

**Some new Ostrowski type fractional integral
inequalities for generalized relative
semi- $(r; m, h)$ -preinvex mappings via Caputo
 k -fractional derivatives**

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Abstract

In the present paper, the notion of generalized relative semi- $(r; m, h)$ -preinvex mappings is introduced and some new integral inequalities for the left-hand side of Gauss-Jacobi type quadrature formula involving generalized relative semi- $(r; m, h)$ -preinvex mappings are given. Moreover, some new generalizations of Ostrowski type integral inequalities to generalized relative semi- $(r; m, h)$ -preinvex mappings that are $(n + 1)$ -differentiable via Caputo k -fractional derivatives are established. Some applications to special means are also obtain. It is pointed out that some new special cases can be deduced from main results of the article.

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1. Introduction

The following notations are used throughout this paper. We use I to denote an interval on the real line $\mathbf{R} = (-\infty, +\infty)$ and I° to denote the interior of I . The set of continuous differentiable functions of order n on the interval $[a, b]$ is denoted by $C^n[a, b]$.

The following result is known in the literature as the Ostrowski inequality [41], which gives an upper bound for the approximation of the integral average

$$\frac{1}{b-a} \int_a^b f(t) dt \text{ by the value } f(x) \text{ at point } x \in [a, b].$$

Theorem 1.1. *Let $f : I \rightarrow \mathbf{R}$ be a mapping differentiable on I° and let $a, b \in I^\circ$ with $a < b$. If $|f'(x)| \leq M$ for all $x \in [a, b]$, then*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq M(b-a) \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right], \quad \forall x \in [a, b].$$

Ostrowski inequality is playing a very important role in all the fields of mathematics, especially in the theory of approximations. Thus such inequalities were studied extensively by many researches and numerous generalizations, extensions and variants of them for various kind of functions like bounded variation, synchronous, Lipschitzian, monotonic, absolutely, continuous and n -times differentiable mappings etc. appeared in a number of papers, see [[2]-[4],[7],[13]-[15],[17],[18],[20], [23]-[32],[38],[39],[41],[43],[45], [48],[50],[51],[57],[59],[67],[69],[72],[74]]. In recent years, one more dimension has been added to this studies, by introducing a number of integral inequalities involving various fractional operators like Riemann-Liouville, Erdelyi-Kober, Katugampola, conformable fractional integral operators etc. by many authors, see [[1],[36],[37],[55],[60]-[65]]. Riemann-Liouville fractional integral operators are the most central between these fractional operators.

In numerical analysis many quadrature rules have been established to approximate the definite integrals, see [[16],[22],[44],[46],[47],[52],[56],[68],[70]]. Ostrowski inequality provides the bounds for many numerical quadrature

rules. In recent decades Ostrowski inequality is studied in fractional calculus point of view by many mathematicians, see [[8]-[12],[27],[29],[32]-[35],[42],[53],[58]].

Let us recall some special functions and evoke some basic definitions as follows.

Definition 1.2. The Euler beta function is defined for $a, b > 0$ as

$$\beta(a, b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

Definition 1.3. [73] A set $M_\varphi \subseteq \mathbf{R}^n$ is said to be a relative convex (φ -convex) set, if and only if, there exists a function $\varphi : \mathbf{R}^n \longrightarrow \mathbf{R}^n$ such that,

$$(1.2) \quad t\varphi(x) + (1-t)\varphi(y) \in M_\varphi, \quad \forall x, y \in \mathbf{R}^n : \varphi(x), \varphi(y) \in M_\varphi, t \in [0, 1].$$

Definition 1.4. [73] A function f is said to be a relative convex (φ -convex) function on a relative convex (φ -convex) set M_φ , if and only if, there exists a function $\varphi : \mathbf{R}^n \longrightarrow \mathbf{R}^n$ such that,

$$(1.3) \quad f(t\varphi(x) + (1-t)\varphi(y)) \leq tf(\varphi(x)) + (1-t)f(\varphi(y)),$$

$$\forall x, y \in \mathbf{R}^n : \varphi(x), \varphi(y) \in M_\varphi, t \in [0, 1].$$

Definition 1.5. [22] A function $f : [0, +\infty) \longrightarrow \mathbf{R}$ is said to be s -convex in the second sense, if

$$(1.4) \quad f(\lambda x + (1-\lambda)y) \leq \lambda^s f(x) + (1-\lambda)^s f(y)$$

$$\forall x, y \geq 0, \lambda \in [0, 1] \text{ and } s \in (0, 1].$$

It is clear that a s -convex function must be convex on $[0, +\infty)$ as usual. The s -convex functions in the second sense have been investigated in [22].

Definition 1.6. [16] A non-negative function $f : I \subseteq \mathbf{R} \longrightarrow [0, +\infty)$ is said to be P -function or P -convex, if

$$f(tx + (1-t)y) \leq f(x) + f(y), \quad \forall x, y \in I, t \in [0, 1].$$

Definition 1.7. [5] A set $K \subseteq \mathbf{R}^n$ is said to be invex with respect to the mapping $\eta : K \times K \longrightarrow \mathbf{R}^n$, if $x + t\eta(y, x) \in K$, $\forall x, y \in K$ and $t \in [0, 1]$.

Notice that every convex set is invex with respect to the mapping $\eta(y, x) = y - x$, but the converse is not necessarily true, see [[5],[54],[71],[73]].

Definition 1.8. [54] The function f defined on the invex set $K \subseteq \mathbf{R}^n$ is said to be preinvex with respect η , if for every $x, y \in K$ and $t \in [0, 1]$, we have that

$$f(x + t\eta(y, x)) \leq (1 - t)f(x) + tf(y).$$

The concept of preinvexity is more general than convexity since every convex function is preinvex with respect to the mapping $\eta(y, x) = y - x$, but the converse is not true.

Definition 1.9. [44] Let $h : [0, 1] \longrightarrow \mathbf{R}$ be a non-negative function and $h \neq 0$. The function f on the invex set K is said to be h -preinvex with respect to η , if

$$(1.5) \quad f(x + t\eta(y, x)) \leq h(1 - t)f(x) + h(t)f(y)$$

$\forall x, y \in K$ and $t \in [0, 1]$ where $f(\cdot) > 0$.

Definition 1.10. [70] Let $h : J \subseteq \mathbf{R} \rightarrow \mathbf{R}$ be a positive function, $h \neq 0$. We say that $f : I \subseteq \mathbf{R} \rightarrow \mathbf{R}$ is h -convex, if f is non-negative and for all $x, y \in I$ and $t \in (0, 1)$, one has

$$(1.6) \quad f(tx + (1 - t)y) \leq h(t)f(x) + h(1 - t)f(y).$$

Definition 1.11. [68] Let $f : K \subseteq \mathbf{R} \longrightarrow \mathbf{R}$ be a non-negative function, we say that $f : K \longrightarrow \mathbf{R}$ is a tgs -convex function on K if the inequality

$$(1.7) \quad f((1 - t)x + ty) \leq t(1 - t)[f(x) + f(y)]$$

holds $\forall x, y \in K$ and $t \in (0, 1)$. We say that f is tgs -concave if $(-f)$ is tgs -convex.

