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Graceful centers of graceful graphs and universal graceful graphs

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Abstract

In this paper we define graceful center of a graceful graph. We proved any graph G which admits α-labeling has at least four graceful centers. We also defined a new strong concept of universal graceful graph. Some results on ring sum of two graphs for their graceful labeling are proved.

Key words: *Graceful center of a graceful graph, universal graceful labeling, ring sum of two graphs.*

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1. Introduction

In this paper a (p, q) graph G, we mean $|V(G)| = p$, $|E(G)| = q$ and it is a finite, undirected simple graph. Terms not defined here are used from Harary [3]. Rosa [1] introduced the notion of graceful labeling (β-valuation) and α-labeling of a graph. Any graph G, which admits α-labeling is necessarily a bipartite graph; such graph is known as α-graceful graph.

A Graph $G = (V, E)$ is said to be a graceful graph if G admits a function $f : V(G) \longrightarrow \{0, 1, \dots, q\}$ is injective and the induce edge function $f^\star : E(G) \longrightarrow \{1, 2, \dots, q\}$ denoted by $f^\star(e = uv) =\mid f(u) - f(v) \mid$ is bijective, $\forall\ e = uv \in E$. Here function f is called graceful labeling of graph G.

Let G be a graceful graph with a graceful labeling $f : V(G) \longrightarrow \{0, 1, 2, \dots, q\}$. A vertex $v \in V(G)$ is called a graceful center of G if $f(v) = 0$ or $f(v) = q$. Any graceful graph G with graceful labeling f has at least two graceful centers. It is obvious that $q - f$ is also a graceful labeling for G and it produce same graceful centers for G. If a graph G has precisely two graceful centers, then they are adjacent in G, as they produce the edge label q under f.

A graph G is said to be a universal graceful graph if for any $v \in V(G)$, v is a graceful center for G with respect to some graceful labeling of G. Also we call G is a universal α-graceful graph if for any $v \in V(G)$, v is a graceful center for G with respect to some α-graceful labeling of G.

Every cycle C_n ($n \equiv 0 \pmod 4$), star $K_{1,n}$ are universal graceful graphs as well they are universal α-graceful graphs. While C_n ($n \equiv 3 \pmod 4$) and wheel W_n are universal graceful graphs, but they are not universal α-graceful graphs, as symmetric structure of above said graphs and their graceful labeling are given in Rosa [1], Hoede and Kuiper [2].

Ring sum of two graphs G_1 and G_2 denoted $G_1 \oplus G_2$, where $G_1 \oplus G_2 = \Big(V(G_1) \cup V(G_2),\ E(G_1) \cup E(G_2) - E(G_1) \cap E(G_2)\Big)$. Throughout this paper we consider the ring sum of a graceful graph G with $K_{1,n}$ by considering one vertex v which is a graceful center of G and the apex vertex of $K_{1,n}$ as a common vertex. Rest vertex of G and $K_{1,n}$ are distinct. If $H = G \oplus K_{1,n}$ then $H = \Big(V(G) \cup V(K_{1,n}),\ E(G) \cup E(K_{1,n})\Big)$, as $E(G) \cap E(K_{1,n}) = \phi$. Thus, $|V(H)| = |V(G)| + n$ and $|E(H)| = |E(G)| + n$.

2. Main Results

Theorem - 1 : Any α-graceful graph G has atleast four graceful centers.

Proof : Let G be an α-graceful graph and $f : V(G) \longrightarrow \{0, 1, 2, \ldots, q\}$ be an α-labeling for G. Since, f is α-labeling for G then $\exists$ an integer k $(0 \leq k < q)$ such that for any $uv \in E(G)$, $\min\{f(u), f(v)\} \leq k < \max\{f(u), f(v)\}$. Thus, $V(G)$ can be partitioned into two parts

$\mathrm{V}_1 = \{v \in V(G)/f(v) \leq k\}$ and
$V_2 = V(G) - V_1 = \{v \in V(G)/f(v) > k\}$.

