Weakly convex hypersurfaces of pseudo-Euclidean spaces satisfying the condition

$$L_k H_{k+1} = \lambda H_{k+1}$$

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Received: 04 June 2019 | Accepted: 24 December 2020

Abstract:

In this paper, we try to give a classification of spacelike hypersurfaces of the Lorentz-Minkowski space-time E_1^{n+1} , whose mean curvature vector field of order $(k+1)$ is an eigenvector of the k th linearized operator L_k , for a non-negative integer k less than n . The operator L_k is defined as the linear part of the first variation of the $(k+1)$ th mean curvature of a hypersurface arising from its normal variations. We show that any spacelike hypersurface of E_1^{n+1} satisfying the condition $L_k H_{k+1} = \lambda H_{k+1}$ (where $0 \leq k \leq n-1$) belongs to the class of L_k -biharmonic, L_k -1-type or L_k -null-2-type hypersurface. Furthermore, we study the above condition on a well-known family of spacelike hypersurfaces of Lorentz-Minkowski spaces, named the weakly convex hypersurfaces (i.e. on which all of principle curvatures are nonnegative). We prove that, on any weakly convex spacelike hypersurface satisfying the above condition for an integer k (where, $0 \leq r \leq n-1$), the $(k+1)$ th mean curvature will be constant. As an interesting result, any weakly convex spacelike hypersurfaces, having assumed to be L_k -biharmonic, has to be k -maximal.

Keywords: Spacelike hypersurfaces; L_k -biharmonic; k -maximal; Weakly convex.

MSC (2020): 53-02, 53C40, 53C42.

Cite this article as (IEEE citation style):

F. Pashaie, "Weakly convex hypersurfaces of pseudo-Euclidean spaces satisfying the condition $L_k H_{k+1} = \lambda H_{k+1}$ ", *Proyecciones (Antofagasta, On line)*, vol. 40, no. 3, pp. 711-719, 2021, doi: 10.22199/issn.0717-6279-3584

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1. Introduction

The biharmonic functions as the solution of some well-known partial differential equations frequently appear in mathematical physics. Especially, when it becomes very difficult to find harmonic maps, the biharmonic ones are helpful. From geometric points of view, the role of biharmonic surfaces in elasticity and fluid mechanics can be considered as a physical motivation for the theory of biharmonicity. In nineteen eighties, B. Y. Chen has investigated the properties of biharmonic submanifolds in the Euclidean spaces (with position vector field $x : M^n \rightarrow E^{n+k}$ satisfying the condition $\Delta^2 x = 0$, where Δ is the Laplace operator). He introduced some open problems and conjectures in [5], among them, a longstanding conjecture says that every biharmonic submanifold in a Euclidean space is minimal. Chen himself has proved the conjecture for surfaces in E^3 . Later on, I. Dimitrić has verified Chen conjecture in several different cases such as special curves, submanifolds of constant mean curvature and also, hypersurfaces of the Euclidean spaces with at most two distinct principal curvatures. T. Hasanis and T. Vlachos [10] verified the conjecture for hypersurfaces in E^4 . Assuming the completeness, Akutagawa and Maeta [1] gave an affirmative answer to the global version of Chen's conjecture for biharmonic submanifolds in Euclidean spaces. Recently, in [8], it is proved that only biharmonic hypersurfaces in space forms with three distinct principal curvatures are minimal ones. An equivalent condition for the biharmonicity of an Euclidean hypersurfaces can be expressed as $\Delta H = 0$, where H is the ordinary mean curvature vector field on the hypersurface. In 1988, Chen has introduced a natural extension of this condition by assuming H to be an eigenvector of Δ , associated to an arbitrary constant real eigenvalue. In [7], Defever has proved that the hypersurfaces of E^4 satisfying the condition $\Delta H = \lambda H$ have constant mean curvature.