Definition 1.12. [47] A function: $I \subseteq \mathbf{R} \longrightarrow \mathbf{R}$ is said to be m -MT-convex, if f is positive and for $\forall x, y \in I$, and $t \in (0, 1)$, with $m \in [0, 1]$, satisfies the following inequality

$$(1.8) \quad f(tx + m(1 - t)y) \leq \frac{\sqrt{t}}{2\sqrt{1 - t}}f(x) + \frac{m\sqrt{1 - t}}{2\sqrt{t}}f(y).$$

Definition 1.13. [53] Let $K \subseteq \mathbf{R}$ be an open m -invex set with respect to $\eta : K \times K \times (0, 1] \longrightarrow \mathbf{R}$. A function $f : K \longrightarrow \mathbf{R}$, $h_1, h_2 : [0, 1] \longrightarrow [0, +\infty)$, if

$$(1.9) \quad f(mx + t\eta(y, x, m)) \leq mh_1(t)f(x) + h_2(t)f(y)$$

is valid $\forall x, y \in K$ and $t \in [0, 1]$, with $m \in (0, 1]$, then we say that $f(x)$ is a generalized (m, h_1, h_2) -preinvex function with respect to η . If the inequality (1.9) reverses, then f is said to be (m, h_1, h_2) -preincave on K .

$$\eta : A \times A \times (0, 1] \rightarrow \mathbf{R}$$

The Gauss-Jacobi type quadrature formula has the following

$$(1.10) \quad \int_a^b (x-a)^p(b-x)^q f(x)dx = \sum_{k=0}^{+\infty} B_{m,k} f(\gamma_k) + R_m^*|f|,$$

for certain $B_{m,k}, \gamma_k$ and rest $R_m^*|f|$, see [66].

Recently, Liu [40] obtained several integral inequalities for the left-hand side of (1.10) under the Definition 1.6 of P -function. Also in [49], Özdemir et al. established several integral inequalities concerning the left-hand side of (1.10) via some kinds of convexity.

Now, let us evoke some other definitions.

Definition 1.14. For $k \in \mathbf{R}^+$ and $x \in \mathbf{C}$, the k -gamma function is defined by

$$(1.11) \quad \Gamma_k(x) = \lim_{n \rightarrow \infty} \frac{n!k^n(nk)^{\frac{x}{k}-1}}{(x)_{n,k}}.$$

Its integral representation is given by

$$(1.12) \quad \Gamma_k(\alpha) = \int_0^\infty t^{\alpha-1} e^{-\frac{t^k}{k}} dt.$$

One can note that

$$\Gamma_k(\alpha + k) = \alpha \Gamma_k(\alpha).$$

For $k = 1$, (1.12) gives integral representation of gamma function.

Definition 1.15. For $k \in \mathbf{R}^+$ and $x, y \in \mathbf{C}$, the k -beta function with two parameters x and y is defined as

$$(1.13) \quad \beta_k(x, y) = \frac{1}{k} \int_0^1 t^{\frac{x}{k}-1} (1-t)^{\frac{y}{k}-1} dt.$$

For $k = 1$, (1.13) gives integral representation of beta function.

Theorem 1.16. Let $x, y > 0$, then for k -gamma and k -beta function the following equality holds:

$$(1.14) \quad \beta_k(x, y) = \frac{\Gamma_k(x) \Gamma_k(y)}{\Gamma_k(x+y)}.$$

Definition 1.17. [35] Let $\alpha > 0$ and $\alpha \notin \{1, 2, 3, \dots\}$, $n = [\alpha] + 1$, $f \in C^n[a, b]$ such that $f^{(n)}$ exists and are continuous on $[a, b]$. The Caputo fractional derivatives of order α are defined as follows:

$$(1.15) \quad {}^c D_{a+}^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{f^{(n)}(t)}{(x-t)^{\alpha-n+1}} dt, \quad x > a$$

and

$$(1.16) \quad {}^c D_{b-}^\alpha f(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^b \frac{f^{(n)}(t)}{(t-x)^{\alpha-n+1}} dt, \quad x < b.$$

If $\alpha = n \in \{1, 2, 3, \dots\}$ and usual derivative of order n exists, then Caputo fractional derivative $({}^c D_{a+}^\alpha f)(x)$ coincides with $f^{(n)}(x)$. In particular we have

$$(1.17) \quad ({}^c D_{a+}^0 f)(x) = ({}^c D_{b-}^0 f)(x) = f(x)$$

where $n = 1$ and $\alpha = 0$.

Definition 1.18. [21] Let $\alpha > 0$, $k \geq 1$ and $\alpha \notin \{1, 2, 3, \dots\}$, $n = [\alpha] + 1$, $f \in C^n[a, b]$. The Caputo k -fractional derivatives of order α are defined as follows:

$$(1.18) \quad {}^c D_{a+}^{\alpha, k} f(x) = \frac{1}{k \Gamma_k(n - \frac{\alpha}{k})} \int_a^x \frac{f^{(n)}(t)}{(x-t)^{\frac{\alpha}{k}-n+1}} dt, \quad x > a$$

and

$$(1.19) \quad {}^c D_{b-}^{\alpha, k} f(x) = \frac{(-1)^n}{k \Gamma_k(n - \frac{\alpha}{k})} \int_x^b \frac{f^{(n)}(t)}{(t-x)^{\frac{\alpha}{k}-n+1}} dt, \quad x < b.$$

Motivated by these results, in Sect. 2, the notion of generalized relative semi- $(r; m, h)$ -preinvex mapping is introduced and some new integral inequalities for the left-hand side of (1.10) involving generalized relative semi- $(r; m, h)$ -preinvex mappings are given. In Sect. 3, some generalizations of Ostrowski type inequalities to generalized relative semi- $(r; m, h)$ -preinvex mappings that are $(n+1)$ -differentiable via Caputo k -fractional derivatives are given. It is pointed out that some new special cases will be deduced from main results of the article. In Sect. 4, some applications to special means are also obtain. In Sect. 5, some conclusions and future research are given.

2. New integral inequalities

Definition 2.1. [19] A set $K \subseteq \mathbf{R}^n$ is said to be m -invex with respect to the mapping $\eta : K \times K \times (0, 1] \rightarrow \mathbf{R}^n$ for some fixed $m \in (0, 1]$, if $mx + t\eta(y, mx) \in K$ holds for each $x, y \in K$ and any $t \in [0, 1]$.

Remark 2.2. In Definition 2.1, under certain conditions, the mapping $\eta(y, mx)$ could reduce to $\eta(y, x)$. For example when $m = 1$, then the m -invex set degenerates an invex set on K .

We next give new definition, to be referred as generalized relative semi- $(r; m, h)$ -preinvex mapping.

Definition 2.3. Let $K \subseteq \mathbf{R}$ be an open m -invex set with respect to the mapping $\eta : K \times K \times (0, 1] \rightarrow \mathbf{R}$. Also, let $h : [0, 1] \rightarrow [0, +\infty)$ and $\varphi : I \rightarrow K$ are continuous functions. A mapping $f : K \rightarrow (0, +\infty)$ is said to be generalized relative semi- $(r; m, h)$ -preinvex, if

$$(2.1) \quad f\left(m\varphi(x) + t\eta(\varphi(y), \varphi(x), m)\right) \leq M_r(h(t); f(x), f(y), m)$$

holds $\forall x, y \in I$ and $t \in [0, 1]$ and for some fixed $m \in (0, 1]$, where

$$M_r(h(t); f(x), f(y), m) := \begin{cases} \left[mh(1-t)f^r(x) + h(t)f^r(y) \right]^{\frac{1}{r}}, & \text{if } r \neq 0; \\ [f(x)]^{mh(1-t)} [f(y)]^{h(t)}, & \text{if } r = 0, \end{cases}$$

is the weighted power mean of order r for positive numbers $f(x)$ and $f(y)$.

Remark 2.4. Let us discuss some special cases in Definition 2.3 as follows.

(I) Taking $h(t) = t$, then we get generalized relative semi- m -preinvex mappings.

(II) Taking $h(t) = t^s$ for $s \in (0, 1]$, then we get generalized relative semi- (m, s) -Breckner-preinvex mappings.

(III) Taking $h(t) = t^{-s}$ for $s \in (0, 1]$, then we get generalized relative semi- (m, s) -Godunova-Levin-Dragomir-preinvex mappings.

(IV) Taking $h(t) = 1$, then we get generalized relative semi- (m, P) -preinvex mappings.

(V) Taking $h(t) = t(1 - t)$, then we get generalized relative semi- (m, tgs) -preinvex mappings.

(VI) Taking $h(t) = \frac{\sqrt{t}}{2\sqrt{1-t}}$, then we get generalized relative semi- m - MT -preinvex mappings.

It is worth to mention here that to the best of our knowledge all the special cases discussed above are new in the literature.

