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On the uniform ergodic theorem in invariant subspaces

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Abstract

Let T be a bounded linear operator on a Banach space X into itself.
In this paper, we study the uniform ergodicity of the operator $T|_Y$ when Y is a closed subspace invariant under T. We show that if T satisfies $\lim\limits_{n\to\infty} \frac{\|T^n\|}{n} = 0$, then T is uniformly ergodic on X if and only if the restriction of T to some closed subspace $Y \subset X$, invariant under T and $R[(I-T)^k] \subset Y$ for some integer $k \geq 1$, is uniformly ergodic. Consequently, we obtain other equivalent conditions concerning the theorem of Mbekhta and Zemànek [9, theorem 1], also to the theorem of the Gelfand-Hille type.

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1. Introduction

Throughout this paper, $\mathbf{B}(X)$ denotes the Banach algebra of all bounded linear operators on a Banach space X into itself. For $T \in \mathbf{B}(X)$, we denote by $R(T)$, $N(T)$, $\sigma(T)$, $r(T)$ and $\rho(T)$, the range, the kernel, the spectrum, the spectral radius and the resolvent set of T, respectively. A closed subspace Y of X is called invariant under T or shortly T-invariant if $T(Y) \subset Y$.

By $M_n(T) = \frac{1}{n}\sum_{k=0}^{n-1} T^k$, for $n \in N$, denote the sequence of the arithmetic means of the powers of T. An operator $T \in \mathbf{B}(X)$ is called uniformly ergodic (resp. mean ergodic) if $\{M_n(T)\}$ is uniformly (resp. strongly) convergent in $\mathbf{B}(X)$.

Recall that, $T \in \mathbf{B}(X)$ is called Cesàro bounded if $\sup_n \|M_n(T)\| < \infty$. Then a every uniformly ergodic operator T is necessarily Cesàro bounded, and $\frac{\|T^n\|}{n} \to 0$ when $n \to \infty$, by the following identity:

$$T^n = (n+1)M_{n+1}(T) - M_n(T).$$

In 1938, Yosida [12] showed that when $\{M_n(T)\}$ converge strongly to $P \in \mathbf{B}(X)$ (i.e. T is mean ergodic) then P is the projection onto the space $N(I-T)$ along $\overline{R(I-T)}$, corresponding to the ergodic decomposition (see [7, Theorem 2.1.3.])

$$X = N(I-T) \oplus \overline{R(I-T)}.$$

Much of ergodic operator theory is concerned with determining when, conversely, the convergence to 0 of $\{\frac{T^n}{n}\}$ and/or the boundedness of $\{M_n(T)\}$ implies the convergence of $\{M_n(T)\}$ in the operator topology considered. A fundamental theorem due to M. Lin [8] (see also [7, Theorem 2.2.1. p.87]) says that when $\frac{\|T^n\|}{n} \to 0$, T is uniformly ergodic if and only if $R(I-T)$ is closed. In this case $R[(I-T)^k]$ is closed for every integer $k \geq 1$.

Mbekhta and Zemànek [9], still assuming $\frac{\|T^n\|}{n} \to 0$ when $n \to \infty$, have showed that T is uniformly ergodic if and only if $R[(I-T)^k]$ is closed for some integer $k \geq 1$. The case $k = 2$, is due to Dunford [4]. On the other hand, Koliha in [6, Theorem 4] proved that $\{T^n, n \geq 0\}$ converge uniformly if and only if $r(T) \leq 1$, $\sigma(T) \cap \Gamma \subset \{1\}$ and 1 is a pole of the resolvent of order at most 1 (Γ is the unit circle). The latter condition is equivalent to uniform ergodicity (see [4]). In the same case, Mbekhta and Zemànek give another condition equivalent to Koliha's theorem [6], presented in the following theorem:

Theorem 1.1. *[9, Corollary 3] Let T be a bounded linear operator on a complex Banach space X. Then the following conditions are equivalent:*

1. *$\{T^n\}_{n\geq 0}$ converge uniformly in $\mathbf{B}(X)$,*
2. *T is uniformly ergodic and $\sigma(T)\cap\Gamma\subset\{1\}$,*
3. *$\lim\limits_{n\to\infty}\frac{\|T^n\|}{n}=0$, $\sigma(T)\cap\Gamma\subset\{1\}$ and $R[(I-T)^k]$ is closed for some integer $k\geq 1$,*
4. *$\|T^n-T^{n+1}\|\to 0$ when $n\to\infty$ and $R[(I-T)^k]$ is closed for some integer $k\geq 1$.*

M. E. Becker [2] gives an example of a subspace of X which is invariant under T denoted by $X_1 := \{x\in X : \lim\limits_{n\to\infty}\sum_{k=1}^{n}\frac{T^k x}{k}$ *exists* $\}$. By the Hahn-Banach theorem, $X_1\subset\overline{R(I-T)}$. She proved [2, Remark 2] that if $\lim\limits_{n\to\infty}\frac{\|T^n\|}{n}=0$ and $R(I-T)\subset X_1$, then T is uniformly ergodic if and only if X_1 is closed. When T is power-bounded, $R(I-T)\subset X_1$.