On the other hand, the Laplacian operator Δ can be seen as the first one of a sequence of n operators $L_0 = \Delta, L_1, \dots, L_{n-1}$, where L_k stands for the linearized operator of the first variation of the $(k+1)$ th mean curvature arising from normal variations of the hypersurface (see, for instance, [2]). These operators are given by $L_k(f) = \text{tr}(P_k \circ \nabla^2 f)$ for any $f \in C^\infty(M)$, where P_k denotes the k th Newton transformation associated to the second fundamental form of the hypersurface and $\nabla^2 f$ is the hessian of f . From this point of view, as an extension of finite type theory, S.M.B. Kashani [11] has introduced the notion of L_k -finite type hypersurface in the Euclidean space, which can be found in the second edition of Chen's book [6]. Fur-

thermore, in [3], it is proved that every L_k -biharmonic hypersurface in E^m (for arbitrary integer $m > 2$) with at most two distinct principal curvatures is k -minimal, $0 < k < m$.

Recently, in [8], it is proved that only biharmonic hypersurfaces with three distinct principal curvatures in E^5 are minimal ones. This result is generalized to L_1 -biharmonic hypersurfaces of E^5 in [14]. In [16], we have proved that every L_k -biharmonic spacelike hypersurface in E_1^4 with three distinct principal curvatures is k -maximal. In this paper, we try to classify all Riemannian hypersurfaces of pseudo-Euclidean space E_1^{n+1} whose $(k+1)$ th mean curvature vector field satisfies the condition $L_k H_{k+1} = \lambda H_{k+1}$. Also, we study this condition on a smaller category of such hypersurfaces named the weakly convex ones. Here is our main results:

Theorem 1.1. *Let M^n be a connected orientable spacelike hypersurface of the Lorentz-Minkowski spacetime E_1^{n+1} , isometrically immersed by $x : M^n \rightarrow E_1^{n+1}$. Then its $(k+1)$ th mean curvature vector field, H_{k+1} , is an eigenvector of L_k if and only if M^n satisfies at least one of the following properties:*

- (i) L_k -biharmonic,
- (ii) L_k -1-type,
- (iii) L_k -null-2-type.

Theorem 1.2. *Let $x : M^n \rightarrow E_1^{n+1}$ be an isometrically immersed spacelike hypersurface into the Lorentz-Minkowski space. x satisfies the condition $L_{n-1} H_n = \lambda H_n$, if and only if the n th mean curvature of M^n , H_n , is constant. Moreover, if $H_n \neq 0$ then $H_1 = \frac{-\lambda}{nH_n}$.*

Theorem 1.3. *Let $x : M^n \rightarrow E_1^{n+1}$ be a weakly convex spacelike hypersurface satisfying $L_k H_{k+1} = \lambda H_{k+1}$. Then, the $(n+1)$ th mean curvature of M^n , H_{k+1} , is constant.*

Theorem 1.4. *Assume that $x : M^n \rightarrow E_1^{n+1}$ is a weakly convex L_k -biharmonic hypersurface in E_1^{n+1} . Then M^n is k -maximal.*

2. Preliminaries

In this section, we recall some prerequisites from [4, 12, 15, 16]. By E_p^m , we mean the Euclidean space R^m with the scalar product $\langle x, y \rangle := -\sum_{i=1}^p x_i y_i + \sum_{j>p} x_j y_j$ (where, $0 \leq p < m$). Especially, $E_0^m = E^m$, and E_1^m is the Lorentz-Minkowski space of dimension m . Let $x : M^n \rightarrow E_1^{n+1}$

