we can show that $T_{12} = 0$ by applying P_2 instead of P_1 in above.

Claim 5. *For each $A_{11} \in \mathcal{A}_{11}, A_{12} \in \mathcal{A}_{12}, A_{21} \in \mathcal{A}_{21}$ and $A_{22} \in \mathcal{A}_{22}$ we have*

1.

$$\Phi(A_{11} + A_{12} + A_{21}) = \Phi(A_{11}) + \Phi(A_{12}) + \Phi(A_{21}).$$

2.

$$\Phi(A_{12} + A_{21} + A_{22}) = \Phi(A_{12}) + \Phi(A_{21}) + \Phi(A_{22}).$$

We show that

$$T = \Phi(A_{11} + A_{12} + A_{21}) - \Phi(A_{11}) - \Phi(A_{12}) - \Phi(A_{21}) = 0.$$

So, we have

$$\begin{aligned}
&\Phi(A_{11} + A_{12} + A_{21}) \diamond X_{21} + (A_{11} + A_{12} + A_{21}) \diamond \Phi(X_{21}) \\
&= \Phi((A_{11} + A_{12} + A_{21}) \diamond X_{21}) = \Phi(A_{11} \diamond X_{21}) + \Phi(A_{12} \diamond X_{21}) + \Phi(A_{21} \diamond X_{21}) \\
&= (\Phi(A_{11}) + \Phi(A_{12}) + \Phi(A_{21})) \diamond X_{21} + (A_{11} + A_{12} + A_{21}) \diamond \Phi(X_{21}).
\end{aligned}$$

It follows that $T \diamond X_{21} = 0$. Since $T = T_{11} + T_{12} + T_{21} + T_{22}$ we have

$$T_{22}^* X_{21} + T_{21}^* X_{21} - X_{21}^* T_{22} - C_{21}^* T_{21} = 0.$$

Therefore, $T_{22} = T_{21} = 0$.
From Claim 4, we obtain

$$\begin{aligned}
&\Phi(A_{11} + A_{12} + A_{21}) \diamond X_{12} + (A_{11} + A_{12} + A_{21}) \diamond \Phi(X_{12}) \\
&= \Phi((A_{11} + A_{12} + A_{21}) \diamond X_{12}) = \Phi((A_{11} + A_{12}) \diamond X_{12}) + \Phi(A_{21} \diamond X_{12}) \\
&= \Phi(A_{11} \diamond X_{12}) + \Phi(A_{12} \diamond X_{12}) + \Phi(A_{21} \diamond X_{12}) \\
&= (\Phi(A_{11}) + \Phi(A_{12}) + \Phi(A_{21})) \diamond X_{12} + (A_{11} + A_{12} + A_{21}) \diamond \Phi(X_{12}).
\end{aligned}$$

Hence,

$$T_{11}^* X_{12} + T_{12}^* X_{12} - X_{12}^* T_{11} - X_{12}^* T_{12} = 0.$$

Then $T_{11} = T_{12} = 0$. Similarly

$$\Phi(A_{12} + A_{21} + A_{22}) = \Phi(A_{12}) + \Phi(A_{21}) + \Phi(A_{22}).$$

Claim 6. *For each $A_{11} \in \mathcal{A}_{11}, A_{12} \in \mathcal{A}_{12}, A_{21} \in \mathcal{A}_{21}$ and $A_{22} \in \mathcal{A}_{22}$ we have*

$$\Phi(A_{11} + A_{12} + A_{21} + A_{22}) = \Phi(A_{11}) + \Phi(A_{12}) + \Phi(A_{21}) + \Phi(A_{22}).$$

We show that

$$T = \Phi(A_{11} + A_{12} + A_{21} + A_{22}) - \Phi(A_{11}) - \Phi(A_{12}) - \Phi(A_{21}) - \Phi(A_{22}) = 0.$$

From Claim 5, we have

$$\begin{aligned}
&\Phi(A_{11} + A_{12} + A_{21} + A_{22}) \diamond X_{12} + (A_{11} + A_{12} + A_{21} + A_{22}) \diamond \Phi(X_{12})\\
&= \Phi((A_{11} + A_{12} + A_{21} + A_{22}) \diamond X_{12})\\
&= \Phi((A_{11} + A_{12} + A_{21}) \diamond X_{12}) + \Phi(A_{22} \diamond X_{12})\\
&= \Phi(A_{11} \diamond X_{12}) + \Phi(A_{12} \diamond X_{12}) + \Phi(A_{21} \diamond X_{12}) + \Phi(A_{22} \diamond X_{12})\\
&= (\Phi(A_{11}) + \Phi(A_{12}) + \Phi(A_{21}) + \Phi(A_{22})) \diamond X_{12}\\
&+(A_{11} + A_{12} + A_{21} + A_{22}) \diamond \Phi(X_{12}).
\end{aligned}$$

So, $T \diamond X_{12} = 0$. It follows that

$$T_{11}^* X_{12} + T_{12}^* X_{12} - X_{12}^* T_{11} - X_{12}^* T_{12} = 0.$$

Then $T_{11} = T_{12} = 0$.