This paper is organized as follows. In section 2, we give some definitions and fundamental properties of ergodic operator theory. In section 3, we show that for every operator $T\in\mathbf{B}(X)$ which satisfies $\lim\limits_{n\to\infty}\frac{\|T^n\|}{n}=0$, T is uniformly ergodic if and only if there exists a closed subspace $Y\subset X$, T-invariant, contains $R[(I-T)^k]$ for some integer $k\geq 1$ and $T|_Y$ is uniformly ergodic. This result was also obtained for an operator satisfying $\lim\limits_{n\to\infty}\frac{T^n x}{n}=0$ for all $x\in X$, in the weak operator topology.

2. Preliminaries

In this section, we briefly review the definitions and some basic properties which we will need in the sequel.

For $T\in\mathbf{B}(X)$, Recall that the ascent of T is the smallest non negative integer n such that $N(T^n)=N(T^{n+1})$, if no such n exists, we write $asc(T)=\infty$. Similarly, the descent of T is the smallest nonnegative integer n such that $R(T^n)=R(T^{n+1})$, if there is no such n, we write $des(T)=\infty$ (see e.g. [1, Definition 3.1] or [3] p.10). It may be instructive to note that one may have $des(T)\leq n<\infty$ without $R(T^n)$ begin closed (see example at the end of [5]). We mention the following characterization:

Lemma 2.1. [6, *Lemma 1.1*] *Given a non-negative integer d and $T \in$ $\mathbf{B}(X)$, we have*
(i) $asc(T) \leq d < \infty$ if and only if $R(T^d) \cap N(T^m) = \{0\}$, for some (equivalently, all) integer $m \geq 1$.
(ii) $des(T) \leq d < \infty$ if and only if $R(T^m) + N(T^d) = X$, for some (equivalently, all) integer $m \geq 1$.

If both $asc(T)$ and $des(T)$ are finite, then they are equal, and $X = R(T^d) \oplus N(T^d)$

where $d = asc(T) = des(T)$. For more details see Lemma 1.4.6, Lemma 3.2.4 and Proposition 1.4.3 from [3] or p.330 from [11].

Lemma 2.2. *Let T be a bounded linear operator on a real or complex Banach space X. Let $Y \subset X$ be a closed subspace which is T-invariant and $T|_Y$ be the restriction of T to Y.*

If T is uniformly ergodic (resp. mean ergodic), then $T|_Y$ is uniformly ergodic (resp. mean ergodic).

Proof:
Let $Y \subset X$ be a closed subspace which is T-invariant, and denote $S = T|_Y$. Since Y is T-invariant, it is also T^n-invariant for each $n \in N$, thus it is also invariant under $S_n = (T^n|_Y)$. Using that Y is closed, we get that both kind of limits are inside of Y.

Thus, S is uniformly ergodic. □

Proposition 2.1. *[5, Remark 1.4] Let T be a bounded linear operator on a complex Banach space X. If T satisfies either of the following assumptions:*
(i) T is Cesàro bounded operator, or
(ii) $\frac{T^n}{n}$ converge weakly to 0.

Then the spectral radius of T is not greater than 1. Moreover, $N(I - T) \cap R(I - T) = \{0\}$, which in turn yields $asc(I - T) \leq 1$.

The next lemma can be considered a version of the Gelfand-Hille theorem [13].

Lemma 2.3. *[9, Corollary 2] Let T be a bounded linear operator on a complex Banach space X such that $\sigma(T) = \{1\}$. If T is uniformly ergodic, then $T = I$.*

3. Main results

The first main result of this paper is the following theorem:

Theorem 3.1. *Let T be a bounded linear operator on a real or complex Banach space X such that $\lim\limits_{n\to\infty} \frac{\|T^n\|}{n} = 0$. Let $T|_Y$ be the restriction of T to a closed subspace $Y \subset X$, which is invariant under T, and $R(I-T) \subset Y$.*

If $T|_Y$ is uniformly ergodic, then T is uniformly ergodic.

Proof:
Let Y be a closed subspace of X (not trivial) which is invariant under T, assume that $R(I-T)$ is included in Y, and suppose that $T|_Y$ is uniformly ergodic.

