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On generalized δ_ω−closed sets

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Abstract:

A new class of sets called generalized δ_ω−closed sets in topological spaces is introduced and some of their basic properties are investigated. This new class of sets lies between the class of δ_ω−closed and generalized closed sets in (X, τ). Moreover, we provide several relatively new decompositions of continuity. Several examples are provided to illustrate the behavior of the new sets.

Keywords: δ_ω−open sets; δ_ω−compact space; Generalized δ_ω−closed; Generalized closed.

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1. Introduction

Throughout this work a space will always mean a topological space in which no separation axioms is assumed unless explicitly stated. If A is a subset of a space (X, τ) then the closure of A, the interior of A and the relative topology on A in (X, τ) will be denoted by $Cl(A)$, $Int(A)$ and τ_A, respectively.

Let A be a subset of a space (X, τ). A subset A is called a regular open subset of (X, τ) if $A = Int(Cl(A))$. The family of all regular open subsets of (X, τ) is denoted by $RO(X, \tau)$. The complement of a regular open set is called regular closed. A subset A is called $\delta-$open [14] if and only if for each $x \in A$ there exists a regularly open set G such that $x \in G \subseteq A$. It is well know that the collection of all $\delta-$open sets in a topological space (X, τ) forms a topology τ_δ weaker than τ [12]. The space (X, τ_δ) is also called the semigeneralization topology of (X, τ) [12]. The complement of a $\delta-$open set is called $\delta-$closed [14]. A point $x \in X$ is called a $\delta-$cluster point of A if and only if $Int(Cl(V)) \cap A = \phi$, for each open set V containing x. The set of all $\delta-$cluster points of A is called the $\delta-$closure of A [14], which is denoted by $Cl_\delta(A)$. A space (X, τ) is said to be semi$-$regular [12] if $\tau_\delta = \tau$. Any regular space is semi-regular, but the converse is false. A family $\{A_\alpha\}_{\alpha \in \Delta}$ of subsets of a topological space X is locally finite [8] if for every point $x \in X$ there exists a neighbourhood U such that the set $\{\alpha \in \Delta : U \cap A_\alpha = \phi\}$ is finite.

Let A be any subset of a space (X, τ). Then a point $x \in X$ is called a condensation point [9] of A if for each $U \in \tau$ with $x \in U$, the set $U \cap A$ is uncountable. The set A is called $\omega-$closed if it contains all its condensation points. The complement of an $\omega-$closed set is called $\omega-$open or equivalently A is $\omega-$open [2] if for each $x \in A$, there exists an open set U containing x such that $U - A$ is countable. The family of all $\omega-$open subsets of a space (X, τ), denoted by τ_ω, forms a topology on X finer that τ. The $\omega-$closure of a subset A of a space (X, τ) is the closure of A in the space (X, τ_ω), and it is denoted by $Cl_\omega(A)$. A space (X, τ) is called anti$-$locally countable [2] if each non-empty open subset of (X, τ) is uncountable.

Let A be a subset of a space (X, τ). A subset A is called a regular $\omega-$open [13] subset of (X, τ) if $A = Int(Cl_\omega(A))$. The family of all regular $\omega-$open subsets of (X, τ) is denoted by $R\omega O(X)$. The complement of a regular $\omega-$open set is called regular $\omega-$closed. The class of $R\omega-$open sets forms a base for some a topology on X denoted by $\tau_{\delta-\omega}$. A point $x \in X$ is called a $\delta_\omega-$cluster point of A [1] if and only if $Int(Cl_\omega(V)) \cap A = \phi$, for each open set V containing x. The set of all $\delta_\omega-$cluster points of A is called the $\delta_\omega-$closure of A [1], which is denoted by $Cl_{\delta_\omega}(A)$. A subset A of a space X is called $\delta_\omega-$closed [1] if and only if $A = Cl_{\delta_\omega}(A)$ and it is called $\delta_\omega-$open if and only if it is complement is $\delta_\omega-$closed and the set of all $\delta_\omega-$open sets form a topology denoted by τ_{δ_ω} and equal to $\tau_{\delta-\omega}$. In [1], we show that a subset A is $\delta_\omega-$open if and only if for each $x \in A$ there exists a regularly $\omega-$open set G such that $x \in G \subseteq A$.

Generalized semiclosed [3](resp., $\alpha-$generalized closed [11], $\theta-$generalized closed [6], $\delta-$generalized closed [5]) sets are defined by replacing the closure operator in Livine's original [10] by the semiclosure (resp., $\alpha-$closure, $\theta-$closure, $\delta-$closure) operator.

In section 2 of this work, we follows a similar line to introduce $\delta_\omega-$generalized closed sets by using the $\delta_\omega-$closure operator. Also we define generalized $\delta_\omega-$closed and $\delta_\omega^{\delta\omega}-$generalized closed, then we study some of relationship between them. In section 3, we introduce and study some of topological properties of $\delta_\omega-$generalized closed sets. In section 4, we introduce $\delta_\omega-$continuity and $\delta_\omega-$irresoluteness and study some of its characterizations. We introduce $\delta_\omega g-$continuity and $\delta_\omega g-$irresoluteness by using $\delta_\omega g-$closed sets and study some of their fundamental properties.

In this paper $\mathbf{R}, \mathbf{Q}$ and $\mathbf{N}$ denote, respectively the set of real numbers, the set of rational numbers and the set of natural numbers.

Now we begin with some notations, definitions, and result will be used in this work.

Proposition 1.1. *[1] A topological space (X,τ) is connected if and only if (X,τ_{δ_ω}) is connected.*

Theorem 1.2. *[1] Let (X,τ) be a topological space. Then:*

1. *$\tau_\delta \subseteq \tau_{\delta_\omega} \subseteq \tau$.*
2. *If (X,τ) is regular, then $\tau_\delta = \tau_{\delta_\omega} = \tau$.*

Proposition 1.3. *[1] Let (X,τ) be a topological space and let $A \subseteq X$. Then:*

1. *For each $A \in \tau_\omega$, $Cl_{\delta\omega}(A) = Cl(A)$.*
2. *For each $A \in \tau$, $Cl_\delta(A) = Cl_{\delta\omega}(A) = Cl(A)$.*

Lemma 1.4. *[1] Let (X,τ) be an anti-locally countable space, then $(\tau_{\delta_\omega})_{\delta\omega} = \tau_{\delta_\omega}$.*

Theorem 1.5. *[1] Let (X,τ) and (Y,σ) be two topological spaces. Then $(\tau\times\sigma)_{\delta_\omega}\subseteq\tau_{\delta_\omega}\times\sigma_{\delta_\omega}$*

Definition 1.6. *[4] A function $f:(X,\tau)\to(Y,\sigma)$ is called $g-$continuous if $f^{-1}(V)$ is $g-$closed in (X,τ) for every closed set V of (Y,σ).*

Theorem 1.7. *[8] For every locally finite family $\{A_\alpha\}_{\alpha\in\Delta}$ we have the equality $Cl\left(\bigcup\limits_{\alpha\in\Delta}A\alpha\right)=\bigcup\limits_{\alpha\in\Delta}Cl\,(A\alpha)$.*

Recall that a subset A of a space (X,τ) is called generalized closed [10] (resp. $\delta-$generalized closed [5]) if $Cl(A)\subseteq U$ (resp. $Cl_\delta(A)\subseteq U$) whenever $A\subseteq U$ and U is open in (X,τ) and we will denoted by $g-$closed and $\delta g-$closed; respectively. A topological space (X,τ) is called $T_{\frac{1}{2}}$-space [10] if every $g-$closed set is closed, (equivalently, every singleton is either open or closed [7]).