Similarly, by applying X_{21} instead of X_{12} in above, we obtain $T_{21} = T_{22} = 0$.

Claim 7. *For each $A_{ij}, B_{ij} \in \mathcal{A}_{ij}$ such that $i \neq j$, we have*

$$\Phi(A_{ij} + B_{ij}) = \Phi(A_{ij}) + \Phi(B_{ij}).$$

It is easy to show that

$$(P_i + A_{ij})(P_j + B_{ij}) - (P_j + B_{ij}^*)(P_i + A_{ij}^*) = A_{ij} + B_{ij} - A_{ij}^* - B_{ij}^*.$$

So, we can write

$$\begin{aligned}
&\Phi(A_{ij} + B_{ij}) + \Phi(-A_{ij}^* - B_{ij}^*) = \Phi((P_i + A_{ij}^*) \diamond (P_j + B_{ij}))\\
&= \Phi(P_i + A_{ij}^*) \diamond (P_j + B_{ij}) + (P_i + A_{ij}^*) \diamond \Phi(P_j + B_{ij})\\
&= (\Phi(P_i) + \Phi(A_{ij}^*)) \diamond (P_j + B_{ij}) + (P_i + A_{ij}^*) \diamond (\Phi(P_j) + \Phi(B_{ij}))\\
&= \Phi(P_i) \diamond B_{ij} + P_i \diamond \Phi(B_{ij}) + \Phi(A_{ij}^*) \diamond P_j + A_{ij}^* \diamond \Phi(P_j)\\
&= \Phi(P_i \diamond B_{ij}) + \Phi(A_{ij}^* \diamond P_j)\\
&= \Phi(B_{ij}) + \Phi(-B_{ij}^*) + \Phi(A_{ij}) + \Phi(-A_{ij}^*).
\end{aligned}$$

Therefore, we show that

$$\Phi(A_{ij}+B_{ij})+\Phi(-A_{ij}^*-B_{ij}^*)=\Phi(A_{ij})+\Phi(B_{ij})+\Phi(-A_{ij}^*)+\Phi(-B_{ij}^*) \tag{2.9}$$

By an easy computation, we can write

$(P_i+A_{ij})(iP_j+iB_{ij})-(-iP_j-iB_{ij}^*)(P_i+A_{ij}^*)=iA_{ij}+iB_{ij}+iA_{ij}^*+iB_{ij}^*.$

Then, we have

$$\begin{aligned}
&\Phi(iA_{ij}+iB_{ij})+\Phi(iA_{ij}^*+iB_{ij}^*)=\Phi((P_i+A_{ij}^*)\diamond(iP_j+iB_{ij}))\\
&=\Phi(P_i+A_{ij}^*)\diamond(iP_j+iB_{ij})+(P_i+A_{ij}^*)\diamond\Phi(iP_j+iB_{ij})\\
&=(\Phi(P_i)+\Phi(A_{ij}^*))\diamond(iP_j+iB_{ij})+(P_i+A_{ij}^*)(\Phi(iP_j)+\Phi(iB_{ij}))\\
&=\Phi(P_i)\diamond iB_{ij}+P_i\diamond\Phi(iB_{ij})+\Phi(A_{ij}^*)\diamond iP_j+A_{ij}^*\diamond\Phi(iP_j)\\
&=\Phi(P_i\diamond iB_{ij})+\Phi(A_{ij}^*\diamond iP_j)\\
&=\Phi(iB_{ij})+\Phi(iB_{ij}^*)+\Phi(iA_{ij})+\Phi(iA_{ij}^*).
\end{aligned}$$

We showed that

$\Phi(iA_{ij}+iB_{ij})+\Phi(iA_{ij}^*+iB_{ij}^*)=\Phi(iB_{ij})+\Phi(iB_{ij}^*)+\Phi(iA_{ij})+\Phi(iA_{ij}^*).$

From Claims 2, 3 and the above equation, we have

$$\Phi(A_{ij}+B_{ij})-\Phi(-A_{ij}^*-B_{ij}^*)=\Phi(B_{ij})-\Phi(-B_{ij}^*)+\Phi(A_{ij})-\Phi(-A_{ij}^*) \tag{2.10}$$