Put $Z := \overline{R(I-T)}$. Then $Z \subset Y$, so $T|_Z$ is uniformly ergodic on Z. The limit is 0 on $R(I-T)$, so by continuity of the limit $\|M_n(T|_Z)\| \to 0$. Hence $(I_Z - T|_Z)Z = Z$. Then

$$R(I-T) \subset Z = (I_Z - T|_Z)Z \subset R(I-T),$$

which implies $R(I-T) = Z$, so $R(I-T)$ is closed and T is uniformly ergodic by [8].□

Corollary 3.1. *Let T be a bounded linear operator on a complex Banach space X such that $\lim\limits_{n\to\infty} \frac{\|T^n\|}{n} = 0$. Let $Y \subset X$ be a closed subspace T-invariant such that $R[(I-T)^k] \subset Y$, for some integer $k \geq 1$. If $T|_Y$ is uniformly ergodic, then T is uniformly ergodic.*

Proof:
Let $Y \subset X$ be a closed subspace T-invariant such that $R[(I-T)^k] \subset Y$, for some integer $k \geq 1$. Let's denote $S = T|_Y$, and assume that S is uniformly ergodic. By uniform ergodicity, $Y = R(I-S) \oplus N(I-S)$ with $R(I-S)$ closed [8].

We will show $R[(I-T)^k] = R(I-S)$. Since $R[(I-S)^k] \subset R[(I-T)^k]$, we need to prove only $R[(I-T)^k] \subset R[(I-S)^k]$. Since $R[(I-T)^k] \subset Y = R(I-S) \oplus N(I-S)$ by assumption, for $x \in X$ write $(I-T)^k x = (I-S)y + z$ with $y, z \in Y$ and $Tz = Sz = z$. Then

$$M_n(T)(I-T)^k x = M_n(S)(I-S)y + z,$$

and $\|T^n\|/n \to 0$ yields that $z = 0$, so $R[(I-T)^k] \subset R(I-S)$. But by uniform ergodicity $I - S$ is invertible on $R(I-S)$, which is closed, so $R[(I-S)^k] = R(I-S)$.

Hence $R[(I-T)^k] \subset R[(I-S)^k]$ and equality holds, so $R[(I-T)^k] = R(I-S)$ is closed.

Hence T is uniformly ergodic by [9]. $\square$

In the following theorem, we give a generalization of Theorem 3.1 and Corollary 3.1 where X is a real or complex Banach space and $T \in \mathbf{B}(X)$ satisfying $\lim\limits_{n\to\infty} \frac{T^n x}{n} = 0$, for all $x \in X$, in the weak operator topology.

Theorem 3.2. *Let T be a bounded linear operator on a real or complex Banach space X such that $\lim\limits_{n\to\infty} \frac{T^n x}{n} = 0$, for all $x \in X$, in the weak operator topology. Let $Y \subset X$ be a closed subspace T-invariant such that $R[(I-T)^k] \subset Y$, for some integer $k \geq 1$.*

If $T|_Y$ is uniformly ergodic, then T is uniformly ergodic.

Proof:

Let $Y \subset X$ be a closed subspace T-invariant such that $R[(I-T)^k] \subset Y$, for some integer $k \geq 1$. Let's denote $S = T|_Y$, and assume that S is uniformly ergodic.

Using the technique employed in the proof of Corollary 3.1, we get $R[(I-T)^k] = R(I-S)$. Now, we prove that if $k > 1$, then also $R[(I-T)^{k-1}] = R(I-S)$.

Let $x \in X$, then $(I-T)^k x = (I-S)y = (I-T)y$ for some $y \in Y$, and by uniform ergodicity of S, we can take $y \in R(I-S)$. Then $(I-T)\Big[(I-T)^{k-1}x - y] = 0$, so $(I-T)^{k-1}x = y + z$ with $y \in R(I-S)$ and $z \in N(I-T)$.

Applying $M_n(T)$ to both sides and using $T^n/n \to 0$ weakly, we get $z = 0$, which implies $R[(I-T)^{k-1}] \subset R(I-S)$. We proceed by induction $R(I-T) = R(I-S)$.

Then $M_n(T|_{R(I-T)}) = M_n(S|_{R(I-S)})$, yields that $\lim\limits_{n\to\infty} \frac{\|T^n\|}{n} = 0$ by uniform ergodicity of S.
Since $R(I-T) = R(I-S)$ is closed, T is uniformly ergodic by [8].

□

Theorem 3.3. *Let T be a Cesàro bounded operator on a real or complex Banach space X,*
and Y be a closed subspace of X which is invariant under T.
If there exists $k \geq 1$ such that $R[(I-T)^k] \subset Y$ and $T|_Y$ is uniformly ergodic, then $R(I-T)$ is closed and $X = R(I-T) \oplus N(I-T)$. Moreover, T is uniformly ergodic.