2. $\delta_\omega-$generalized closed sets

Definition 2.1. *A subset A of a space (X,τ) is called generalized $\delta_\omega-$closed (reps. $\delta_\omega-$generalized closed, $\delta_\omega^{\delta\omega}-$generalized closed) if $Cl(A)\subseteq U$ (resp. $Cl_{\delta_\omega}(A)\subseteq U$, $Cl_{\delta_\omega}(A)\subseteq U$) whenever $A\subseteq U$ and U is $\delta_\omega-$open (reps. U is open, U is δ_ω-open). Respectively, we will briefly it by $g\delta_\omega-$closed, $\delta_\omega g-$closed and $\delta_\omega^{\delta\omega}g-$closed.*

We denote the family of all generalized $\delta_\omega-$closed (reps. $\delta_\omega-$generalized closed, $\delta_\omega^{\delta\omega}-$generalized closed, $\delta g-$closed, $g-$closed) subsets of a space (X,τ) by $G\delta_\omega C(X,\tau)$ (resp. $\delta_\omega GC(X,\tau)$, $\delta_\omega^{\delta_\omega}GC(X,\tau)$, $\delta GC(X,\tau)$, $GC(X,\tau)$). It is note that $\delta_\omega^{\delta_\omega}GC(X,\tau)=GC(X,\tau_{\delta_\omega})$.

Observe that if (X,τ) is a locally countable space or regular space, then $\tau_{\delta_\omega}=\tau$ and so $GC(X,\tau)=G\delta_\omega C(X,\tau)=\delta_\omega GC(X,\tau)=\delta_\omega^{\delta_\omega}GC(X,\tau)$.

The following implications follows from the definitions and the fact that for any space (X,τ), $\tau_\delta\subseteq\tau_{\delta_\omega}\subseteq\tau$.

$$\begin{array}{ccccc}
\text{closed} & \to & GC(X,\tau) & \to & G\delta_\omega C(X,\tau)\\
\uparrow & & \uparrow & & \uparrow\\
\delta_\omega-\text{closed} & \to & \delta_\omega GC(X,\tau) & \to & \delta_\omega^{\delta_\omega}GC(X,\tau)\\
\uparrow & & \uparrow & & \\
\delta-\text{closed} & \to & \delta GC(X,\tau) & &
\end{array}$$

Example 2.2. *Let $X = \{1, 2, 3\}$ with the topology $\tau = \{\phi, X, \{1, 2\}\}$ and let $A = \{1, 3\}$. Since the only open superset of A is X, then $A \in \delta_\omega GC(X, \tau)$. But A is not δ_ω-closed.*

Example 2.3. *Let $X = \mathbf{R}$ with the topology $\tau = \{\phi\} \cup \{U \subseteq X : \mathbf{R} - \mathbf{Q} \subseteq U\}$ and let $A = \{1\}$. Then $U = (\mathbf{R} - \mathbf{Q}) \cup \{1\}$ is an open set in (X, τ) such that $A \subseteq U$ and $Cl_{\delta_\omega}(A) = \mathbf{R} \subseteq U$. So $A \in GC(X, \tau) - \delta_\omega GC(X, \tau)$.*

Example 2.4. *Let $X = \mathbf{R}$ with the topology $\tau = \{\phi, \mathbf{R}, \mathbf{R} - \mathbf{Q}\}$ and let $A = \{\sqrt{2}\}$. Then $A \in G\delta_\omega C(X, \tau)$. On the other hand, $A \notin GC(X, \tau)$ since $A \subseteq \mathbf{R} - \mathbf{Q} \in \tau$ and $Cl(A) = X \subseteq \mathbf{R} - \mathbf{Q}$.*

Example 2.5. *Let $X = \mathbf{R}$ with the topology $\tau = \{\phi\} \cup \{U \subseteq X : 0 \in U\}$ and let $A = \mathbf{R} - \mathbf{Q}$. Note that $\tau_\omega = \tau_{dis}$ and so $\tau_{\delta_\omega} = \tau$. On the other hand, $\tau_\delta = \tau_{ind}$. Thus $A \in \delta_\omega GC(X, \tau)$, but $A \notin \delta GC(X, \tau)$.*

Example 2.6. *Let $X = \mathbf{R}$ with the topology $\tau = \{\phi, \mathbf{R}, \{0\}, [0, \infty)\}$ and let $A = \{1\}$. Then $A \in \delta_\omega^{\delta_\omega} GC(X, \tau)$. But A is neither $g-$closed in (X, τ) nor $\delta_\omega g-$closed in (X, τ). Note that $A \subseteq [0, \infty) \in \tau$ and $Cl_{\delta_\omega}(A) = Cl(A) = \mathbf{R} - \{0\} \subseteq [0, \infty)$.*

In the following theorem we will show what the additional conditions that make the reversal of previous relationships is true.

Theorem 2.7. *Let (X, τ) be a space and A be an open subset of X. Then the following are equivalent:*

1. *$A \in \delta_\omega GC(X, \tau)$.*
2. *$A \in \delta GC(X, \tau)$.*
3. *$A \in GC(X, \tau)$.*

Proof. The proof follows from the Proposition 1.3. □

It is note that for an element $x \in X$, the set $X - \{x\}$ is $\delta_\omega g-$closed or open. To show that suppose $X - \{x\}$ is not open. Then the only open set containing $X - \{x\}$ is X. Hence $X - \{x\}$ is $\delta_\omega g-$closed set in X.

Theorem 2.8. *Let (X, τ) be an anti-locally countable space. Then $A \in \delta_\omega GC(X, \tau_{\delta_\omega})$ if and only if $A \in \delta_\omega^{\delta_\omega} GC(X, \tau)$.*

Proof. The proof follows immediately from Lemma 1.4. □

Theorem 2.9. *A space (X, τ) is a $T_{\frac{1}{2}}$-space if and only if every $\delta_\omega-$generlized closed set in (X, τ) is closed in (X, τ).*

Proof. Necessity. Let $A \subseteq X$ be $\delta_\omega-$generalized closed. Since (X, τ) is a $T_{\frac{1}{2}}$-space and every $\delta_\omega-$generalized closed set is $g-$closed so A is closed.

Sufficiency. Let $x \in X$. If $\{x\}$ is not closed, then $B = X - \{x\}$ is not open and thus the only superset of B is X. Trivially, B is $\delta_\omega-$generlized closed. By assumption, B is closed or, equivalently, $\{x\}$ is open. Thus, every singleton in (X, τ) is open or closed. Hence, (X, τ) is a $T_{\frac{1}{2}}$-space. □

Theorem 2.10. *Let (X, τ) be $T_{\frac{1}{2}}$ space. Then the collection of $\delta_\omega g-$closed in (X, τ) coincided with the collection of $\delta_\omega-$closed sets in (X, τ).*

Proof. Let $x \in Cl_{\delta_\omega}(A)$. Since (X, τ) is $T_{\frac{1}{2}}$, so either $\{x\}$ is open or closed in (X, τ). If $\{x\}$ is open, then $x \in A$. Now if $\{x\}$ is closed in (X, τ), then $X - \{x\} \in \tau_{\delta_\omega}$. Suppose that $x \notin A$. Then $A \subseteq X - \{x\}$. As A is $\delta_\omega g-$closed in (X, τ) so $Cl_{\delta_\omega}(A) \subseteq X - \{x\}$, which contradicts the assumption. Therefore, A is $\delta_\omega-$closed set in (X, τ). □

Proposition 2.11. *If every $\{x\}$ is $\delta_\omega-$ closed in (X, τ) or $R\omega O(X)$ then every $\delta_\omega g-$closed in (X, τ) is closed in (X, τ).*

Proof. Let $A \in \delta_\omega GC(X, \tau)$ and suppose that $x \in Cl(A)$. If $\{x\} \in R\omega O(X)$, then $x \in A$. Suppose that $\{x\}$ is $\delta_\omega-$ closed in (X, τ), and $x \notin A$. Then $x \in Cl(A) - A \subseteq Cl_{\delta_\omega}(A) - A$, which is impossible. Thus $Cl(A) = A$. □

Proposition 2.12. *If $A \in \delta_\omega GC(X, \tau)$, then $Cl_{\delta_\omega}(A) - A$ does not contain a nonempty closed set in (X, τ).*