be an isometrically immersion of the spacelike hypersurface M^n into the Lorentz-Minkowski space E_1^{n+1} . The symbols ∇ and $\bar{\nabla}$ denote the Levi-Civita connections on M^n and E_1^{n+1} , respectively. The Weingarten formula is $\bar{\nabla}_V W = \nabla_V W - \langle SV, W \rangle N$, for every $V, W \in \chi(M)$. Here, $\chi(M)$ stands for the set of all vector fields on M^n and S is the shape operator of M^n associated to a timelike unit normal vector field N on M^n . Since M^n is spacelike, S can be diagonalized. We can choose a local orthonormal frame field $\{e_i\}_{1 \leq i \leq n+1}$ on M^n , where $e_1, \dots, e_n$ are eigenvectors of S and $e_{n+1} = N$ is the unit normal time-oriented vector field N . As usual, we denote the eigenvalues of S (the principal curvatures of M) by the functions $\lambda_1, \dots, \lambda_n$ on M . The elementary symmetric function is defined as $s_k := \sum_{1 \leq i_1 < \dots < i_k \leq n} \lambda_{i_1} \dots \lambda_{i_k}$, so, the k th mean curvature H_k of M is given by $\binom{n}{k} H_k = (-1)^k s_k$. The hypersurface M^n in E_1^{n+1} is called k -maximal, if its $(k+1)$ th mean curvature H_{k+1} is identically zero. A 0-maximal hypersurface is nothing but a maximal hypersurface in E_1^{n+1} .

The classical Newton transformations $P_k : \chi(M) \rightarrow \chi(M)$ are defined inductively by the shape operator S as $P_0 = I$ and $P_k = \binom{n}{k} H_k I + S \circ P_{k-1}$ for $k = 1, \dots, n$ where I denotes the identity transformation in $\chi(M)$. This map has an explicit formula as $P_k = \sum_{j=0}^k \binom{n}{k-j} H_{k-j} S^j$, which gives, by the Cayley-Hamilton theorem (stating that any operator is annihilated by its characteristic polynomial), that $P_n = 0$. Since at each point $p \in M^n$, the self-adjoint linear operators S and P_k commute and can be simultaneously diagonalized, then there exists a suitable basis $\{e_1, \dots, e_n\}$ on $T_p M$ consisting of the unit eigenvectors of $S(p)$ and $P_k(p)$, with corresponding eigenvalues $\lambda_1(p), \dots, \lambda_n(p)$ for $S(p)$. The eigenvalues of $P_k(p)$ with respect to the above basis are given by

$$\mu_{i,k}(p) = (-1)^k \sum_{1 \leq i_1 < \dots < i_k \leq n, i_j \neq i} \lambda_{i_1}(p) \cdots \lambda_{i_k}(p),$$

for $i = 1, \dots, n$. By a direct computation one can find the following useful equality

$$\lambda_i \mu_{i,j} = \mu_{i,j+1} - (-1)^{j+1} s_{j+1} = \mu_{i,j+1} - \binom{n}{j+1} H_{j+1}. \quad (2.1)$$

Also, there exist useful formulae for Newton transformations as follow [16]

$$\text{tr}(P_k) = c_k H_k, \quad \text{tr}(S \circ P_k) = -c_k H_{k+1},$$

$$\operatorname{tr}(S^2 \circ P_k) = \binom{n}{k+1} (nH_1H_{k+1} - (n-k-1)H_{k+2}),$$

where $c_k = (n-k)\binom{n}{k} = (k+1)\binom{n}{k+1}$. Now, we recall the operator $L_k : C^\infty(M) \rightarrow C^\infty(M)$ defined by

$$L_k(f) = \operatorname{tr}(P_k \circ \nabla^2 f),$$

where, $\nabla^2 f : \chi(M) \rightarrow \chi(M)$ denotes the self-adjoint linear operator metrically equivalent to the Hessian of f and is given by $\langle \nabla^2 f(X), Y \rangle = \langle \nabla_X(\nabla f), Y \rangle$, where $X, Y \in \chi(M)$, ∇f is the gradient of f and ∇ is the Levi-Civita connections on M . Also, for a vector field $F = (f_1, \dots, f_n)$ on a coordinate system in M , we define $L_k F := (L_k(f_1), \dots, L_k(f_n))$ as a vector field.

Now we recall the definition of an L_k -finite type hypersurface from [13], which is a basic notion in the paper.