By adding equations (2.8) and (2.9), we obtain

$$\Phi(A_{ij}+B_{ij})=\Phi(A_{ij})+\Phi(B_{ij}).$$

Claim 8. *For each $A_{ii}, B_{ii}\in\mathcal{A}_{ii}$ such that $1\le i\le 2$, we have*

$$\Phi(A_{ii}+B_{ii})=\Phi(A_{ii})+\Phi(B_{ii}).$$

We show that

$$T=\Phi(A_{ii}+B_{ii})-\Phi(A_{ii})-\Phi(B_{ii})=0.$$

We can write

$$\begin{aligned}
&\Phi(A_{ii}+B_{ii})\diamond P_j+(A_{ii}+B_{ii})\diamond\Phi(P_j)=\Phi((A_{ii}+B_{ii})\diamond P_j)\\
&=\Phi(A_{ii}\diamond P_j)+\Phi(B_{ii}\diamond P_j)\\
&\Phi(A_{ii})\diamond P_j+A_{ii}\diamond\Phi(P_j)+\Phi(B_{ii})\diamond P_j+B_{ii}\diamond\Phi(P_j)\\
&=(\Phi(A_{ii})+\Phi(B_{ii}))\diamond P_j+(A_{ii}+B_{ii})\diamond\Phi(P_j).
\end{aligned}$$

So, we have

$$T \diamond P_j = 0.$$

Therefore, we obtain $T_{ij} = T_{ji} = T_{jj} = 0$.
On the other hand, for every $X_{ij} \in \mathcal{A}_{ij}$, we have

$$\begin{aligned}
&\Phi(A_{ii} + B_{ii}) \diamond X_{ij} + (A_{ii} + B_{ii}) \diamond \Phi(X_{ij}) = \Phi((A_{ii} + B_{ii}) \diamond X_{ij}) \\
&= \Phi(A_{ii} \diamond X_{ij}) + \Phi(B_{ii} \diamond X_{ij}) = \Phi(A_{ii}) \diamond X_{ij} + A_{ii} \diamond \Phi(X_{ij}) \\
&+\Phi(B_{ii}) \diamond X_{ij} + B_{ii} \diamond \Phi(X_{ij}) \\
&= (\Phi(A_{ii}) + \Phi(B_{ii})) \diamond X_{ij} + (A_{ii} + B_{ii}) \diamond \Phi(X_{ij}).
\end{aligned}$$

So,

$$(\Phi(A_{ii} + B_{ii}) - \Phi(A_{ii}) - \Phi(B_{ii})) \diamond X_{ij} = 0.$$

It follows that $T \diamond X_{ij} = 0$ or $T_{ii}X_{ij} = 0$. By knowing that $\mathcal{A}$ is prime, we have $T_{ii} = 0$.
Hence, the additivity of Φ comes from the above claims.

In the rest of this paper we show that Φ is $*$-derivation.

Theorem 2.2. *With notaion of the previous theorem, if $\Phi(\alpha I)$ is self-adjoint operator for $\alpha \in \{1, i\}$ then Φ is $*$-derivation.*

Proof. We present the proof of the above theorem by several claims.

Claim 9. $\Phi(iI) = \Phi(I) = 0$.

Consider $\Phi(I \diamond iI) = \Phi(I) \diamond iI + I \diamond \Phi(iI)$ that imply

$$2\Phi(iI) = i\Phi(I)^* + i\Phi(I) + \Phi(iI) - \Phi(iI)^* = i\Phi(I). \tag{2.11}$$

By taking the adjoint of above equation we have $\Phi(iI) = \Phi(I) = 0$

Claim 10. Φ *preserves star.*

Since $\Phi(I) = 0$ then we can write

$$\Phi\,(I \diamond A) = I \diamond \Phi(A).$$

Then

$$\Phi(A - A^*) = \Phi(A) - \Phi(A)^*.$$

So, we showed that Φ preserves star.

Claim 11. *We prove that Φ is derivation.*

For every $A, B \in \mathcal{A}$ we have

$$\begin{aligned}\Phi(AB - B^*A^*) &= \Phi(A^* \diamond B)\\ &= \Phi(A^*) \diamond B + A^* \diamond \Phi(B)\\ &= \Phi(A^*)^*B - \Phi(B)^*A^* - B^*\Phi(A^*) + A\Phi(B).\end{aligned}$$

On the other hand, since Φ preserves star, we have

$$\Phi(AB - B^*A^*) = \Phi(A)B + A\Phi(B) - B^*\Phi(A^*) - \Phi(B)^*A^*. \tag{2.12}$$

So, from (2.11), we have

$$\begin{aligned}&\Phi(i(AB + B^*A^*)) = \Phi(A(iB) - (iB)^*A^*)\\ &= \Phi(A)(iB) + A\Phi(iB) - (iB)^*\Phi(A^*) - \Phi(iB)^*A^*.\end{aligned}$$

Therefore, from claims 2 and 9 we have

$$\Phi(AB + B^*A^*) = \Phi(A)B + A\Phi(B) + B^*\Phi(A^*) + \Phi(B^*)A^*. \tag{2.13}$$

By adding equations (2.11) and (2.12), we have

$$\Phi(AB) = \Phi(A)B + A\Phi(B).$$

This completes the proof.

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