Proof. Assume that F is a closed subset of (X, τ) such that $F \subseteq Cl_{\delta_\omega}(A) - A$ and so $A \subseteq (X - F) \in \tau$. Thus $Cl_{\delta_\omega}(A) \subseteq (X - F)$. It follows that $F \subseteq X - Cl_{\delta_\omega}(A)$. Therefore, $F \subseteq (X - Cl_{\delta_\omega}(A)) \cap (Cl_{\delta_\omega}(A))$. Thus $F = \phi$. □

Corollary 2.13. *If $A \in \delta_\omega GC(X, \tau)$ and $Cl_{\delta_\omega}(A) - A$ is closed then A is $\delta_\omega-$closed.*

Proof. Since $Cl_{\delta_\omega}(A) - A$ is closed and $Cl_{\delta_\omega}(A) - A \subseteq Cl_{\delta_\omega}(A) - A$ so by Proposition 2.12, $Cl_{\delta_\omega}(A) - A = \phi$. Thus $Cl_{\delta_\omega}(A) = A$ and so A is $\delta_\omega-$closed. □

Theorem 2.14. *Let (X, τ) be an antilocally countable space. Then (X, τ) is a T_1-space if and only if every $\delta_\omega g-$closed set in (X, τ) is $\delta\omega-$closed in (X, τ).*

Proof. Necessity. Let $A \subseteq X$ be $\delta_\omega-$generalized closed and let $x \in Cl_{\delta\omega}(A)$. Since (X, τ) is T_1, then $\{x\}$ is closed and thus by Proposition 2.12, $x \notin Cl_{\delta_\omega}(A) - A$. Since $x \in Cl_{\delta_\omega}(A)$, then $x \in A$. This show that A is $\delta_\omega-$closed set in (X, τ).

Sufficiency. Let $x \in X$ and suppose that $\{x\}$ is not closed. Then $B = X - \{x\}$ is not open, and thus B is $\delta_\omega g-$closed in (X, τ). Therefore, by assumption, B is $\delta_\omega-$closed, and thus $\{x\}$ is $\delta_\omega-$open. So there exists $U \in \tau$ such that $x \in U \subseteq Int(Cl_\omega(U)) \subseteq \{x\}$. It follows that U ia a nonempty countable open subset of (X, τ), a contradiction. □

The proof of the following lemma is clear.

Lemma 2.15. *Let (X, τ) be any space such that (X, τ_δ) is a T_1-space. Then A is $\delta_\omega-$closed in (X, τ) if and only if $A \in \delta_\omega^{\delta_\omega} GC(X, \tau)$.*

3. Some properties of $\delta_\omega-$generalized closed sets

In this section we introduce and study some of topological properties of $\delta_\omega-$generalized closed sets.

From the definition of $\delta_\omega-$generalized closed sets we can get the following definition.

Definition 3.1. *A subset A of a space (X, τ) is called $\delta_\omega-$generalized open(briefly $\delta_\omega g-$open) if its complement $X - A$ is $\delta_\omega-$generalized closed.*

Proposition 3.2. *A subset A of a space (X, τ) is $\delta_\omega-$generalized open if and only if $F \subseteq Int_{\delta_\omega}(A)$, whenever $F \subseteq A$ and F is closed in (X, τ).*

Proof. The proof follows immediately from the definition. □

Theorem 3.3. *Let A be $\delta_\omega-$generalized closed in a space (X, τ). If $B \subseteq X$ such that $A \subseteq B \subseteq Cl_{\delta_\omega}(A)$, then $B \in \delta_\omega GC(X, \tau)$.*

Proof. Let $U \in \tau$ such that $B \subseteq U$. Then $A \subseteq B \subseteq U$. Since $A \in \delta_\omega GC(X,\tau)$, $Cl_{\delta_\omega}(A) \subseteq Cl_{\delta_\omega}(B) \subseteq Cl_{\delta_\omega}(Cl_{\delta_\omega}(A)) = Cl_{\delta_\omega}(A) \subseteq U$. Hence $B \in \delta_\omega GC(X,\tau)$. □

The following Theorem and example show that the finite union of $\delta_\omega-$generalized closed sets is $\delta_\omega-$generalized closed but the countable union of $\delta_\omega-$generalized closed sets need not be $\delta_\omega-$generalized closed.

Theorem 3.4. *Union of two $\delta_\omega-$generalized closed sets in (X,τ) is $\delta_\omega-$generalized closed set.*

Proof. Let A and B be two $\delta_\omega-$generalized closed sets in a space (X,τ). Let $U \in \tau$ such that $A \cup B \subseteq U$. Then $A \subseteq U$ and $B \subseteq U$. Since $A, B \in \delta_\omega GC(X,\tau)$, $Cl_{\delta_\omega}(A \cup B) = Cl_{\delta_\omega}(A) \cup Cl_{\delta_\omega}(B) \subseteq U$. Thus $A \cup B$ is $\delta_\omega-$generalized closed. □

Corollary 3.5.
1. *Finite union of $\delta_\omega-$generalized closed sets in (X,τ) is $\delta_\omega-$generalized closed set.*
2. *Finite intersection of $\delta_\omega-$generalized open sets in (X,τ) is $\delta_\omega-$generalized open set.*

To show that a countable union of $\delta_\omega-$generalized closed sets need not be $\delta_\omega-$generalized closed set we consider the following example.

Example 3.6. *Let $X = \mathbf{R}$ with the usual topology. For each $n \in \mathbf{N}$, put $A_n = [\frac{1}{n}, 1]$ and $A = \bigcup_{n\in\mathbf{N}} A_n$. Then for each $n \in \mathbf{N}, A_n \in \delta_\omega GC(X,\tau)$, so A is a countable union of $\delta_\omega-$generalized closed sets but $A \notin \delta_\omega GC(X,\tau)$ since $U = (0,5) \in \tau, A \subseteq U$ and $Cl_{\delta_\omega}(A) \subseteq U$. Note that $0 \in Cl_{\delta_\omega}(A)$ but $0 \notin U$.*

The following example shows that the intersection of two $\delta_\omega-$generalized closed sets in (X,τ) may fail to be $\delta_\omega-$generalized closed set.

Example 3.7. *Let $X = \{1,2,3,4,5\}$ and $\tau = \{\phi, X, \{1,2\}, \{3\}, \{1,2,3\}\}$. Set $A = \{1,3,4\}$ and $B = \{2,3,5\}$. It is easily to proof A and B are two $\delta_\omega-$generalized closed sets. But $A \cap B = \{3\} \notin \delta_\omega GC(X,\tau)$.*

Theorem 3.8. *The intersection of a $\delta_\omega-$generalized closed set in (X,τ) and $\delta_\omega-$closed is always $\delta_\omega-$generalized closed.*

Proof. Let $A \in \delta_\omega GC(X,\tau)$ and B be $\delta_\omega-$closed in (X,τ). Let U be an open set in (X,τ) such that $A \cap B \subseteq U$. Then $A \subseteq U \cup (X-B) \in \tau$. Since $A \in \delta_\omega GC(X,\tau)$, $Cl_{\delta_\omega}(A) \subseteq U \cup (X-B)$. Now, $Cl_{\delta_\omega}(A \cap B) \subseteq Cl_{\delta_\omega}(A) \cap Cl_{\delta_\omega}(B) = Cl_{\delta_\omega}(A) \cap B \subseteq (U \cup (X-B)) \cap B \subseteq U$. Hence $A \cap B \in \delta_\omega GC(X,\tau)$. □

Let $\{(X_\alpha, \tau_\alpha) : \alpha \in \Delta\}$ be a collection of topological spaces such that $X_\alpha \cap X_\beta = \phi$ for each $\alpha = \beta$. Let $X = \underset{\alpha\in\Delta}{\cup} X_\alpha$ be topologized by $\tau_s = \{G \subseteq X : G \cap X_\alpha \in \tau_\alpha$ for each $\alpha \in \Delta\}$. Then (X, τ_s) is called the sum of the spaces $\{(X_\alpha, \tau_\alpha) : \alpha \in \Delta\}$ and we write $X = \underset{\alpha\in\Delta}{\oplus} X_\alpha$.