Definition 2.1. A isometrically immersed hypersurface $x : M^n \rightarrow E_1^{n+1}$ is said to be of L_k -finite type if x has a finite decomposition $x = \sum_{i=0}^m x_i$, for a positive integer m satisfying the condition $L_k x_i = \kappa_i x_i$, where each $\kappa_i \in \mathbb{R}$ is a constant real number and each $x_i : M^n \rightarrow E_1^{n+1}$ is a smooth non-constant map for $i = 1, \dots, m$ and $x_0 \in E_1^{n+1}$ is a constant vector. If all κ_i 's are mutually different, then M^n is said to be of L_k - m -type. An L_k - m -type hypersurface is said to be null if κ_i is null for some i (where $1 \leq i \leq m$).

3. Spacelike hypersurfaces in pseudo-Euclidean spaces

Let $x : M^n \rightarrow E_1^{n+1}$ be a connected time-oriented spacelike hypersurface of the Lorentz-Minkowski spacetime, with a timelike Gauss map N . Taking $H_{k+1} := H_{k+1}N$, the well known equality $L_k x = c_k H_{k+1}$ (where $c_k = (n-k)\binom{n}{k} = (k+1)\binom{n}{k+1}$), gives that, for each non-negative $k < n$, any k -maximal hypersurface of E_1^{n+1} has to be L_k -biharmonic (use [2, 16]). By definition, a hypersurface M^n of E_1^{n+1} is said to be L_k -biharmonic if its position vector field x satisfies the condition $L_k^2 x = 0$, or equivalently, $L_k(H_{k+1}) = 0$ (see [3, 16]). Clearly, k -maximal immersions are L_k -biharmonic. Naturally, the L_k -biharmonicity condition can be extended by considering as

$$(3.1) \quad L_k H_{k+1} = \lambda H_{k+1},$$

for a constant real coefficient λ . Theorem 1.1 classifies hypersurfaces of E_1^{n+1} satisfying (3.1).

Proof of Theorem 1.1. Condition (3.1) can be considered in two different cases depending on λ is null or not. If $\lambda = 0$, then M^n is a L_k -biharmonic hypersurface, which gives (i). Now, assume that $\lambda \neq 0$. Taking $x_p = \frac{1}{\lambda}L_k x$ and $x_0 = x - x_p$, we find

$$L_k x_p = \frac{1}{\lambda} L_k^2 x = \frac{c_k}{\lambda} L_k H_{k+1} = c_k H_{k+1} = L_k x.$$

Hence, M is either of L_k -1-type or of L_k -null-2-type, depending on x_0 is a constant or non-constant. The converse is easy to verify. $\square$

Using the formulae computed in [2, 16], we have

$$(3.2) \quad \begin{aligned} L_k H_{k+1} &= 3 \binom{n}{k+1} H_{k+1} \nabla H_{k+1} - 2 P_{k+1} \nabla H_{k+1} \\ &+ \left(L_k H_{k+1} - H_{k+1} (n \binom{n}{k+1} H_1 H_{k+1} + c_{k+1} H_{k+2}) \right) \mathbf{N}. \end{aligned}$$

By Using (3.2), one obtains necessary and sufficient conditions for the $(k+1)$ th mean curvature vector field H_{k+1} being an eigenvector of L_k , namely

$$(3.3) \quad \begin{aligned} (i) \quad P_{k+1}(\nabla H_{k+1}) &= \frac{3}{2} \binom{n}{k+1} H_{k+1} \nabla H_{k+1}, \\ (ii) \quad L_k H_{k+1} &= H_{k+1} (\lambda + n \binom{n}{k+1} H_1 H_{k+1} + c_{k+1} H_{k+2}) \end{aligned}$$

Proof of Theorem 1.2. Take $k = n-1$. Since $P_n = 0$, we get by (3.3-part (i)) $\nabla H_n^2 = 0$. Therefore H_n is constant. Also, by (3.3-part (ii)) we have $H_n(\lambda + n H_1 H_n) = 0$. Then, having assumed H_n to be non-null, we get $H_1 = \frac{-\lambda}{n H_n}$, so nothing remains to prove. $\square$

4. Weakly convexity

Recently, in [12], the ordinary biharmonicity condition is verified on the hypersurfaces of space forms of non-positive sectional curvature with an additional condition named weakly convexity. A hypersurfaces of a space form is said to be *weakly convex* if all of its principal curvatures are nonnegative. Here, we study the L_k -biharmonicity condition and in general the condition (3.1) on weakly convex spacelike hypersurfaces $x : M^n \rightarrow E_1^{n+1}$ of pseudo-Euclidean space E_1^{n+1} . We prove Theorem 1.3.

