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Spline collocation approach to study Brachistochrone problem

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Abstract

In this paper authors discussed a problem of quickest descent, the Brachistochrone curve. Spline collocation method is used to solve the non-linear boundary value problem. The numerical results obtained are compared with the transformation method to show effectiveness and accuracy of this method.

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1. Introduction

Johann Bernoulli posed the problem of the Brachistochrone to the readers of *ActaEruditorum* in June; 1696. He has also published its solution in the next year. After deriving the differential equation for the curve five mathematicians responded with solutions Isaac Newton, Jacob Bernoulli, Gottfried Leibniz, Ehrenfried Walther von Tschirnhaus and Guillaume de l'Hopital. (Dunham[8]). Solution of the classical Brachistochrone problem using techniques of calculus of variations, (Gelfand and Fomin [11]) or geometrical optics (Erlichson [9]). Problem of finding Brachistochrone curve with coulomb friction lying on a vertical plane in a uniform gravitational field has been discussed by Ashby et al.[2], Hayen [13], Heijden and Diepstraten [19] and further generalized to a curve with friction lying on a cylinder by Covic and Veskovic[5] and Vratanaar and Saje [21] studied related problem of finding the Brachistochrone curve in a non conservative resistance field. Problem of finding Brachistochrone curve on cylinders in uniform fields has been solved by Yamini and Mulhem [22]. Brachistochrone curve on spheres mentioned in the unpublished work of Palmieri [14]. A generalization to non-uniform fields has been discussed by several authors (Arvind[1], Denman [7]). Venezian[20] successfully obtained solution of Brachistochrone for linear radial fields. Parnovsky[15] and Tee[18] got solutions for inverse square radial fields. Further, generalizations of this problem to include special relativistic effects have been initiated by Farina[10], Goldstein and Bender[12], Scarpello and Ritelli[16]. A study of point-to-point optimal control problem where the objective is to transfer the state of a dynamical system with minimum cost from one point to another point has been discussed by Sussmann and Willem[17]

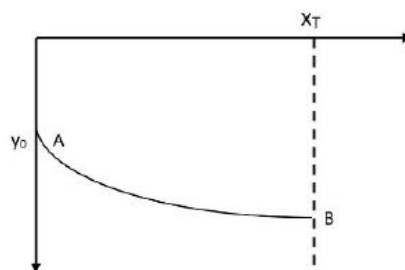


FIGURE 1. Brachistochrone Problem

Cal de Boor[6] was first person to introduced bicubic splines method. Bickley[3] brought forward a useful aspect of spline functions in linear two point boundary value problem. In continuation of this study, Blue[4] discussed the applicability of spline functions to non-linear differential equations. In this paper, spline collocation method has been applied to generate approximate solutions of the governing equations of the Brachistochrone problem.

2. Formulation of Brachistochrone Problem

Consider two points A and B in a space containing a constant gravitational forced field as shown in figure. Point A is the initial point at $(0, y_0)$ a fixed point whereas point B lies anywhere on the terminal line $x = x_T$ will the restraint that $\frac{dy}{dx} = 0$ at a point. It is the purpose of the methods to find a frictionless path from A to B along which a particle will slide in minimum time. Since gravitational force I the only force acting on the mass, the time of descent T is given by,

$$(2.1) \quad T = \int_0^{x_T} \frac{ds}{V} = \int_0^{x_T} \sqrt{\frac{1 + y'^2}{2gy}} dx$$

the integrand is defined by

$$(2.2) \quad F = \sqrt{\frac{1 + y'^2}{2gy}} dx$$

we get,

$$(2.3) \quad T = \int_0^{x_T} F dx$$

To minimize the time of descent T ,Eulers equation is applied to the integrand F ,

$$(2.4) \quad \frac{\partial F}{\partial y} - \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) = 0$$

Equation (2.4) can be reduce to,

$$(2.5) \quad 2y \frac{d^2 y}{dx^2} + \left[1 + \left(\frac{dy}{dx} \right)^2 \right] = 0$$

with boundary conditions,

$$(2.6) \quad y(0) = y_0, \frac{dy(x_T)}{dx} = 0$$

Equation (2.5) subject to the boundary condition (2.6), is the two point non linear boundary value problem.