Theorem 3.9. *[1] For any collection of spaces $\{(X_\alpha, \tau_\alpha) : \alpha \in \Delta\}$, we have $(\tau_s)_{\delta_\omega} = (\tau_{\alpha_{\delta_\omega}})_s$.*

Theorem 3.10. *Let $\{(X_\alpha, \tau_\alpha) : \alpha \in \Delta\}$ be a collection of spaces and $A = \underset{\alpha\in\Delta}{\cup} A_\alpha$ such that $A_{\alpha_\circ} \subseteq X_{\alpha_\circ}$, then:*

1. *$Cl_{(\tau_{\alpha_\circ})_{\delta_\omega}}(A_{\alpha_\circ}) = Cl_{(\tau_s)_{\delta_\omega}}(A_{\alpha_\circ})$.*
2. *$\underset{\alpha\in\Delta}{\cup} Cl_{(\tau_\alpha)_{\delta_\omega}}(A_\alpha) = Cl_{(\tau_s)_{\delta_\omega}}(A)$.*

Proof. (1) Let $x \in Cl_{(\tau_{\alpha_\circ})_{\delta_\omega}}(A_{\alpha_\circ})$ and let $W \in (\tau_s)_{\delta_\omega}$ such that $x \in W$. Then by Theorem 3.9, $W \in (\tau_{\alpha_{\delta_\omega}})_s$, i.e. $W \cap X_{\alpha_\circ} \in \tau_{(\alpha_\circ)_{\delta_\omega}}$ and so $\phi = W \cap X_{\alpha_\circ} \cap A_{\alpha_\circ} = W \cap A_{\alpha_\circ}$. Therefore, $x \in Cl_{(\tau_s)_{\delta_\omega}}(A_{\alpha_\circ})$. Conversely, let $x \in Cl_{(\tau_s)_{\delta_\omega}}(A_{\alpha_\circ})$ and let $W \in (\tau_{\alpha_\circ})_{\delta_\omega}$ such that $x \in W$. So for each $\alpha = \alpha_\circ$, $W \cap X_\alpha = \phi$ and so by Theorem 3.9, $W \in (\tau_{\alpha_{\delta_\omega}})_s = (\tau_s)_{\delta_\omega}$. Therefore, $W \cap A_{\alpha_\circ} = \phi$. Thus $x \in Cl_{(\tau_{\alpha_\circ})_{\delta_\omega}}(A_{\alpha_\circ})$.
(2) Since $(\tau_s)_{\delta_\omega}$ is a topology on X , so $\underset{\alpha\in\Delta}{\cup} Cl_{(\tau_\alpha)_{\delta_\omega}}(A_\alpha) = \underset{\alpha\in\Delta}{\cup} Cl_{(\tau_s)_{\delta_\omega}}(A_\alpha) \subseteq Cl_{(\tau_s)_{\delta_\omega}}(\underset{\alpha\in\Delta}{\cup} A_\alpha) = Cl_{(\tau_s)_{\delta_\omega}}(A)$. Conversely, let $x \in Cl_{(\tau_s)_{\delta_\omega}}(A)$. Then there exists $\alpha_\circ \in \Delta$ such that $x \in A_{\alpha_\circ}$. Let $W \in (\tau_{\alpha_\circ})_{\delta_\omega}$ such that $x \in W$. Then by Theorem 3.9, $W \in (\tau_s)_{\delta_\omega}$ and since $x \in Cl_{(\tau_s)_{\delta_\omega}}(A)$, $\phi = A \cap W = (\underset{\alpha\in\Delta}{\cup} A_\alpha)' \cap W = W \cap A_{\alpha_\circ}$. Thus $x \in Cl_{(\tau_{\alpha_\circ})_{\delta_\omega}}(A_{\alpha_\circ}) \subseteq \underset{\alpha\in\Delta}{\cup} Cl_{(\tau_\alpha)_{\delta_\omega}}(A_\alpha)$.
□

Theorem 3.11. *Let $\{(X_\alpha, \tau_\alpha) : \alpha \in \Delta\}$ be a collection of spaces and $A = \underset{\alpha\in\Delta}{\cup} A_\alpha$ such that $A_\alpha \subseteq X_\alpha$ for each $\alpha \in \Delta$. Then A_α is $\delta_\omega-$generalized closed in (X, τ_α) for each $\alpha \in \Delta$ if and only if A is $\delta_\omega-$generalized closed in (X, τ_s).*

Proof. Let $W \in \tau_s$ such that $A \subseteq W$. For each $\alpha \in \Delta$, $A_\alpha = A \cap X_\alpha \subseteq W \cap X_\alpha$ and $W \cap X_\alpha \in \tau_\alpha$. Since A_α is $\delta_\omega-$generalized closed in (X, τ_α), $Cl_{(\tau_\alpha)_{\delta_\omega}}(A_\alpha) \subseteq W \cap X_\alpha$. Hence, $Cl_{(\tau_s)_{\delta_\omega}}(A) = \cup Cl_{(\tau_\alpha)_{\delta_\omega}}(A_\alpha) \subseteq \cup(W \cap X_\alpha) = W$. Therefore, A is $\delta_\omega-$ generalized closed in (X, τ_s) . Conversely, Fix $\alpha_\circ \in \Delta$ and let $W_\circ \in \tau_{\alpha_\circ}$ such that $A_{\alpha_\circ} \subseteq W_\circ$. Then $W = W_\circ \cup (\bigcup\limits_{\alpha = \alpha_\circ} X_\alpha)$ is an open set in (X, τ_s) such that $A \subseteq W$. Then $Cl_{(\tau_s)_{\delta_\omega}}(A) \subseteq W$. By Theorem 3.10, $\cup Cl_{(\tau_\alpha)_{\delta_\omega}}(A_\alpha) \subseteq W$. To show that $Cl_{(\tau_{\alpha_\circ})_{\delta_\omega}}(A_{\alpha_\circ}) \subseteq W_\circ$, let $x_\circ \in Cl_{(\tau_{\alpha_\circ})_{\delta_\omega}}(A_{\alpha_\circ})$. Then $x_\circ \in W$. Since $x_\circ \in X_{\alpha_\circ}$, then $x_\circ \notin X_\alpha$ for all $\alpha = \alpha_\circ$ and so $x_\circ \in W_\circ$. Therefore, $Cl_{(\tau_{\alpha_\circ})_{\delta_\omega}}(A_{\alpha_\circ}) \subseteq W_\circ$. Thus $A_{\alpha_\circ}$ is $\delta_\omega-$ generalized closed in $(X_{\alpha_\circ}, \tau_{\alpha_\circ})$. $\square$

Theorem 3.12. *Let $\{A_\alpha\}_{\alpha \in \Delta}$ be locally finite family in $(X, \tau_{\delta_\omega})$ such that $A_\alpha \in \delta_\omega GC(X, \tau)$ for each $\alpha \in \Delta$. Then $A = \bigcup\limits_{\alpha \in \Delta} A_\alpha \in \delta_\omega GC(X, \tau)$.*

Proof. Let $A \subseteq U$ and U be an open set in (X, τ). Since $A_\alpha \in \delta_\omega GC(X, \tau)$ and $A_\alpha \subseteq A \subseteq U$ for each $\alpha \in \Delta$, $Cl_{\delta_\omega}(A_\alpha) \subseteq U$. Since $\{A_\alpha\}_{\alpha \in \Delta}$ is locally finite in $(X, \tau_{\delta_\omega})$, by using Theorem 1.7, $Cl_{\delta_\omega} \bigcup\limits_{\alpha \in \Delta}(\cup A_\alpha) = \bigcup\limits_{\alpha \in \Delta} Cl_{\delta_\omega}(A_\alpha) \subseteq \bigcup\limits_{\alpha \in \Delta} U = U$. Therefore, $A = \bigcup\limits_{\alpha \in \Delta} A_\alpha \in \delta_\omega GC(X, \tau)$. $\square$

Theorem 3.13. *For a topological space (X, τ) the following conditions are equivalent:*

1. *The open sets in (X, τ) are clopen in (X, τ).*
2. *If $A \subseteq X$, then $A \in \delta_\omega GC(X, \tau)$.*

Proof. $(1 \to 2)$ Let $A \subseteq U$, where $U \in \tau$. Then U is clopen in (X, τ) and so it is clopen in $(X, \tau_{\delta_\omega})$ by Proposition 1.1. Therefore, $Cl_{\delta_\omega} A \subseteq Cl_{\delta_\omega} U = U$.
$(2 \to 1)$ Let $U \subseteq X$ be open. Since $U \in \delta_\omega GC(X, \tau)$, $Cl_{\delta_\omega} U \subseteq U$, so $Cl_{\delta_\omega} U = U$. Thus U is $\delta_\omega-$closed and so it is closed. $\square$

The following is nontrivial example that demonstrate the above theorem.