In this section, in order to prove our main results regarding some new integral inequalities for the left-hand side of (1.10) involving generalized relative semi- $(r; m, h)$ -preinvex mappings, we need the following new lemma.

Lemma 2.5. Let $\varphi : I \longrightarrow K$ be a continuous function. Assume that $f : K = [m\varphi(a), m\varphi(a) + \eta(\varphi(b), \varphi(a), m)] \longrightarrow \mathbf{R}$ is a continuous mapping on K° with respect to $\eta : K \times K \times (0, 1] \longrightarrow \mathbf{R}$, where $\eta(\varphi(b), \varphi(a), m) > 0$. Then for some fixed $m \in (0, 1]$ and any fixed $p, q > 0$, we have

$$\begin{aligned} & \int_{m\varphi(a)}^{m\varphi(a) + \eta(\varphi(b), \varphi(a), m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x) dx \\ &= \eta^{p+q+1}(\varphi(b), \varphi(a), m) \int_0^1 t^p (1-t)^q f(m\varphi(a) + t\eta(\varphi(b), \varphi(a), m)) dt. \end{aligned}$$

Proof. It is easy to observe that

$$\begin{aligned}
 & \int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x) dx \\
 &= \eta(\varphi(b), \varphi(a), m) \int_0^1 (m\varphi(a) + t\eta(\varphi(b), \varphi(a), m) - m\varphi(a))^p \\
 & \quad \times (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - m\varphi(a) - t\eta(\varphi(b), \varphi(a), m))^q \\
 & \quad \times f(m\varphi(a) + t\eta(\varphi(b), \varphi(a), m)) dt \\
 &= \eta^{p+q+1}(\varphi(b), \varphi(a), m) \int_0^1 t^p (1-t)^q f(m\varphi(a) + t\eta(\varphi(b), \varphi(a), m)) dt.
 \end{aligned}$$

This completes the proof of the lemma. $\square$

By using Lemma 2.5, one can extend to the following results.

Theorem 2.6. Suppose $h : [0, 1] \longrightarrow [0, +\infty)$ and $\varphi : I \longrightarrow K$ are continuous functions. Assume that $f : K = [m\varphi(a), m\varphi(a) + \eta(\varphi(b), \varphi(a), m)] \longrightarrow (0, +\infty)$ is a continuous mapping on K° with respect to $\eta : K \times K \times (0, 1] \longrightarrow \mathbf{R}$, where $\eta(\varphi(b), \varphi(a), m) > 0$. Let $k > 1$ and $0 < r \leq 1$. If $f^{\frac{k}{k-1}}$ is generalized relative semi- $(r; m, h)$ -preinvex mappings on an open m -invex set K for some fixed $m \in (0, 1]$, then for any fixed $p, q > 0$, we have

$$\begin{aligned}
 & \int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x) dx \\
 & \leq \eta^{p+q+1}(\varphi(b), \varphi(a), m) \beta^{\frac{1}{k}}(kp+1, kq+1) \left[m f^{\frac{rk}{k-1}}(a) + f^{\frac{rk}{k-1}}(b) \right]^{\frac{k-1}{rk}} \\
 & \quad \times \left(\int_0^1 h^{\frac{1}{r}}(t) dt \right)^{\frac{k-1}{k}}.
 \end{aligned}$$

Proof. Since $f^{\frac{k}{k-1}}$ is generalized relative semi- $(r; m, h)$ -preinvex mappings on K , combining with Lemma 2.5, Hölder inequality and Minkowski inequality, we get

$$\begin{aligned}
& \int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x-m\varphi(a))^p (m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x) dx \\
& \leq \eta^{p+q+1}(\varphi(b),\varphi(a),m) \left[\int_0^1 t^{kp} (1-t)^{kq} dt \right]^{\frac{1}{k}} \\
& \quad \times \left[\int_0^1 f^{\frac{k}{k-1}}(m\varphi(a)+t\eta(\varphi(b),\varphi(a),m)) dt \right]^{\frac{k-1}{k}} \\
& \leq \eta^{p+q+1}(\varphi(b),\varphi(a),m) \beta^{\frac{1}{k}}(kp+1,kq+1) \\
& \quad \times \left[\int_0^1 \left[mh(1-t)f^{\frac{rk}{k-1}}(a) + h(t)f^{\frac{rk}{k-1}}(b) \right]^{\frac{1}{r}} dt \right]^{\frac{k-1}{k}} \\
& \leq \eta^{p+q+1}(\varphi(b),\varphi(a),m) \beta^{\frac{1}{k}}(kp+1,kq+1) \\
& \quad \times \left\{ \left(\int_0^1 m^{\frac{1}{r}} f^{\frac{k}{k-1}}(a) h^{\frac{1}{r}}(1-t) dt \right)^r + \left(\int_0^1 f^{\frac{k}{k-1}}(b) h^{\frac{1}{r}}(t) dt \right)^r \right\}^{\frac{k-1}{rk}} \\
& = \eta^{p+q+1}(\varphi(b),\varphi(a),m) \beta^{\frac{1}{k}}(kp+1,kq+1) \left[mf^{\frac{rk}{k-1}}(a) + f^{\frac{rk}{k-1}}(b) \right]^{\frac{k-1}{rk}} \\
& \quad \times \left(\int_0^1 h^{\frac{1}{r}}(t) dt \right)^{\frac{k-1}{k}}.
\end{aligned}$$

So, the proof of this theorem is completed. $\square$

We point out some special cases of Theorem 2.6.

Corollary 2.7. Under the same conditions as in Theorem 2.6 for $r = 1$ and $h(t) = t^s$ where $s \in [0, 1]$, we get the following inequality for generalized relative semi- (m, s) -Breckner-preinvex mappings:

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x) dx$$

$$\leq \eta^{p+q+1}(\varphi(b), \varphi(a), m) \beta_{\frac{1}{k}}(kp + 1, kq + 1) \left[\frac{mf^{\frac{k}{k-1}}(a) + f^{\frac{k}{k-1}}(b)}{s + 1} \right]^{\frac{k-1}{k}}.$$

Corollary 2.8. Under the same conditions as in Theorem 2.6 for $r = 1$ and $h(t) = t^{-s}$ where $s \in [0, 1]$, we get the following inequality for generalized relative semi- (m, s) -Godunova-Levin-Dragomir-preinvex mappings:

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x) dx$$

$$\leq \eta^{p+q+1}(\varphi(b), \varphi(a), m) \beta_{\frac{1}{k}}(kp + 1, kq + 1) \left[\frac{mf^{\frac{k}{k-1}}(a) + f^{\frac{k}{k-1}}(b)}{1 - s} \right]^{\frac{k-1}{k}}.$$

Corollary 2.9. Under the same conditions as in Theorem 2.6 for $r = 1$ and $h(t) = t(1 - t)$, we get the following inequality for generalized relative semi- (m, tgs) -preinvex mappings:

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x) dx$$

$$\leq \eta^{p+q+1}(\varphi(b), \varphi(a), m) \beta_{\frac{1}{k}}(kp + 1, kq + 1) \left[\frac{mf^{\frac{k}{k-1}}(a) + f^{\frac{k}{k-1}}(b)}{6} \right]^{\frac{k-1}{k}}.$$

Corollary 2.10. Under the same conditions as in Theorem 2.6 for $r = 1$ and $h(t) = \frac{\sqrt{t}}{2\sqrt{1-t}}$, we get the following inequality for generalized relative semi- m - MT -preinvex mappings:

$$\begin{aligned} & \int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x) dx \\ & \leq \left(\frac{\pi}{4}\right)^{\frac{k-1}{k}} \eta^{p+q+1}(\varphi(b), \varphi(a), m) \beta^{\frac{1}{k}}(kp+1, kq+1) \left[m f^{\frac{k}{k-1}}(a) + f^{\frac{k}{k-1}}(b) \right]^{\frac{k-1}{k}}. \end{aligned}$$