Let us introduce following transformations,

$$(2.7) \quad \eta = 1 - \frac{x}{x_T}, f = \frac{y}{x_T}$$

then equation (2.5) reduces to,

$$(2.8) \quad 2f \frac{d^2 f}{d\eta^2} + \left[1 + \left(\frac{df}{d\eta} \right)^2 \right] = 0$$

subject to the boundary conditions

$$(2.9) \quad \frac{df(0)}{d\eta} = 0, f(1) = 3$$

3. General Spline Collocation Method

Use of Spline functions with moments for the solution of nonlinear differential equation was suggested by Blue (1969).

Consider a linear two point boundary value problem

$$(3.1) \quad y''(x) + p(x)y'(x) + q(x)y(x) = r(x)$$

subject to the boundary conditions

$$(3.2) \quad G_1[y(a), y'(a)] = 0 \quad \text{at } x = a$$

$$(3.3) \quad G_2[y(b), y'(b)] = 0 \quad \text{at } x = b$$

Since $s(x)$ is a cubic Spline interpolating $y(x)$ given by equation (3.1), we have $s(x_i) = y(x_i)$, and $s''(x)$ is a linear function.

Let us define $s''(x)$ in the subinterval $[x_i, x_{i+1}]$ of $[a, b]$ as follows:

$$(3.4) \quad s''(x) = y''_{i+1} \frac{x - x_i}{h_{i+1}} + y''_i \frac{x_{i+1} - x}{h_{i+1}}, i = 0, 1, 2, \dots, n-1$$

where $h_{i+1} = x_{i+1} - x_i$

On applying two successive integrations on equation (3.4) produced a cubic spline $s(x)$ in $[x_i, x_{i+1}]$ given by

$$(3.5) \quad s(x) = y''_{i+1} \frac{(x - x_i)^3}{6h_{i+1}} + y''_i \frac{(x_{i+1} - x)^3}{6h_{i+1}} + A_1 \frac{x_{i+1} - x}{h_{i+1}} + B_1 \frac{x - x_i}{h_{i+1}}$$

where A_1 and B_1 are constants to be determined.

Equation (3.5) can be written as

$$(3.6) \quad s(x) = y''_{i+1} \frac{(x - x_i)^3}{6h_{i+1}} + y''_i \frac{(x_{i+1} - x)^3}{6h_{i+1}} + \left(y_i - \frac{h_{i+1}^2}{6} y''_i \right) \frac{x_{i+1} - x}{h_{i+1}} + \left(y_{i+1} - \frac{h_i^2}{6} y''_{i+1} \right) \frac{(x - x_i)}{h_{i+1}}$$

Similarly $s(x)$ can be obtained in the interval $[x_{i-1}, x_i]$ as

$$(3.7) \quad s(x) = y''_i \frac{(x - x_{i-1})^3}{6h_i} + y''_{i-1} \frac{(x_i - x)^3}{6h_i} + \left(y_i - \frac{h_i^2}{6} y''_i \right) \frac{x - x_{i-1}}{h_i} + \left(y_{i-1} - \frac{h_i^2}{6} y''_{i-1} \right) \frac{(x_i - x)}{h_i}$$

From equation (3.5), we obtained $s'(x)$ at $x = x_i$ denoted by $s'(x_{i+})$, arrived at

$$(3.8) \quad s'(x_{i+}) = -\frac{h_{i+1}}{3} y''_i - \frac{h_{i+1}}{6} y''_i + \frac{y_{i+1} - y_i}{h_{i+1}}, \quad i = 0, 1, 2, \dots, n-1.$$

and

$$(3.9) \quad s'(x_{i-}) = \frac{h_i}{6} y''_{i-1} - \frac{h_i}{3} y''_i + \frac{y_i - y_{i-1}}{h_i}, \quad i = 1, 2, \dots, n.$$

Continuity of $s''(x)$ at $x = x_i$ requires that $s''(x_{i+}) = s''(x_{i-})$ so that,

$$h_i y_{i+1} - (h_i + h_{i+1}) y_i + h_{i+1} y_{i-1} = h_i h_{i+1} \left(\frac{h_i}{6} y''_{i-1} + \frac{h_i + h_{i+1}}{6} y''_i + \frac{h_{i+1}}{6} y''_{i+1} \right),$$

(3.10) $i = 1, 2, \dots, n-1.$

Equation (3.10) gives a system of $(n-1)$ equations in $(n+1)$ variables $y_i, i = 0, 1, 2, \dots, n$ to be determined. Therefore, the moments $y''_i, i = 0, 1, 2, \dots$ can be obtained from equation (3.10) if a curve is initially fitted to the data.

Let us express the equation (3.1) in the form

$$(3.11) \quad y''(x) = f(x, y, y')$$

subject to the boundary conditions (3.2) and (3.3). From these boundary conditions, it can be seen that there are four pairs of boundary conditions as possible combinations viz.