Example 3.14. *Let $X = \mathbf{R}$ with topology $\tau = \{\phi, X, \mathbf{Q}, \mathbf{R} - \mathbf{Q}\}$. Then $\tau_{\delta_\omega} = \tau$. For any $A \subseteq X$, $A = H \cup L$ where $H \subseteq \mathbf{Q}$ and $L \subseteq \mathbf{R} - \mathbf{Q}$. If $H = \phi$ and $L = \phi$, then the only open set containing A is X and so $A \in \delta_\omega GC(X, \tau)$. Now, suppose $L = \phi$ and let $U \in \tau$ such that $A \subseteq U$.*

Then $U = \mathbf{Q}$ or $U = X$. For $U = \mathbf{Q}$, the $Cl_{\delta_\omega}(A) = Cl(A) = \mathbf{Q} \subseteq \mathbf{Q}$. Therefore, $A \in \delta_\omega GC(X,\tau)$. By the same way, we show that if $H = \phi$, then $A \in \delta_\omega GC(X,\tau)$.

Proposition 3.15. *[1] Let (X,τ) be a topological space. If $A \in \tau$, then $(\tau_{\delta_\omega})_A = (\tau_A)_{\delta_\omega}$.*

The relationship between the $\delta_\omega-$generalized closed sets and the $\delta_\omega-$generalized closed sets of the subspace will be given in the next theorem.

Theorem 3.16. *Let (Y,τ_Y) be an open subspace of a space (X,τ) and $A \subseteq Y$. Then the following hold:*

1. *If $A \in \delta_\omega GC(X,\tau)$, then $A \in \delta_\omega GC(Y,\tau_Y)$.*
2. *If $A \in \delta_\omega GC(Y,\tau_Y)$ and Y is $\delta_\omega-$closed, then $A \in \delta_\omega GC(X,\tau)$.*

Proof. (1) Let $V \in \tau_Y$ such that $A \subseteq V$. Then $V = U \cap Y$ for some $U \in \tau$. Since $A \in \delta_\omega GC(X,\tau)$ and $A \subseteq U$, $Cl_{\delta_\omega}(A) \subseteq U$. It follows that $Cl_{\delta_\omega}(A) \cap Y \subseteq U \cap Y = V$. Since Y is open in X so, by Proposition 3.15, $Cl_{(\tau_Y)_{\delta_\omega}}(A) = Cl_{(\delta_\omega)_{\tau_Y}}(A) = Cl_{\delta_\omega}(A) \cap Y \subseteq V$. Thus $A \in \delta_\omega GC(Y,\tau_Y)$.
(2) Let U be an open set in (X,τ) such that $A \subseteq U$. Then $A \subseteq U \cap Y \in \tau_Y$. Since $A \in \delta_\omega GC(Y,\tau_Y)$ and Y is open in (X,τ), by Proposition 3.15, $Cl_{(\tau_Y)_{\delta_\omega}}(A) = Cl_{(\delta_\omega)_{\tau_Y}}(A) = Cl_{\delta_\omega}(A) \cap Y \subseteq U \cap Y$. As Y is $\delta_\omega-$closed in (X,τ) so $Cl_{\delta_\omega}(A) = Cl_{\delta_\omega}(A \cap Y) \subseteq Cl_{\delta_\omega}(A) \cap Cl_{\delta_\omega}(Y) = Cl_{\delta_\omega}(A) \cap Y \subseteq U \cap Y \subseteq U$. Therefore, $A \in \delta_\omega GC(X,\tau)$. □

The condition that Y is $\delta_\omega-$closed in (X,τ) in Theorem 3.16, can not be dropped as we see in the following example.

Example 3.17. *Let X be an uncountable set and let A be a subset of X such that A and $X - A$ are uncountable. Let $\tau = \{\phi, A, X\}$. If $Y = A$, then $A \in \delta_\omega GC(Y,\tau_Y) - \delta_\omega GC(X,\tau)$.*

Theorem 3.18. *Let (X,τ) and (Y,σ) be two spaces. If $A \times B$ is a $\delta_\omega g-$open subset of $(X \times Y, \tau \times \sigma)$, then A is a $\delta_\omega g-$open set in (X,τ) and B is a $\delta_\omega g-$open in (Y,σ).*

Proof. Let F_A be a closed subset of (X, τ) and let F_B be a closed subset of (Y, σ) such that $F_A \subseteq A$ and $F_B \subseteq B$. Then $F_A \times F_B$ is closed in $(X \times Y, \tau \times \sigma)$ such that $F_A \times F_B \subseteq A \times B$. By assumption, $A \times B \in \delta_\omega GC(X \times Y, \tau \times \sigma)$ and so $F_A \times F_B \subseteq Int_{(\tau\times\sigma)_{\delta_\omega}}(A \times B) \subseteq Int_{\tau_{\delta_\omega}}(A) \times Int_{\sigma_{\delta_\omega}}(B)$ by using Theorem 1.5. Therefore, $F_A \subseteq Int_{\tau_{\delta_\omega}}(A)$ and $F_B \subseteq Int_{\sigma_{\delta_\omega}}(B)$. Thus $A \in \delta_\omega g-$open set in (X, τ) and $B \in \delta_\omega g-$open set in (Y, σ). □

Theorem 3.19. *Let (X, τ) be a normal space. If $F \cap A = \phi$, where F is closed and $A \in \delta_\omega GC(X, \tau)$, then there exist two disjoint open sets G and H in (X, τ) such that $F \subseteq G$ and $A \subseteq H$.*

Proof. Suppose that(X, τ) is normal and $F \cap A = \phi$. Then, $A \subseteq X - F$ and $X - F$ is open. Since $A \in \delta_\omega GC(X, \tau)$, $Cl_{\delta_\omega}(A) \subseteq X - F$. That is $Cl_{\delta_\omega}(A) \cap F = \phi$, this implies that $Cl_{\delta_\omega}(A)$ and F are disjoint closed sets in the normal space (X, τ). Then there exist disjoint open sets G and H such that $F \subseteq G$ and $A \subseteq Cl_{\delta_\omega}(A) \subseteq H$. □

4. $\delta_\omega g-$continuous functions and $\delta_\omega g-$irresolute functions

In this section we introduce $\delta_\omega g-$continuity and $\delta_\omega g-$irresoluteness by using $\delta_\omega g-$closed sets and we study some of their fundamental properties.

Definition 4.1. *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be*

1. *$\delta_\omega g-$continuous if $f^{-1}(V)$ is $\delta_\omega g-$closed in (X, τ) for every closed set V of (Y, σ).*
2. *$\delta_\omega g-$irresolute if $f^{-1}(V)$ is $\delta_\omega g-$closed in (X, τ) for every $\delta_\omega g-$closed set V of (Y, σ).*

It follows from the definitions that a function $f : (X, \tau) \to (Y, \sigma)$ is $\delta_\omega g-$continuous ($\delta_\omega g-$irresolute) if and only if $f^{-1}(V)$ is $\delta_\omega g-$open in (X, τ) for every open ($\delta_\omega g-$open) subset V of (Y, σ).

The following two examples show that for any function $f : (X, \tau) \to (Y, \sigma)$, the $\delta_\omega g-$continuous and $\delta_\omega g-$irresolute are independent notions.