Theorem 2.11. Suppose $h : [0, 1] \longrightarrow [0, +\infty)$ and $\varphi : I \longrightarrow K$ are continuous functions. Assume that $f : K = [m\varphi(a), m\varphi(a) + \eta(\varphi(b), \varphi(a), m)] \longrightarrow (0, +\infty)$ is a continuous function on K° with respect to $\eta : K \times K \times (0, 1] \longrightarrow \mathbf{R}$, where $\eta(\varphi(b), \varphi(a), m) > 0$. Let $l \geq 1$ and $0 < r \leq 1$. If f^l is generalized relative semi- $(r; m, h)$ -preinvex mappings on an open m -invex set K for some fixed $m \in (0, 1]$, then for any fixed $p, q > 0$, we have

$$\begin{aligned} & \int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x) dx \\ & \leq \eta^{p+q+1}(\varphi(b), \varphi(a), m) \beta^{\frac{l-1}{l}}(p+1, q+1) \\ & \quad \times \left[m f^{rl}(a) I^r(h(t); r, p, q) + f^{rl}(b) I^r(h(t); r, q, p) \right]^{\frac{1}{rl}}, \end{aligned}$$

where

$$I(h(t); r, p, q) := \int_0^1 t^p (1-t)^q h^{\frac{1}{r}}(1-t) dt.$$

Proof. Since f^l is generalized relative semi- $(r; m, h)$ -preinvex mappings on K , combining with Lemma 2.5, the well-known power mean inequality and Minkowski inequality, we get

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x) dx$$

$$\begin{aligned}
 &= \eta^{p+q+1}(\varphi(b), \varphi(a), m) \int_0^1 \left[t^p(1-t)^q \right]^{\frac{l-1}{t}} \left[t^p(1-t)^q \right]^{\frac{1}{t}} \\
 &\quad \times f(m\varphi(a) + t\eta(\varphi(b), \varphi(a), m)) dt \\
 &\leq \eta^{p+q+1}(\varphi(b), \varphi(a), m) \left[\int_0^1 t^p(1-t)^q dt \right]^{\frac{l-1}{t}} \\
 &\quad \times \left[\int_0^1 t^p(1-t)^q f^l(m\varphi(a) + t\eta(\varphi(b), \varphi(a), m)) dt \right]^{\frac{1}{t}} \\
 &\leq \eta^{p+q+1}(\varphi(b), \varphi(a), m) \beta^{\frac{l-1}{t}}(p+1, q+1) \\
 &\quad \times \left[\int_0^1 t^p(1-t)^q \left[mh(1-t)f^{rl}(a) + h(t)f^{rl}(b) \right]^{\frac{1}{r}} dt \right]^{\frac{1}{t}} \\
 &\leq \eta^{p+q+1}(\varphi(b), \varphi(a), m) \beta^{\frac{l-1}{t}}(p+1, q+1) \\
 &\quad \times \left\{ \left(\int_0^1 m^{\frac{1}{r}} f^l(a) t^p(1-t)^q h^{\frac{1}{r}}(1-t) dt \right)^r + \left(\int_0^1 f^l(b) t^p(1-t)^q h^{\frac{1}{r}}(t) dt \right)^r \right\}^{\frac{1}{rl}} \\
 &= \eta^{p+q+1}(\varphi(b), \varphi(a), m) \beta^{\frac{l-1}{t}}(p+1, q+1) \\
 &\quad \times \left[m f^{rl}(a) I^r(h(t); r, p, q) + f^{rl}(b) I^r(h(t); r, q, p) \right]^{\frac{1}{rl}}.
 \end{aligned}$$

So, the proof of this theorem is completed. $\square$

We point out some special cases of Theorem 2.11.

Corollary 2.12. *Under the same conditions as in Theorem 2.11 for $r = 1$ and $h(t) = t^s$ where $s \in [0, 1]$, we get the following inequality for generalized relative semi- (m, s) -Breckner-preinvex mappings:*

$$\int_{m\varphi(a)}^{m\varphi(a) + \eta(\varphi(b), \varphi(a), m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x) dx$$

$$\leq \eta^{p+q+1}(\varphi(b), \varphi(a), m) \beta^{\frac{l-1}{t}} (p+1, q+1) \\ \times \left[m f^l(a) \beta(p+1, q+s+1) + f^l(b) \beta(q+1, p+s+1) \right]^{\frac{1}{t}}.$$

Corollary 2.13. Under the same conditions as in Theorem 2.11 for $r = 1$ and $h(t) = t^{-s}$ where $s \in (0, 1]$, we get the following inequality for generalized relative semi- (m, s) -Godunova-Levin-Dragomir-preinvex mappings:

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b), \varphi(a), m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x) dx \\ \leq \eta^{p+q+1}(\varphi(b), \varphi(a), m) \beta^{\frac{l-1}{t}} (p+1, q+1) \\ \times \left[m f^l(a) \beta(p+1, q-s+1) + f^l(b) \beta(q+1, p-s+1) \right]^{\frac{1}{t}}.$$

Corollary 2.14. Under the same conditions as in Theorem 2.11 for $r = 1$ and $h(t) = t(1-t)$, we get the following inequality for generalized relative semi- (m, tgs) -preinvex mappings:

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b), \varphi(a), m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x) dx \\ \leq \eta^{p+q+1}(\varphi(b), \varphi(a), m) \beta^{\frac{l-1}{t}} (p+1, q+1) \beta^{\frac{1}{t}} (p+2, q+2) \\ \times \left[m f^l(a) + f^l(b) \right]^{\frac{1}{t}}.$$

Corollary 2.15. Under the same conditions as in Theorem 2.11 for $r = 1$ and $h(t) = \frac{\sqrt{t}}{2\sqrt{1-t}}$, we get the following inequality for generalized relative semi- m -MT-preinvex mappings:

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b), \varphi(a), m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x) dx$$

$$\leq \left(\frac{1}{2}\right)^{\frac{1}{t}} \eta^{p+q+1}(\varphi(b), \varphi(a), m) \beta^{\frac{l-1}{t}} (p+1, q+1) \\ \times \left[m f^l(a) \beta \left(p + \frac{1}{2}, q + \frac{3}{2}\right) + f^l(b) \beta \left(q + \frac{1}{2}, p + \frac{3}{2}\right) \right]^{\frac{1}{t}}.$$

3. Some new Ostrowski type fractional integral inequalities

In this section, in order to present some new Ostrowski type integral inequalities for generalized relative semi- $(r; m, h)$ -preinvex mappings via Caputo k -fractional derivatives, we need the following lemma.

Lemma 3.1. *Let $\alpha > 0$, $k \geq 1$ and $\alpha \notin \{1, 2, 3, \dots\}$, $n = [\alpha] + 1$. Suppose $h : [0, 1] \rightarrow [0, +\infty)$ and $\varphi : I \rightarrow K$ are continuous functions. Suppose $K = [m\varphi(a), m\varphi(a) + \eta(\varphi(b), \varphi(a), m)] \subseteq \mathbf{R}$ be an open m -invex subset with respect to $\eta : K \times K \times (0, 1] \rightarrow \mathbf{R}$ for some fixed $m \in (0, 1]$. Assume that $f : K \rightarrow \mathbf{R}$ is a mapping on K° such that $f \in C^{n+1}(K)$, where $\eta(\varphi(b), \varphi(a), m) > 0$. Then the following equality for Caputo k -fractional derivatives holds:*

$$\frac{\eta^{n-\frac{\alpha}{k}}(\varphi(x), \varphi(a), m) f^{(n)}(m\varphi(a) + \eta(\varphi(x), \varphi(a), m))}{\eta(\varphi(b), \varphi(a), m)} \\ - \frac{\eta^{n-\frac{\alpha}{k}}(\varphi(x), \varphi(b), m) f^{(n)}(m\varphi(b) + \eta(\varphi(x), \varphi(b), m))}{\eta(\varphi(b), \varphi(a), m)} \\ + (-1)^{n+1} \frac{(nk - \alpha) \Gamma_k(n - \frac{\alpha}{k})}{\eta(\varphi(b), \varphi(a), m)} \\ \times \left[{}^c D_{(m\varphi(a) + \eta(\varphi(x), \varphi(a), m))}^{\alpha, k} f(m\varphi(a)) - {}^c D_{(m\varphi(b) + \eta(\varphi(x), \varphi(b), m))}^{\alpha, k} f(m\varphi(b)) \right] \\ = \frac{\eta^{n-\frac{\alpha}{k}+1}(\varphi(x), \varphi(a), m)}{\eta(\varphi(b), \varphi(a), m)} \int_0^1 t^{n-\frac{\alpha}{k}} f^{(n+1)}(m\varphi(a) + t\eta(\varphi(x), \varphi(a), m)) dt \\ (3.1) \frac{\eta^{n-\frac{\alpha}{k}+1}(\varphi(x), \varphi(b), m)}{\eta(\varphi(b), \varphi(a), m)} \int_0^1 t^{n-\frac{\alpha}{k}} f^{(n+1)}(m\varphi(b) + t\eta(\varphi(x), \varphi(b), m)) dt.$$