- (i) $y(a) = K$; $y(b) = L$
- (ii) $y(a) = K$; $y'(b) = L$
- (iii) $y'(a) = K$; $y(b) = L$
- (iv) $y'(a) = K$; $y'(b) = L$

For case (iii) equation (3.10) reduces in general form,

$$\begin{aligned} y_0 - y_1 &= \frac{h_1^2}{6} (2y''_0 + y''_1) - h_1 y'_0 \\ h_i y_{i+1} - (h_i + h_{i+1}) y_i + h_{i+1} y_{i+1} &= \\ h_i h_{i+1} \left(\frac{h_i}{6} y''_{i-1} + \frac{h_i + h_{i+1}}{3} y''_i + \frac{h_{i+1}}{6} y''_{i+1} \right). & \quad i = 1, 2, 3, \dots, n-1. \\ y_{n-2} - 2y_{n-1} &= -\frac{h_n^2}{6} (y''_{n-2} + 4y''_{n-1} + y''_n) - y_n \end{aligned}$$

Similarly, we are able to deal with remaining cases. Now in order to obtain a solution to equation (2.8) with the boundary conditions given in equations (2.9) we fit a straight line $y(x) = mx + c$ through the boundary points, this is an initial guess, the moments are calculated from the relation (3.11) at the nodal points $x = x_i$. These moments y''_i are now utilized to evaluate $y_i, i = 0, 1, 2, \dots, n$, through the relation along with the two additional equations from boundary conditions. This can be furnished by solving merely a tridiagonal system of equations. The results so obtained can again be improved by continuing the same process till the desired solutions are found or two successive iterations produce the same values.

4. Solution of Brachistochrone Problem by Spline

Rewrite the differential equation (2.8) in the form of

$$\frac{d^2 f}{d\eta^2} = - \frac{\left[1 + \left(\frac{df}{d\eta} \right)^2 \right]}{2f}$$

To find the Spline approximation $s(x)$ of $f(\eta)$ described in equation (2.9) satisfying the boundary conditions, a line $f(\eta) = u\eta + v$ is assumed to be the first approximation to start with the iterative scheme. The straight line $f(\eta) = 3$ can be fitted through points $x = 0$ and $x = 1$. The calculation of, $y_i, i = 0, 1, 2, \dots, N$ is to be carried out through the solution of tridiagonal system of $(N + 1)$ equations. The solution obtained the tridiagonal system of $(N + 1)$ equation is given below:

Values of η	Solution using Spline Method	Solution using Transformation Method
0	3.08457122	3.0864
0.092	3.083877691	3.0763
0.185	3.081796739	3.0742
0.277	3.078327057	3.0707
0.369	3.073466128	3.0658
0.462	3.067210162	3.0596
0.554	3.059554079	3.0519
0.646	3.050491483	3.0429
0.738	3.04001463	3.0324
0.831	3.02811439	3.0205
0.923	3.014780198	3.0071
1	3	3

FIGURE 2. Comparison between Spline Solution and Trasformstion Method

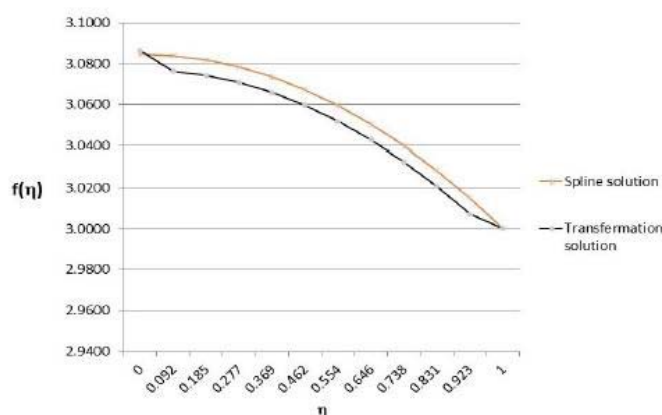


FIGURE 3. Graphical representation of Spline Solution and Transformation Method

5. Conclusion

The brachistochrone problem can be fit well by an equally space knots. The spline collocation method is considered to find the approximate solution of the brachistochrone problem. An analysis is presented without doing any linearization for solving highly non-linear governing equation. To validate the present system, the result determined by the above analysis has been compared with solution obtained by transform method. The result indicates that spline collocation method gives the best approximation for the non-linear problems without any assumption and linearization. The method is easy to implement and yield very accurate results.

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