Proof of Theorem 1.3. Define $U := \{p \in M : \nabla H_{k+1}^2(p) \neq 0\}$, it is enough to show that $U = \emptyset$. Using the local orthonormal tangent frame

field $\{e_1, \dots, e_n\}$ of principal directions on M associated to nonnegative principal curvatures $\lambda_1, \dots, \lambda_n$, we have $P_k e_i = \mu_{i,k} e_i$ for $1 \leq i \leq n$. By polar decomposition we have

$$\nabla H_{k+1} = \sum_{i=1}^n \langle \nabla H_{k+1}, e_i \rangle e_i,$$

so from (3.3 – (i)) and (2.1) we obtain

$$\langle \nabla H_{k+1}, e_i \rangle \left(\lambda_i \mu_{i,k} - \frac{1}{2} \binom{n}{k+1} H_{k+1} \right) = 0,$$

and then

$$\langle \nabla H_{k+1}, e_i \rangle (-1)^k \left(\lambda_i \sum_{1 \leq i_1 < \dots < i_k \leq n, i_j \neq i} \lambda_{i_1} \dots \lambda_{i_k} + \frac{1}{2} s_{k+1} \right) = 0,$$

on U , for every $i = 1, \dots, n$. Now, for each i , having assumed $\langle \nabla H_{k+1}, e_i \rangle \neq 0$ on U , we get

$$\lambda_i \sum_{1 \leq i_1 < \dots < i_k \leq n, i_j \neq i} \lambda_{i_1} \dots \lambda_{i_k} + \frac{1}{2} s_{k+1} = 0,$$

which, by the assumption that M^n is weakly convex, gives $H_{k+1} = 0$ locally on U , which is a contradiction with the definition of U . Hence, $U = \emptyset$. $\square$

Using the idea of the last proof, we prove Theorem 1.4 as follows.

Proof of Theorem 1.4. Putting $\bar{H}_i := (-1)^i H_i$ for $i = 1, 2, \dots, n$, we have $\bar{H}_{k+1}$ is constant, by Theorem 1.3. It is always true that

$$\bar{H}_{i-1} \bar{H}_{i+1} \leq \bar{H}_i^2$$

and

$$\bar{H}_1 \geq \bar{H}_2^{1/2} \geq \bar{H}_3^{1/3} \geq \dots \geq \bar{H}_i^{1/i} \quad (1 \leq i < n),$$

provided $\bar{H}_1, \bar{H}_2, \dots, \bar{H}_i$ are nonnegative, (see [4, 9]). Then, we obtain

$$(4.1) \quad \begin{aligned} H \bar{H}_{k+1} - \bar{H}_{k+2} &\geq \frac{\bar{H}_{k+1}}{\bar{H}_k} (\bar{H} \bar{H}_k - \bar{H}_{k+1}) \\ &\geq \frac{\bar{H}_{k+1}}{\bar{H}_k} (\bar{H} \bar{H}_k - (\bar{H}_k)^{\frac{k+1}{k}}) \geq \bar{H}_{k+1} (\bar{H} - \bar{H}_k^{\frac{1}{k}}) \geq 0. \end{aligned}$$

On the other hand, since H_{k+1} is a constant and M^n is assumed to be L_k -biharmonic, by formula (3.3) we get

$$n \bar{H} \bar{H}_{k+1} = -(n - k - 1) \bar{H}_{k+2},$$

so, when $k = n - 1$, we have $\bar{H} = 0$ therefore from the above inequalities, we get $\bar{H}_n = 0$. Also, in the case $k < n - 1$, formula (4.1) and the last equation give $\bar{H}_{k+1} = 0$. $\square$

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