Example 4.2. *Let $X = \{1, 2, 3\}$ with the topologies $\tau = \{\phi, X, \{1\}, \{3\}, \{1, 3\}\}$ and $\sigma = \{\phi, X, \{1\}\}$. Let $f : (X, \tau) \to (X, \sigma)$ be the function defined by*

$$f(x)=\begin{cases} 1 & ,x=1,3 \\ 2 & ,x=2 \end{cases}.$$

Then f is $\delta_\omega g-$continuous but it is not $\delta_\omega g-$irresolute.

Example 4.3. *Let (X,τ) and (Y,σ) be spaces that defined in Example 4.2 and define the function $f:(X,\tau)\rightarrow(Y,\sigma)$ as*

$f(x)=\begin{cases} 1 & ,x=2 \\ 2 & ,x=1,3 \end{cases}$. *Then f is $\delta_\omega g-$irresolute but it is not $\delta_\omega g-$continuous.*

Theorem 4.4. *Let $f:(X,\tau)\rightarrow(Y,\sigma)$ be a $\delta_\omega g-$continuous. Then f is $\delta_\omega g-$irresolute, if one of the following holds:*

1. *f is bijective and open function.*
2. *f is closed function.*

Proof. (1) Let $V\in\delta_\omega GC(Y,\sigma)$ and let $U\in\tau$ such that $f^{-1}(V)\subseteq U$. Clearly $V\subseteq f(U)$. Since $f(U)\in\sigma$, and $V\in\delta_\omega GC(Y,\sigma)$, then $Cl_{\delta_\omega}(V)\subseteq f(U)$ and $f^{-1}(Cl_{\delta_\omega}(V))\subseteq U$. Since f is $\delta_\omega g-$ continuous and $Cl_{\delta_\omega}(V)$ is a closed subset of Y, then $Cl_{\delta_\omega}(f^{-1}(Cl_{\delta_\omega}(V)))\subseteq U$ and $Cl_{\delta_\omega}(f^{-1}(V))\subseteq U$. Thus, f is $\delta_\omega g-$irresolute.
(2) Let A be a $\delta_\omega g-$open subset of (Y,σ) and $C\subseteq f^{-1}(A)$, where C is a closed set in (X,τ). Then $f(C)$ is closed in (Y,σ) such that $f(C)\subseteq A$. Since A is $\delta_\omega g-$open in (Y,σ), $f(C)\subseteq Int_{\sigma_{\delta_\omega}}(A)$ and thus $C\subseteq f^{-1}(Int_{\sigma_{\delta_\omega}}(A))$. Since f is $\delta_\omega g-$continuous and $Int_{\sigma_{\delta_\omega}}(A)$ is open in (Y,σ), then $f^{-1}(Int_{\sigma_{\delta_\omega}}(A))$ is $\delta_\omega g-$open in (X,τ). Since $C\subseteq Int_{\tau_{\delta_\omega}}(f^{-1}(Int_{\sigma_{\delta_\omega}}(A)))\subseteq Int_{\tau_{\delta_\omega}}(f^{-1}(A))$, then f is $\delta_\omega g-$irresolute. □
Example 4.2 shows that the condition that f is bijective in part (1) and closed in part (2) in Theorem 4.4 can not be dropped.

Corollary 4.5. *Under the same assumptions of Theorem 4.4 part* (1), *If (X,τ) is $T_{\frac{1}{2}}$, then (Y,σ) is $T_{\frac{1}{2}}$.*

Proof. Let $V\in\delta_\omega GC(Y,\sigma)$. Since f is $\delta_\omega g-$irresolute, then $f^{-1}(V)\in\delta_\omega GC(X,\tau)$. But (X,τ) is $T_{\frac{1}{2}}$, therefore, by Theorem 2.9, $f^{-1}(V)$ is closed in (X,τ). Thus $f(f^{-1}(V))=V$ is closed in (Y,σ) since f is bijective and open function. □

Let $f:(X,\tau)\rightarrow(Y,\sigma)$ be a function. Then a function $f^{\delta_\omega}_{\delta_\omega}:(X,\tau_{\delta_\omega})\rightarrow(Y,\sigma_{\delta_\omega})$ (resp., $f_{\delta_\omega}:(X,\tau_{\delta_\omega})\rightarrow(Y,\sigma)$, $f^{\delta_\omega}:(X,\tau)\rightarrow(Y,\sigma_{\delta_\omega})$) associated with f is defined as follows: $f_{\delta_\omega}^{\delta_\omega}(x) = f(x)$ (resp., $f_{\delta_\omega}(x) = f(x)$, $f^{\delta_\omega}(x) = f(x)$) for each $x \in X$.

The proof of the following results follow immediately from the Definition 2.1.

Theorem 4.6. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function:*

1. *If f_{δ_ω} is continuous, then $f_{\delta_\omega}^{\delta_\omega}$ is continuous.*
2. *If $f_{\delta_\omega}^{\delta_\omega}$ is continuous, then f^{δ_ω} is $\delta_\omega g-$continuous.*
3. *If f_{δ_ω} is continuous, then f is $\delta_\omega g-$continuous.*
4. *If f is $\delta_\omega g-$continuous, then f is $g-$continuous.*
5. *If f is $\delta_\omega g-$irresolute, then f^{δ_ω} is $\delta_\omega g-$continuous.*

The next examples will show the reverse implications are not necessarily true.

Example 4.7. (1) *Let $X = \mathbf{R}$ with the topologies $\tau = \{\phi, \mathbf{R}, \mathbf{R} - \mathbf{Q}\}$ and $\sigma = \{\phi, \mathbf{R}, \{0\}, [0, \infty)\}$. Let $f : (X, \tau) \to (Y, \sigma)$ and $g : (X, \tau) \to (X, \tau)$ be the identity functions. Then f^{δ_ω} is $\delta_\omega g-$continuous but $f_{\delta_\omega}^{\delta_\omega}$ is not continuous since $\tau_{\delta_\omega} = \tau_{ind}$. On the other hand $g_{\delta_\omega}^{\delta_\omega}$ is continuous but g_{δ_ω} is not continuous.*
(2) *Let $X = \mathbf{R}$, with the topologies $\tau = \{U : \mathbf{R} - \mathbf{Q} \subseteq U\} \cup \{\phi\}$ and $\sigma = \{\phi, X, \mathbf{R} - \{1\}\}$. Let $f : (X, \tau) \to (X, \sigma)$ be the identity function. One can easily check that $\tau_{\delta\omega} = \tau_{ind}$ and f is $g-$continuous but it is not $\delta_\omega g-$continuous.*
(3) *Let $X = \{1,\ 2,\ 3\}$, with the topologies $\tau = \{\phi, X, \{1,\ 2\}\}$ and $\sigma = \{\phi, X, \{2\}\}$. Let $f : (X, \tau) \to (X, \sigma)$ be the identity function. It is easily to observe that f is δ_ω $g-$continuous but f_{δ_ω} is not continuous.*
(4) *Consider the function given in Example 4.2. Then f^{δ_ω} is $\delta_\omega g-$continuous but f is not $\delta_\omega g-$irresolute.*

The proof of Theorems 4.8 and 4.10 follows directly from definitions and Theorem 2.10.

Theorem 4.8. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function such that (X, τ) is $T_{\frac{1}{2}}$.*

1. *If f is $\delta_\omega g-$continuous, then f_{δ_ω} is continuous.*
2. *If f is $\delta_\omega g-$irresolute, then $f_{\delta_\omega}^{\delta_\omega}$ is continuous.*

The following example shows that the condition that (X,τ) is $T_{\frac{1}{2}}$ in Theorem 4.8 (2) cannot be drop.