Proof:

Let T be a Cesàro bounded operator on a real or complex Banach space X, then by Proposition 2.1 we have $R(I-T) \cap N(I-T) = \{0\}$, hence $asc(I-T) \leq 1$.

Let $Y \subset X$ be a closed subspace which is invariant under T and denote by $S = T|_Y$.

Assume that there is an integer $k \geq 1$ such that $R[(I-T)^k] \subset Y$ and S is uniformly ergodic. Since $N(I-S) \subset N(I-T)$, then $R(I-T) \cap N(I-S) = \{0\}$. Since S is uniformly ergodic, then $Y = R(I-S) \oplus N(I-S)$ and $R(I-S)$ is closed. By assumption we infer that $R[(I-T)^k] \subset R(I-S)$.

The uniform ergodicity of S implies that $I-S$ is invertible on $R(I-S)$, so $R[(I-S)^k] = R(I-S)$. Hence

$$R[(I-T^k)] \subset R(I-S) = R[(I-S)^k] \subset R[(I-T)^k],$$

which shows that $R[(I-T)^k] = R(I-S)$ and this yields $R[(I-T)^n] = R(I-S)$ for all $n \geq k$, hence $des(I-T) < \infty$. Thus by [11, Theorem V.6.2] $X = R(I-T) \oplus N(I-T)$.

Therefore, for all $n \geq 1$, $R(I-T) = R[(I-T)^n] = R(I-S)$. Consequently,

$$M_n(T|_{R(I-T)}) = M_n(S|_{R(I-S)})$$

which implies $\frac{\|T^n\|}{n} \to 0$ by uniform ergodicity of S. Since $R(I-T) = R(I-S)$ is closed, then T is uniformly ergodic by [8].□

Corollary 3.2. *Let T be a Cesàro bounded operator on a real or complex Banach space X. T is uniformly ergodic if (and only if) the restriction of T to $\overline{R[(I-T)^k]}$ is uniformly ergodic, for some integer $k \geq 1$.*

Now, we recall the following lemma that was introduced in [10, Theorem 2.2(ii)].

Lemma 3.1. *Let T be a Cesàro bounded operator on a complex Banach space X with $\sigma(T) \cap \Gamma \subset \{1\}$, where Γ is the unit circle, then* $\lim\limits_{n\to\infty} \frac{\|T^n\|}{n} = 0$.

In accordance with the previous lemma, we infer from Theorem 1.1 and Theorem 3.3 the following result:

Corollary 3.3. *Let T be a bounded linear operator on a complex Banach space X.*
Then the following conditions are equivalent:

1. *$\{T^n\}_{n\geq 0}$ converge uniformly in $\mathbf{B}(X)$,*
2. *T is uniformly ergodic and $\sigma(T) \cap \Gamma \subset \{1\}$,*
3. *T is Cesàro bounded, $\sigma(T) \cap \Gamma \subset \{1\}$ and $R[(I-T)^k]$ is closed, for some integer $k \geq 1$,*
4. *T is Cesàro bounded, $\sigma(T) \cap \Gamma \subset \{1\}$ and $T|_Y$ is uniformly ergodic for some closed subspace $Y \subset X$ which is T-invariant and $R[(I-T)^k] \subset Y$, for some integer $k \geq 1$,*
5. *T is Cesàro bounded, $\sigma(T) \cap \Gamma \subset \{1\}$ and the restriction of T to $\overline{R[(I-T)^k]}$, for some integer $k \geq 1$, is uniformly ergodic.*

In analogy with the Gelfand-Hille theorem, we give the following theorem:

Theorem 3.4. *Let T be a bounded linear operator on a complex Banach space X such that $\sigma(T) = \{1\}$. Let $Y \subset X$ be a closed subspace and T-invariant such that $R[(I-T)^k] \subset Y$, for some integer $k \geq 1$. If $T|_Y$ is uniformly ergodic, then $(I-T)^{k+1} = 0$.*

Proof:
Let $S = T|_Y$ and suppose that there is $k \geq 1$ such that $R[(I-T)^k] \subset Y$.

Assume that S is uniformly ergodic, then $Y = R(I-S) \oplus N(I-S)$ and $R(I-S)$ is closed.

Since $\sigma(S) \subset \sigma(T) = \{1\}$, we have $\sigma(S) = \{1\}$. Lemma 2.3 implies $S = I$, which yields $Y = N(I-S)$. Since $R[(I-T)^k] \subset Y = N(I-S) \subset N(I-T)$, then $R[(I-T)^{k+1}] = \{0\}$, which means $(I-T)^{k+1} = 0$.

□

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