We denote

$$I_{f, \eta, \varphi}(x; \alpha, k, n, m, a, b)$$

$$\begin{aligned}
&:= \frac{\eta^{n-\frac{\alpha}{k}+1}(\varphi(x), \varphi(a), m)}{\eta(\varphi(b), \varphi(a), m)} \int_0^1 t^{n-\frac{\alpha}{k}} f^{(n+1)}(m\varphi(a) + t\eta(\varphi(x), \varphi(a), m)) dt \\
&- \frac{\eta^{n-\frac{\alpha}{k}+1}(\varphi(x), \varphi(b), m)}{\eta(\varphi(b), \varphi(a), m)} \int_0^1 t^{n-\frac{\alpha}{k}} f^{(n+1)}(m\varphi(b) + t\eta(\varphi(x), \varphi(b), m)) dt.
\end{aligned}
\tag{3.2}$$

Proof. Integrating by parts, we get

$$\begin{aligned}
I_{f,\eta,\varphi}(x; \alpha, k, n, m, a, b) &= \frac{\eta^{n-\frac{\alpha}{k}+1}(\varphi(x), \varphi(a), m)}{\eta(\varphi(b), \varphi(a), m)} \\
&\times \left[\frac{t^{n-\frac{\alpha}{k}} f^{(n)}(m\varphi(a) + t\eta(\varphi(x), \varphi(a), m))}{\eta(\varphi(x), \varphi(a), m)} \right]_0^1 \\
&- \frac{n - \frac{\alpha}{k}}{\eta(\varphi(x), \varphi(a), m)} \int_0^1 t^{n-\frac{\alpha}{k}-1} f^{(n)}(m\varphi(a) + t\eta(\varphi(x), \varphi(a), m)) dt \\
&- \frac{\eta^{n-\frac{\alpha}{k}+1}(\varphi(x), \varphi(b), m)}{\eta(\varphi(b), \varphi(a), m)} \\
&\times \left[\frac{t^{n-\frac{\alpha}{k}} f^{(n)}(m\varphi(b) + t\eta(\varphi(x), \varphi(b), m))}{\eta(\varphi(x), \varphi(b), m)} \right]_0^1 \\
&- \frac{n - \frac{\alpha}{k}}{\eta(\varphi(x), \varphi(b), m)} \int_0^1 t^{n-\frac{\alpha}{k}-1} f^{(n)}(m\varphi(b) + t\eta(\varphi(x), \varphi(b), m)) dt \\
&= \frac{\eta^{n-\frac{\alpha}{k}}(\varphi(x), \varphi(a), m) f^{(n)}(m\varphi(a) + \eta(\varphi(x), \varphi(a), m))}{\eta(\varphi(b), \varphi(a), m)} \\
&- \frac{\eta^{n-\frac{\alpha}{k}}(\varphi(x), \varphi(b), m) f^{(n)}(m\varphi(b) + \eta(\varphi(x), \varphi(b), m))}{\eta(\varphi(b), \varphi(a), m)}
\end{aligned}$$

$$\begin{aligned}
 & +(-1)^{n+1} \frac{(nk-\alpha) \Gamma_k\left(n-\frac{\alpha}{k}\right)}{\eta(\varphi(b), \varphi(a), m)} \\
 & \times\left[{}^c D_{(m \varphi(a)+\eta(\varphi(x), \varphi(a), m))}^{\alpha, k} f(m \varphi(a))-{}^c D_{(m \varphi(b)+\eta(\varphi(x), \varphi(b), m))}^{\alpha, k} f(m \varphi(b))\right] .
 \end{aligned}$$

This completes the proof of our lemma. $\square$

By using Lemma 3.1, one can extend to the following results.

Theorem 3.2. Let $\alpha > 0$, $k \geq 1$, $0 < r \leq 1$ and $\alpha \notin \{1, 2, 3, \dots\}$, $n = [\alpha] + 1$. Suppose $h : [0, 1] \rightarrow [0, +\infty)$ and $\varphi : I \rightarrow K$ are continuous functions. Suppose $K = [m\varphi(a), m\varphi(a) + \eta(\varphi(b), \varphi(a), m)] \subseteq \mathbf{R}$ be an open m -invex subset with respect to $\eta : K \times K \times (0, 1] \rightarrow \mathbf{R}$ for some fixed $m \in (0, 1]$. Assume that $f : K \rightarrow (0, +\infty)$ is a mapping on K° such that $f \in C^{n+1}(K)$, where $\eta(\varphi(b), \varphi(a), m) > 0$. If $\left(f^{(n+1)}\right)^q$ is generalized relative semi- $(r; m, h)$ -preinvex mappings on K , $q > 1$, $p^{-1} + q^{-1} = 1$, then the following inequality for Caputo k -fractional derivatives holds:

$$\begin{aligned}
 \left|I_{f, \eta, \varphi}(x ; \alpha, k, n, m, a, b)\right| & \leq\left(\frac{1}{\left(n-\frac{\alpha}{k}\right) p+1}\right)^{\frac{1}{p}} \frac{1}{\eta(\varphi(b), \varphi(a), m)}\left(\int_0^1 h^{\frac{1}{r}}(t) d t\right)^{\frac{1}{q}} \\
 & \times\left\{\left|\eta(\varphi(x), \varphi(a), m)\right|^{n-\frac{\alpha}{k}+1}\left[m\left(f^{(n+1)}(a)\right)^{r q}+\left(f^{(n+1)}(x)\right)^{r q}\right]^{\frac{1}{r q}}\right. \\
 (3.3) & \left.+\left|\eta(\varphi(x), \varphi(b), m)\right|^{n-\frac{\alpha}{k}+1}\left[m\left(f^{(n+1)}(b)\right)^{r q}+\left(f^{(n+1)}(x)\right)^{r q}\right]^{\frac{1}{r q}}\right\} .
 \end{aligned}$$