Example 4.9. *Let $X = \mathbf{R}$, with the topologies $\tau = \{\phi, \mathbf{R}, \mathbf{R} - \mathbf{Q}\}$ and $\sigma = \tau_{ind}$. Let $f : (X,\tau) \to (X,\sigma)$ be the identity function. Then $f_{\delta_\omega}^{\delta_\omega}$ is continuous but it is not $\delta_\omega g-$irresolute since $\{\sqrt{2}\}$ is $\delta_\omega-$gclosed in (X,σ) but $\{\sqrt{2}\} = f^{-1}(\{\sqrt{2}\}) \notin \delta_\omega GC(X,\tau)$.*

Theorem 4.10. *Let $f : (X,\tau) \to (Y,\sigma)$ be a function such that (Y,σ) is $T_{\frac{1}{2}}$.*

1. *If f is $\delta_\omega g-$continuous, then f is $\delta_\omega g-$irresolute.*
2. *If f is $\delta_\omega g-$continuous, then $f_{\delta_\omega}^{\delta_\omega}$ is continuous.*
3. *If $f_{\delta_\omega}^{\delta_\omega}$ is continuous, then f is $\delta_\omega g-$irresolute.*

The following example shows that the condition that (X,τ) is $T_{\frac{1}{2}}$ in Theorem 4.10 (3) is essential.

Example 4.11. *Let $X = \mathbf{R}$, with the topologies $\tau = \tau_{ind}$ and $\sigma = \{\phi, \mathbf{R}, \{0\}, [0,\infty)\}$. Let $f : (X,\tau) \to (X,\sigma)$ be the identity function. Then f is $\delta_\omega g-$irresolute but $f_{\delta_\omega}^{\delta_\omega}$ is not continuous. Note that $\{0\} \in \sigma_{\delta_\omega}$, but $\{0\} = f^{-1}(\{0\}) \notin \tau_{\delta_\omega}$.*

Theorem 4.12. *If $f : (X,\tau) \to (Y,\sigma)$ is $\delta_\omega g-$continuous, then for each $x \in X$ and each open set V in (Y,σ) with $f(x) \in V$, there exists a $\delta_\omega g-$open set U in (X,τ) such that $x \in U$ and $f(U) \subseteq V$.*

Proof. Let $x \in X$ and let V be any open set in (Y,σ) containing $f(x)$. Put $U = f^{-1}(V)$. Then, by assumption, U is a $\delta_\omega g-$open set in (X,τ) such that $x \in U$ and $f(U) \subseteq V$, and the result follows. □

The converse of the above theorem is not true in general as the following example shows.

Example 4.13. *Let $X = \mathbf{R}$, with the topologies $\tau = \{\phi, X, \mathbf{R} - \mathbf{Q}\}$ and $\sigma = \{\phi, X, \mathbf{R} - \{\sqrt{2}\}\}$. Let $f : (X, \tau) \to (X, \sigma)$ be the identity function. Then f is not $\delta_\omega g-$continuous. On the other hand, f satisfies the property stated in the above theorem because $x \in U$ is a $\delta_\omega g-$open set in (X, τ) for each $x \in X$.*

Next we offer the following composition theorem and the proof is clear.

Theorem 4.14. *Let $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \upsilon)$ be two functions. Then:*

1. *gof is $\delta_\omega g-$continuous, if g is continuous and f is $\delta_\omega g-$continuous.*
2. *gof is $\delta_\omega g-$irresolute, if g is $\delta_\omega g-$irresolute and f is $\delta_\omega g-$irresolute.*
3. *gof is $\delta_\omega g-$continuous, if g is $\delta_\omega g-$continuous and f is $\delta_\omega g-$irresolute.*
4. *Let (Y, σ) be $T_{\frac{1}{2}}$. Then gof is $\delta_\omega g-$continuous, if f and g are $\delta_\omega g-$continuous.*

The following example shows that the composition of two $\delta_\omega g-$continuous functions need not be $\delta_\omega g-$continuous.

Example 4.15. *Let $X = \mathbf{R}$ with the topologies $\tau = \{U : \mathbf{R} - \mathbf{Q} \subseteq U\} \cup \{\phi\}$, $\sigma = \{\phi, \mathbf{R}, \mathbf{R} - \mathbf{Q}\}$ and $\upsilon = \{\phi, X, \mathbf{R} - \{1\}\}$. Let $f : (X, \tau) \to (X, \sigma)$ and $g : (X, \sigma) \to (X, \upsilon)$ be the identity functions. Note that f and g are $\delta_\omega g-$ continuous, but the composition function gof:$(X, \tau) \to (X, \upsilon)$ is not $\delta_\omega g-$ continuous.*

Theorem 4.16. *Let $f : (X, \tau) \to (Y, \sigma)$ be continuous and suppose that $f_{\delta_\omega}^{\delta_\omega}$ is closed. If $A \in \delta_\omega GC(X, \tau)$, then $f(A) \in \delta_\omega GC(Y, \sigma)$.*

Proof. Let A be $\delta_\omega-$closed in (X, τ). Let $f(A) \subseteq O$, where O is open in (Y, σ). Therefore, $f^{-1}(O)$ is an open set in (X, τ) containing the $\delta_\omega g-$closed set A. Then $Cl_{\delta_\omega}(A) \subseteq f^{-1}(O)$. Thus $f(Cl_{\delta_\omega}(A)) \subseteq O$. Hence $Cl_{\delta_\omega}(f(A)) \subseteq Cl_{\delta_\omega}(f(Cl_{\delta_\omega}(A))) = f(Cl_{\delta_\omega}(A)) \subseteq O$, since $f_{\delta_\omega}^{\delta_\omega}$ is closed. Hence $f(A) \in \delta_\omega GC(Y, \sigma)$. $\square$

The following example shows that the assumption that f is continuous in the above theorem cannot be dropped.

Example 4.17. *Consider the function f as in Example 4.15. Note that $\tau_{\delta_\omega} = \sigma_{\delta_\omega} = \tau_{ind}$. Put $A = \{1\}$. One can easily check that $A \in \delta_\omega GC(X, \tau)$ but $f(A) \notin \delta_\omega GC(Y, \sigma)$.*

Regarding the restriction of a $\delta_\omega g-$continuous function, we have the following.

Theorem 4.18. *Let $f : (X, \tau) \to (Y, \sigma)$ be a $\delta_\omega g-$continuous function and let A be a $\delta_\omega-$closed and open subset of (X, τ). Then, the restriction $f|_A : (A, \tau_A) \to (Y, \sigma)$ is $\delta_\omega g-$continuous .*

Proof. Let F be a closed subset of (Y, σ). Then $(f|_A)^{-1}(F) = f^{-1}(F) \cap A$. Since f is $\delta_\omega g-$continuous, $f^{-1}(F) \in \delta_\omega GC(X, \tau)$ and so, by Theorem 3.8, $f^{-1}(F) \cap A \in \delta_\omega GC(X, \tau)$. Therefore, by Theorem 3.16, $(f|_A)^{-1}(F) \subseteq U \in \delta_\omega GC(A, \tau_A)$ and the result follows. □

The next example shows that we can not drop the condition on A in the previous theorem.

Example 4.19. *Let $X = \{1, 2, 3\}$ with the topology $\tau = \{\phi, X, \{1\}\}$ and let $Y = \{a, b\}$ with the topology $\sigma = \{\phi, Y, a\}$. Define $f : (X, \tau) \to (Y, \sigma)$ as*
$f(x) = \begin{cases} b & , x = 1, 3 \\ a & , x = 2 \end{cases}$ *. Put $A = \{1, 2\}$. Then f is $\delta_\omega g-$continuous but the restriction $f|_A$ is not $\delta_\omega g-$continuous since $(f|_A)^{-1}(\{b\})$ is not $\delta_\omega g-$closed in (X, τ).*

Theorem 4.20. *Let (X, τ) be a topological space such that $X = A \cup B$, where A and B are both open and $\delta_\omega-$closed in (X, τ). Let $f : (X, \tau) \to (Y, \sigma)$ be given such that the restrictions $f|_A$ and $f|_B$ are both $\delta_\omega g-$continuous. Then f is $\delta_\omega g-$continuous.*

Proof. Let F be a closed subset of (Y, σ). Then, $f^{-1}(F) = (f|_A)^{-1}(F) \cup (f|_B)^{-1}(F)$. Since $(f|_A)^{-1}(F) \in \delta_\omega GC(A, \tau_A)$ and A is open and $\delta_\omega-$closed in (X, τ), by Theorem 3.16, $(f|_A)^{-1}(F) \in \delta_\omega GC(X, \tau)$. Similarly, $(f|_B)^{-1}(F) \in \delta_\omega GC(X, \tau)$. By Theorem 3.12, $f^{-1}(F) \in \delta_\omega GC(X, \tau)$. Thus f is $\delta_\omega g-$continuous. □

Definition 4.21. *A subset A of a space (X, τ) is said to be $\delta_\omega g-$compact relative to X if for every collection $\{W_\alpha : \alpha \in \Delta\}$ of $\delta_\omega g-$open subsets of X such that $A \subseteq \cup\{W_\alpha : \alpha \in \Delta\}$ there exists a finite subset $\Delta_\circ$ of Δ such that $A \subseteq \cup\{W_\alpha : \alpha \in \Delta_\circ\}$.*

If $A = X$, then the space (X, τ) is called $\delta_\omega g-$compact. A subset A of a space (X, τ) is called $\delta_\omega g-$compact if the subspace (A, τ_A) is $\delta_\omega g-$compact.