Proof. From Lemma 3.1, generalized relative semi- $(r; m, h)$ -preinvexity of $\left(f^{(n+1)}\right)^q$, Hölder inequality, Minkowski inequality and properties of the modulus, we have

$$\begin{aligned}
 & \left|I_{f, \eta, \varphi}(x ; \alpha, k, n, m, a, b)\right| \\
 & \leq \frac{\left|\eta(\varphi(x), \varphi(a), m)\right|^{n-\frac{\alpha}{k}+1}}{\left|\eta(\varphi(b), \varphi(a), m)\right|} \int_0^1 t^{n-\frac{\alpha}{k}}\left|f^{(n+1)}\left(m \varphi(a)+t \eta(\varphi(x), \varphi(a), m)\right)\right| d t
 \end{aligned}$$

$$\begin{aligned}
& + \frac{|\eta(\varphi(x), \varphi(b), m)|^{n-\frac{\alpha}{k}+1}}{|\eta(\varphi(b), \varphi(a), m)|} \int_0^1 t^{n-\frac{\alpha}{k}} \left| f^{(n+1)}(m\varphi(b) + t\eta(\varphi(x), \varphi(b), m)) \right| dt \\
& \leq \frac{|\eta(\varphi(x), \varphi(a), m)|^{n-\frac{\alpha}{k}+1}}{\eta(\varphi(b), \varphi(a), m)} \left(\int_0^1 t^{(n-\frac{\alpha}{k})p} dt \right)^{\frac{1}{p}} \\
& \quad \times \left(\int_0^1 \left(f^{(n+1)}(m\varphi(a) + t\eta(\varphi(x), \varphi(a), m)) \right)^q dt \right)^{\frac{1}{q}} \\
& \quad + \frac{|\eta(\varphi(x), \varphi(b), m)|^{n-\frac{\alpha}{k}+1}}{\eta(\varphi(b), \varphi(a), m)} \left(\int_0^1 t^{(n-\frac{\alpha}{k})p} dt \right)^{\frac{1}{p}} \\
& \quad \times \left(\int_0^1 \left(f^{(n+1)}(m\varphi(b) + t\eta(\varphi(x), \varphi(b), m)) \right)^q dt \right)^{\frac{1}{q}} \\
& \leq \frac{|\eta(\varphi(x), \varphi(a), m)|^{n-\frac{\alpha}{k}+1}}{\eta(\varphi(b), \varphi(a), m)} \left(\int_0^1 t^{(n-\frac{\alpha}{k})p} dt \right)^{\frac{1}{p}} \\
& \quad \times \left(\int_0^1 \left[mh(1-t) \left(f^{(n+1)}(a) \right)^{rq} + h(t) \left(f^{(n+1)}(x) \right)^{rq} \right]^{\frac{1}{r}} dt \right)^{\frac{1}{q}} \\
& \quad + \frac{|\eta(\varphi(x), \varphi(b), m)|^{n-\frac{\alpha}{k}+1}}{\eta(\varphi(b), \varphi(a), m)} \left(\int_0^1 t^{(n-\frac{\alpha}{k})p} dt \right)^{\frac{1}{p}} \\
& \quad \times \left(\int_0^1 \left[mh(1-t) \left(f^{(n+1)}(b) \right)^{rq} + h(t) \left(f^{(n+1)}(x) \right)^{rq} \right]^{\frac{1}{r}} dt \right)^{\frac{1}{q}} \\
& \leq \frac{|\eta(\varphi(x), \varphi(a), m)|^{n-\frac{\alpha}{k}+1}}{\eta(\varphi(b), \varphi(a), m)} \left(\int_0^1 t^{(n-\frac{\alpha}{k})p} dt \right)^{\frac{1}{p}} \\
& \quad \times \left\{ \left(\int_0^1 m^{\frac{1}{r}} \left(f^{(n+1)}(a) \right)^q h^{\frac{1}{r}}(1-t) dt \right)^r + \left(\int_0^1 \left(f^{(n+1)}(x) \right)^q h^{\frac{1}{r}}(t) dt \right)^r \right\}^{\frac{1}{rq}}
\end{aligned}$$

$$\begin{aligned}
 & + \frac{|\eta(\varphi(x), \varphi(b), m)|^{n-\frac{\alpha}{k}+1}}{\eta(\varphi(b), \varphi(a), m)} \left(\int_0^1 t^{(n-\frac{\alpha}{k})p} dt \right)^{\frac{1}{p}} \\
 & \times \left\{ \left(\int_0^1 m^{\frac{1}{r}} \left(f^{(n+1)}(b) \right)^q h^{\frac{1}{r}}(1-t) dt \right)^r + \left(\int_0^1 \left(f^{(n+1)}(x) \right)^q h^{\frac{1}{r}}(t) dt \right)^r \right\}^{\frac{1}{rq}} \\
 & = \left(\frac{1}{(n-\frac{\alpha}{k})p+1} \right)^{\frac{1}{p}} \frac{1}{\eta(\varphi(b), \varphi(a), m)} \left(\int_0^1 h^{\frac{1}{r}}(t) dt \right)^{\frac{1}{q}} \\
 & \times \left\{ |\eta(\varphi(x), \varphi(a), m)|^{n-\frac{\alpha}{k}+1} \left[m \left(f^{(n+1)}(a) \right)^{rq} + \left(f^{(n+1)}(x) \right)^{rq} \right]^{\frac{1}{rq}} \right. \\
 & \left. + |\eta(\varphi(x), \varphi(b), m)|^{n-\frac{\alpha}{k}+1} \left[m \left(f^{(n+1)}(b) \right)^{rq} + \left(f^{(n+1)}(x) \right)^{rq} \right]^{\frac{1}{rq}} \right\}.
 \end{aligned}$$

So, the proof of this theorem is completed. $\square$

Corollary 3.3. Under the same conditions as in Theorem 3.2, if we choose $m = k = r = 1$, $\eta(\varphi(y), \varphi(x), m) = \varphi(y) - m\varphi(x)$, $\varphi(x) = x$, $\forall x \in I$ and $f^{(n+1)} \leq K$, we get the following inequality for Caputo fractional derivatives:

$$\begin{aligned}
 & \left| \left[\frac{(x-a)^{n-\alpha} - (x-b)^{n-\alpha}}{b-a} \right] f^{(n)}(x) + (-1)^{n+1} \frac{\Gamma(n-\alpha+1)}{b-a} \left[{}^c D_{x-}^{\alpha} f(a) - {}^c D_{x-}^{\alpha} f(b) \right] \right| \\
 (3.4) & \leq \frac{2^{\frac{1}{q}} K}{\left[(n-\alpha)p+1 \right]^{\frac{1}{p}}} \left(\int_0^1 h(t) dt \right)^{\frac{1}{q}} \left[\frac{(x-a)^{n-\alpha+1} + (b-x)^{n-\alpha+1}}{b-a} \right].
 \end{aligned}$$

Remark 3.4. Under the same conditions as in Theorem 3.2, if taking $r = 1$ and $h(t) = t^s$, $s \in [0, 1]$; $h(t) = t^{-s}$, $s \in [0, 1]$; $h(t) = t(1-t)$ or $h(t) = \frac{\sqrt{t}}{2\sqrt{1-t}}$, then we get some special interesting inequalities for Caputo k -fractional derivatives, respectively for generalized relative semi- (m, s) -Breckner-preinvex mappings; generalized relative semi- (m, s) -Godunova-Levin-Dragomir-preinvex mappings; generalized relative semi- (m, tgs) -preinvex mappings and generalized relative semi- m -MT-preinvex mappings.

Theorem 3.5. Let $\alpha > 0$, $k \geq 1$, $0 < r \leq 1$ and $\alpha \notin \{1, 2, 3, \dots\}$, $n = [\alpha] + 1$. Suppose $h : [0, 1] \rightarrow [0, +\infty)$ and $\varphi : I \rightarrow K$ are continuous functions. Suppose $K = [m\varphi(a), m\varphi(a) + \eta(\varphi(b), \varphi(a), m)] \subseteq \mathbf{R}$ be an open m -invex subset with respect to $\eta : K \times K \times (0, 1] \rightarrow \mathbf{R}$ for some fixed $m \in (0, 1]$. Assume that $f : K \rightarrow (0, +\infty)$ is a function on K° such that $f \in C^{n+1}(K)$, where $\eta(\varphi(b), \varphi(a), m) > 0$. If $\left(f^{(n+1)}\right)^q$ is generalized relative semi- $(r; m, h)$ -preinvex mappings on K , $q \geq 1$, then the following inequality for Caputo k -fractional derivatives holds:

$$\begin{aligned}
 & \left| I_{f, \eta, \varphi}(x; \alpha, k, n, m, a, b) \right| \leq \left(\frac{1}{n - \frac{\alpha}{k} + 1} \right)^{1 - \frac{1}{q}} \frac{1}{\eta(\varphi(b), \varphi(a), m)} \\
 & \times \left\{ |\eta(\varphi(x), \varphi(a), m)|^{n - \frac{\alpha}{k} + 1} \left[m \left(f^{(n+1)}(a) \right)^{rq} I^r(h(t); r, \alpha, k, n) \right. \right. \\
 & \quad \left. \left. + \left(f^{(n+1)}(x) \right)^{rq} I^r(h(1-t); r, \alpha, k, n) \right]^{\frac{1}{rq}} \right. \\
 & \quad \left. + |\eta(\varphi(x), \varphi(b), m)|^{n - \frac{\alpha}{k} + 1} \left[m \left(f^{(n+1)}(b) \right)^{rq} I^r(h(t); r, \alpha, k, n) \right. \right. \\
 & \quad \left. \left. + \left(f^{(n+1)}(x) \right)^{rq} I^r(h(1-t); r, \alpha, k, n) \right]^{\frac{1}{rq}} \right\}, \tag{3.5}
 \end{aligned}$$

where

$$I(h(t); r, \alpha, k, n) := \int_0^1 t^{n - \frac{\alpha}{k}} h^{\frac{1}{r}}(1-t) dt.$$