Theorem 4.22. *For a space (X,τ) the following are equivalent:*

1. *(X,τ) is $\delta_\omega g-$compact.*
2. *If $\{F_\alpha : \alpha \in \Delta\}$ is a collection of $\delta_\omega g-$closed subsets of (X,τ) satisfying the finite intersection property, then $\cap\{F_\alpha : \alpha \in \Delta\} = \phi$.*
3. *If $\{F_\alpha : \alpha \in \Delta\}$ is a collection of $\delta_\omega g-$closed subsets of (X,τ) such that $\cap\{F_\alpha : \alpha \in \Delta\} = \phi$, then there exists a finite subset $\Delta_\circ$ of Δ such that $\cap\{F_\alpha : \alpha \in \Delta_\circ\} = \phi$.*

Proof. $(1 \to 2)$ Suppose that (X,τ) is $\delta_\omega g-$compact and let $\{F_\alpha : \alpha \in \Delta\}$ be a collection of $\delta_\omega g-$closed subsets of (X,τ) which satisfying the finite intersection property. Now, suppose by contrary that $\cap\{F_\alpha : \alpha \in \Delta\} = \phi$. Then, the collection $\{X - F_\alpha : \alpha \in \Delta\}$ is a $\delta_\omega g-$open cover of the $\delta_\omega g-$compact space (X,τ) and so there exists a finite subset $\Delta_\circ$ of Δ such that $X = \cup\{X - F_\alpha : \alpha \in \Delta_\circ\}$. Therefore, $\cap\{F_\alpha : \alpha \in \Delta_\circ\} = \phi$ which is a contradiction.
$(2 \to 3)$ Follows from the definition.
$(3 \to 1)$ Suppose by contrary that (X,τ) is not $\delta_\omega g-$compact. Then, there exists a $\delta_\omega g-$open cover $\{W_\alpha : \alpha \in \Delta\}$ of X which has no finite subcover. For each $\alpha \in \Delta$, put $F_\alpha = X - W_\alpha$. Then $\{F_\alpha : \alpha \in \Delta\}$ is a collection of $\delta_\omega g-$ closed subsets of (X,τ) such that $\cap\{F_\alpha : \alpha \in \Delta\} = \phi$ and so by (3), there exists a finite subset $\Delta_\circ$ of Δ such that $\cap\{F_\alpha : \alpha \in \Delta_\circ\} = \phi$. Thus $X = X - \cap\{F_\alpha : \alpha \in \Delta_\circ\} = \cup\{X - F_\alpha : \alpha \in \Delta_\circ\} = \{W_\alpha : \alpha \in \Delta_\circ\}$ which is also a contradiction. □

Proposition 4.23. *Every $\delta_\omega g-$closed subset of a $\delta_\omega g-$compact space (X,τ) is $\delta_\omega g-$compact relative to (X,τ).*

Proof. Let A be a $\delta_\omega g-$closed subset of a $\delta_\omega g-$ compact space (X,τ) and let $\{W_\alpha : \alpha \in \Delta\}$ be a $\delta_\omega g-$open cover of A in (X,τ). Then $\{W_\alpha : \alpha \in \Delta\} \cup \{X - A\}$ is a $\delta_\omega g-$open cover of the $\delta_\omega g-$compact space (X,τ) and so there exists a finite subset $\Delta_\circ$ of Δ such that $X = \cup\{W_\alpha : \alpha \in \Delta_\circ\}$. Thus A is $\delta_\omega g-$compact relative to (X,τ). □

The following two examples show that $\delta_\omega g-$compact and $\delta_\omega g-$compact relative are independent notions.

Example 4.24. *Let $X = A \cup B$ with the topology $\tau = \{U : U \subseteq A\} \cup \{U : A \subseteq U\}$ where A and B are uncountable disjoint sets. Then $\tau_{\delta_\omega} = \{U : U \subseteq A\} \cup \{X\}$ (see Example 4.13 [1]).*

(1) (X,τ) *is* $\delta_\omega g-$*compact space. Let* $x_\circ \in B$ *and let* H *be a* $\delta_\omega g-$*open subset of* (X,τ) *such that* $x_\circ \in H$. *Then* $H = K \cup L$ *where* $K \subseteq A$ *and* $L \subseteq B$. *Now,* L *is a closed subset of* (X,τ) *such that* $L \subseteq H$ *and so* $L \subseteq Int_{\delta_\omega}(H) = K$, *a contradiction. Therefore,* H *must be* X *and so* (X,τ) *is* $\delta_\omega g-$*compact.*
(2) *Put* $H = B$. *Note that the only open set containing* H *is* X *and so* H *is* $\delta_\omega g-$*closed set in* (X,τ). *On the other hand,* $\tau_H = \tau_{dis}$ *and so* (H,τ_H) *is not* $\delta_\omega g-$ *compact.*

Example 4.25. *Let* $X = A\cup B$ *with the topology* $\sigma = \{U : U \subseteq A\}\cup\{X\}$ *where* A *and* B *are uncountable sets such that* $A\cap B = \phi$. *Note that* $\tau_B = \tau_{ind}$ *and so* (B,τ_B) *is* $\delta_\omega g-$*compact. To show that* B *is not* $\delta_\omega g-$*compact relative to* X, *we show that for all* $x \in B$, $A \cup \{x\}$ *is* $\delta_\omega g-$*open set in* (X,τ). *Let* F *be a closed set in* (X,τ) *such that* $F \subseteq A \cup \{x\}$. *Therefore,* $X - F \subseteq A$ *and so* $B \subseteq F$. *This means that the only closed set in* (X,τ) *such that* $F \subseteq A\cup\{x\}$ *is* ϕ *and so* $A\cup\{x\}$ *is* $\delta_\omega g-$*open set in* (X,τ). *Now, it is clear that* B *is not* $\delta_\omega g-$*compact relative to* X *since* B *is uncountable.*

Proposition 4.26. *A* $\delta_\omega g-$*continuous image of a* $\delta_\omega g-$*compact space is compact.*

Proof. Let $f : (X,\tau) \to (Y,\sigma)$ be a $\delta_\omega g-$continuous function from a $\delta_\omega g-$compact space onto a space (Y,σ). Let $\{V_\alpha : \alpha \in \Delta\}$ be an open cover of the space (Y,σ). Since f is $\delta_\omega g-$continuous, then the collection $\{f^{-1}(V_\alpha) : \alpha \in \Delta\}$ is a $\delta_\omega g-$open cover of the $\delta_\omega g-$compact space (X,τ) and so there exists a finite subset $\Delta_\circ$ of Δ such that $X = \cup\{f^{-1}(V_\alpha) : \alpha \in \Delta_\circ\}$. Thus, $Y = f(X) = \cup\{f(f^{-1}(V_\alpha)) : \alpha \in \Delta_\circ\} = \cup\{V_\alpha : \alpha \in \Delta_\circ\}$. Which means that (Y,σ) is compact. □

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