Proof. From Lemma 3.1, generalized relative semi- $(r; m, h)$ -preinvexity of $\left(f^{(n+1)}\right)^q$, the well-known power mean inequality, Minkowski inequality and properties of the modulus, we have

$$\begin{aligned}
 & \left| I_{f, \eta, \varphi}(x; \alpha, k, n, m, a, b) \right| \\
 & \leq \frac{|\eta(\varphi(x), \varphi(a), m)|^{n - \frac{\alpha}{k} + 1}}{|\eta(\varphi(b), \varphi(a), m)|} \int_0^1 t^{n - \frac{\alpha}{k}} \left| f^{(n+1)}(m\varphi(a) + t\eta(\varphi(x), \varphi(a), m)) \right| dt
 \end{aligned}$$

$$\begin{aligned}
& + \frac{|\eta(\varphi(x), \varphi(b), m)|^{n-\frac{\alpha}{k}+1}}{|\eta(\varphi(b), \varphi(a), m)|} \int_0^1 t^{n-\frac{\alpha}{k}} \left| f^{(n+1)}(m\varphi(b) + t\eta(\varphi(x), \varphi(b), m)) \right| dt \\
& \leq \frac{|\eta(\varphi(x), \varphi(a), m)|^{n-\frac{\alpha}{k}+1}}{\eta(\varphi(b), \varphi(a), m)} \left(\int_0^1 t^{n-\frac{\alpha}{k}} dt \right)^{1-\frac{1}{q}} \\
& \quad \times \left(\int_0^1 t^{n-\frac{\alpha}{k}} \left(f^{(n+1)}(m\varphi(a) + t\eta(\varphi(x), \varphi(a), m)) \right)^q dt \right)^{\frac{1}{q}} \\
& \quad + \frac{|\eta(\varphi(x), \varphi(b), m)|^{n-\frac{\alpha}{k}+1}}{\eta(\varphi(b), \varphi(a), m)} \left(\int_0^1 t^{n-\frac{\alpha}{k}} dt \right)^{1-\frac{1}{q}} \\
& \quad \times \left(\int_0^1 t^{n-\frac{\alpha}{k}} \left(f^{(n+1)}(m\varphi(b) + t\eta(\varphi(x), \varphi(b), m)) \right)^q dt \right)^{\frac{1}{q}} \\
& \leq \frac{|\eta(\varphi(x), \varphi(a), m)|^{n-\frac{\alpha}{k}+1}}{\eta(\varphi(b), \varphi(a), m)} \left(\int_0^1 t^{n-\frac{\alpha}{k}} dt \right)^{1-\frac{1}{q}} \\
& \quad \times \left(\int_0^1 t^{n-\frac{\alpha}{k}} \left[mh(1-t) \left(f^{(n+1)}(a) \right)^{rq} + h(t) \left(f^{(n+1)}(x) \right)^{rq} \right]^{\frac{1}{r}} dt \right)^{\frac{1}{q}} \\
& \quad + \frac{|\eta(\varphi(x), \varphi(b), m)|^{n-\frac{\alpha}{k}+1}}{\eta(\varphi(b), \varphi(a), m)} \left(\int_0^1 t^{n-\frac{\alpha}{k}} dt \right)^{1-\frac{1}{q}} \\
& \quad \times \left(\int_0^1 t^{n-\frac{\alpha}{k}} \left[mh(1-t) \left(f^{(n+1)}(b) \right)^{rq} + h(t) \left(f^{(n+1)}(x) \right)^{rq} \right]^{\frac{1}{r}} dt \right)^{\frac{1}{q}} \\
& \leq \frac{|\eta(\varphi(x), \varphi(a), m)|^{n-\frac{\alpha}{k}+1}}{\eta(\varphi(b), \varphi(a), m)} \left(\int_0^1 t^{n-\frac{\alpha}{k}} dt \right)^{1-\frac{1}{q}} \\
& \quad \times \left\{ \left(\int_0^1 m^{\frac{1}{r}} \left(f^{(n+1)}(a) \right)^q t^{n-\frac{\alpha}{k}} h^{\frac{1}{r}}(1-t) dt \right)^r + \left(\int_0^1 \left(f^{(n+1)}(x) \right)^q t^{n-\frac{\alpha}{k}} h^{\frac{1}{r}}(t) dt \right)^r \right\}^{\frac{1}{rq}}
\end{aligned}$$

$$\begin{aligned}
& + \frac{|\eta(\varphi(x), \varphi(b), m)|^{n-\frac{\alpha}{k}+1}}{\eta(\varphi(b), \varphi(a), m)} \left(\int_0^1 t^{n-\frac{\alpha}{k}} dt \right)^{1-\frac{1}{q}} \\
& \times \left\{ \left(\int_0^1 m^{\frac{1}{r}} \left(f^{(n+1)}(b) \right)^q t^{n-\frac{\alpha}{k}} h^{\frac{1}{r}}(1-t) dt \right)^r + \left(\int_0^1 \left(f^{(n+1)}(x) \right)^q t^{n-\frac{\alpha}{k}} h^{\frac{1}{r}}(t) dt \right)^r \right\}^{\frac{1}{rq}} \\
& = \left(\frac{1}{n-\frac{\alpha}{k}+1} \right)^{1-\frac{1}{q}} \frac{1}{\eta(\varphi(b), \varphi(a), m)} \\
& \times \left\{ |\eta(\varphi(x), \varphi(a), m)|^{n-\frac{\alpha}{k}+1} \left[m \left(f^{(n+1)}(a) \right)^{rq} I^r(h(t); r, \alpha, k, n) \right. \right. \\
& \quad \left. \left. + \left(f^{(n+1)}(x) \right)^{rq} I^r(h(1-t); r, \alpha, k, n) \right]^{\frac{1}{rq}} \right. \\
& \quad \left. + |\eta(\varphi(x), \varphi(b), m)|^{n-\frac{\alpha}{k}+1} \left[m \left(f^{(n+1)}(b) \right)^{rq} I^r(h(t); r, \alpha, k, n) \right. \right. \\
& \quad \left. \left. + \left(f^{(n+1)}(x) \right)^{rq} I^r(h(1-t); r, \alpha, k, n) \right]^{\frac{1}{rq}} \right\}.
\end{aligned}$$

So, the proof of this theorem is completed. $\square$

Corollary 3.6. *Under the same conditions as in Theorem 3.5, if we choose $m = k = r = 1$, $\eta(\varphi(y), \varphi(x), m) = \varphi(y) - m\varphi(x)$, $\varphi(x) = x$, $\forall x \in I$ and $f^{(n+1)} \leq K$, we get the following inequality for Caputo fractional derivatives:*

$$\begin{aligned}
& \left| \left[\frac{(x-a)^{n-\alpha} - (x-b)^{n-\alpha}}{b-a} \right] f^{(n)}(x) + (-1)^{n+1} \frac{\Gamma(n-\alpha+1)}{b-a} \left[{}^c D_{x-}^{\alpha} f(a) - {}^c D_{x-}^{\alpha} f(b) \right] \right| \\
& \leq K \left(\frac{1}{n-\alpha+1} \right)^{1-\frac{1}{q}} \left[I(h(t); 1, \alpha, 1, n) + I(h(1-t); 1, \alpha, 1, n) \right]^{\frac{1}{q}} \\
& \times \left[\frac{(x-a)^{n-\alpha+1} + (b-x)^{n-\alpha+1}}{b-a} \right].
\end{aligned}
\tag{3.6}$$

Remark 3.7. Under the same conditions as in Theorem 3.5, if taking $r = 1$ and $h(t) = t^s$, $s \in [0, 1]$; $h(t) = t^{-s}$, $s \in [0, 1]$; $h(t) = t(1 - t)$ or $h(t) = \frac{\sqrt{t}}{2\sqrt{1-t}}$, then we get some special interesting inequalities for Caputo k -fractional derivatives, respectively for generalized relative semi- (m, s) -Breckner-preinvex mappings; generalized relative semi- (m, s) -Godunova-Levin-Dragomir-preinvex mappings; generalized relative semi- (m, tgs) -preinvex mappings and generalized relative semi- m -MT-preinvex mappings.

4. Applications to special means

Definition 4.1. [6] A function $M : \mathbf{R}_+^2 \longrightarrow \mathbf{R}_+$, is called a Mean function if it has the internality property:

$$\min\{x, y\} \leq M(x, y) \leq \max\{x, y\}, \quad \forall x, y \in \mathbf{R}_+.$$

It follows that a mean $M(x, y)$ must have the property $M(x, x) = x$, $\forall x \in \mathbf{R}_+$.

Now, let us consider some means for different positive real numbers α and β .

1. The arithmetic mean:

$$A := A(\alpha, \beta) = \frac{\alpha + \beta}{2},$$

2. The geometric mean:

$$G := G(\alpha, \beta) = \sqrt{\alpha\beta},$$

3. The harmonic mean:

$$H := H(\alpha, \beta) = \frac{2}{\frac{1}{\alpha} + \frac{1}{\beta}},$$

4. The power mean:

$$P_r := P_r(\alpha, \beta) = \left(\frac{\alpha^r + \beta^r}{2} \right)^{\frac{1}{r}}, \quad r \geq 1,$$

5. The identric mean:

$$I := I(\alpha, \beta) = \begin{cases} \frac{1}{e} \left(\frac{\beta^\beta}{\alpha^\alpha} \right), & \alpha \neq \beta; \\ \alpha, & \alpha = \beta. \end{cases}$$

6. The logarithmic mean:

$$L := L(\alpha, \beta) = \frac{\beta - \alpha}{\ln(\beta) - \ln(\alpha)},$$

7. The generalized log-mean:

$$L_p := L_p(\alpha, \beta) = \left[\frac{\beta^{p+1} - \alpha^{p+1}}{(p+1)(\beta - \alpha)} \right]^{\frac{1}{p}}; \quad p \in \mathbf{R} \setminus \{-1, 0\},$$

8. The weighted p -power mean:

$$M_p \left(\begin{matrix} \alpha_1, & \alpha_2, & \cdots, & \alpha_n \\ u_1, & u_2, & \cdots, & u_n \end{matrix} \right) = \left(\sum_{i=1}^n \alpha_i u_i^p \right)^{\frac{1}{p}},$$

where $0 \leq \alpha_i \leq 1$, $u_i > 0$ ($i = 1, 2, \dots, n$) with $\sum_{i=1}^n \alpha_i = 1$. It is well known that L_p is monotonic nondecreasing over $p \in \mathbf{R}$ with $L_{-1} := L$ and $L_0 := I$. In particular, we have the following inequality $H \leq G \leq L \leq I \leq A$. Now, let a and b be positive real numbers such that $a < b$. Consider the function $M := M(\varphi(x), \varphi(y)) : [\varphi(x), \varphi(x) + \eta(\varphi(y), \varphi(x))] \times [\varphi(x), \varphi(x) + \eta(\varphi(y), \varphi(x))] \rightarrow \mathbf{R}_+$, which is one of the above mentioned means, $h : [0, 1] \rightarrow [0, +\infty)$ and $\varphi : I \rightarrow K$ are continuous functions. Therefore one can obtain various inequalities using the results of Sect. 3 for these means as follows: Replace $\eta(\varphi(y), \varphi(x), m)$ with $\eta(\varphi(y), \varphi(x))$ and setting $\eta(\varphi(y), \varphi(x)) = M(\varphi(x), \varphi(y))$ for value $m = 1$ and $\forall x, y \in I$ in (3.3) and (3.5), one can obtain the following interesting inequalities involving means:

$$\begin{aligned} \left| I_{f, M(\cdot, \cdot), \varphi}(x; \alpha, k, n, 1, a, b) \right| &\leq \left(\frac{1}{(n - \frac{\alpha}{k})p + 1} \right)^{\frac{1}{p}} \frac{1}{M(\varphi(a), \varphi(b))} \left(\int_0^1 h^{\frac{1}{r}}(t) dt \right)^{\frac{1}{q}} \\ &\times \left\{ M^{n - \frac{\alpha}{k} + 1}(\varphi(a), \varphi(x)) \left[\left(f^{(n+1)}(a) \right)^{rq} + \left(f^{(n+1)}(x) \right)^{rq} \right]^{\frac{1}{rq}} \right. \end{aligned}$$

$$(4.1) \quad + M^{n-\frac{\alpha}{k}+1}(\varphi(b), \varphi(x)) \left[\left(f^{(n+1)}(b) \right)^{rq} + \left(f^{(n+1)}(x) \right)^{rq} \right]^{\frac{1}{rq}},$$

$$\begin{aligned} |I_{f,M(\cdot,\cdot),\varphi}(x; \alpha, k, n, 1, a, b)| &\leq \left(\frac{1}{n - \frac{\alpha}{k} + 1} \right)^{1-\frac{1}{q}} \frac{1}{M(\varphi(a), \varphi(b))} \\ &\times \left\{ M^{n-\frac{\alpha}{k}+1}(\varphi(a), \varphi(x)) \left[\left(f^{(n+1)}(a) \right)^{rq} I^r(h(t); r, \alpha, k, n) \right. \right. \\ &\quad \left. \left. + \left(f^{(n+1)}(x) \right)^{rq} I^r(h(1-t); r, \alpha, k, n) \right]^{\frac{1}{rq}} \right. \\ &\quad \left. + M^{n-\frac{\alpha}{k}+1}(\varphi(b), \varphi(x)) \left[\left(f^{(n+1)}(b) \right)^{rq} I^r(h(t); r, \alpha, k, n) \right. \right. \\ (4.2) \quad &\quad \left. \left. + \left(f^{(n+1)}(x) \right)^{rq} I^r(h(1-t); r, \alpha, k, n) \right]^{\frac{1}{rq}} \right\}. \end{aligned}$$

Letting $M(\varphi(x), \varphi(y)) := A, G, H, P_r, I, L, L_p, M_p, \forall x, y \in I$, in (4.1) and (4.2), we get the inequalities involving means for a particular choices of $(n+1)$ -differentiable generalized relative semi- $(r; 1, h)$ -preinvex mapping $\left(f^{(n+1)} \right)^q$. The details are left to the interested reader.

5. Conclusions

Motivated by this new interesting class of generalized relative semi- $(r; m, h)$ -preinvex mappings we can indeed see to be vital for fellow researchers and scientists working in the same domain. We conclude that our methods considered here may be a stimulant for further investigations concerning Hermite-Hadamard, Ostrowski and Simpson type integral inequalities for various kinds of preinvex functions involving classical integrals, Riemann-Liouville fractional integrals, k -fractional integrals, local fractional integrals, fractional integral operators, Caputo k -fractional derivatives, q -calculus, (p, q) -calculus, time scale calculus and conformable fractional integrals.

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