Take $|V_1| = l$. It is obvious that $|V_2| = p - l$. Moreover, $\exists$ $w_1, w_2 \in V_1$ such that $f(w_1) = 0, f(w_2) = k$ and $\exists$ $w_3, w_4 \in V_2$ such that $f(w_3) = k + 1$ and $f(w_4) = q$.

Here w_1 and w_4 both are graceful centers for G with respect to α-graceful labeling f.

Define $h : V(G) \longrightarrow \{0, 1, 2, \ldots, q\}$ as follows.

h(u)= k-f(u), $\forall v \in V_1$ and
$h(v) = q + k + 1 - f(v), \forall v \in V_2$.

Note that h is injective, as f is an injective map. Further for any $uv \in E(G)$

$\mathrm{h}^\star(uv) = |h(u) - h(v)|$
$= h(v) - h(u)$, assuming $u \in V_1$
$= q + k + 1 - f(v) - k + f(u)$
$= q + 1 - (f(v) - f(u))$
$= q + 1 - f^\star(uv)$.

Therefore, $h^\star : E(G) \longrightarrow \{1, 2, \ldots, q\}$ is also a bijection, as $f^\star$ is a bijective map. Thus, h is also a graceful labeling for G. Infact h is an α-graceful labeling for G, as $\min\{h(u), h(v)\} \leq k \leq \max\{h(u), h(v)\}$, $\forall$ $uv \in E(G)$. Since, $h(w_2) = 0$ and $h(w_3) = q$, w_2 and w_4 are graceful centers for G with respect to α-labeling h. Thus, G has graceful centers w_1, w_2, w_3 and w_4. So, G admits atlest four graceful centers.

Remark : A α-graceful graph G with α-labeling f admits three more α-labelings $q - f, h$ and $q - h$, as discussed in last theorem.

Theorem-2 : If G is a graceful graph, then $G \oplus K_{1,n}$ is also a graceful graph, for all $n \in N$.

Proof : Let $f : V(G) \longrightarrow \{0, 1, 2, \ldots, q\}$ be a graceful labeling for G and $v \in V(G)$ such that $f(v) = 0$. i.e. v is a graceful center for G with respect to f.

Let $H = G \oplus K_1$ by considering vertex v of G and the apex vertex of $K_{1,n}$ as a common vertex in H. Let $V(G) = \{v_1, v_2, \ldots, v_p = v\}$ and $V(K_{1,n}) = \{v, u_1, u_2, \ldots, u_n\}$ with v is the apex vertex of $K_{1,n}$. It is obvious that $V(H) = V(G) \cup \{u_1, u_2, \ldots, u_n\}$ and $E(H) = E(G) \cup \{vu_i/1 \leq i \leq n\}$. i.e. $|V(H)| = p + n$ and $|E(H)| = q + n$.

Without loss of generality we assume here $f(v) = 0$. Otherwise $f(v) = q$ and in this case $q - f$ is a graceful labeling for G with $(q - f)(v) = 0$. In this case v is also a graceful center for G with respect to $q - f$.

Define $h : V(H) \longrightarrow \{0, 1, 2, \ldots, q + n\}$ as follows.

h(w)=f(w), $\forall w \in V(G)$ and
$h(u_i) = q + i, \forall i = 1, 2, \ldots, n$.

Note that h is an injective map, as f is injective. Also for any $uw \in E(H)$, $h^\star(uw) = f^\star(uw) \in \{1, 2, \ldots, q\}$, if $uw \in E(G)$ and $h^\star(uw) = h^\star(vu_i) = q + i$, $\forall\ i = 1, 2, \ldots, n$, if $uw \in E(K_{1,n})$ (assuming $u = v$ and $w = u_i$, for some $i \in \{1, 2, \ldots, n\}$). Therefore range of $h^\star$ is $\{1, 2, \ldots, q+n\}$ and so, it is a bijective map. Hence, h is a graceful labeling for H and H is a graceful graph, for all $n \in N$.

Corollary - 2.1 : $C_n \oplus K_{1,t}$ is graceful, where $t \in N$ and $n \equiv 0, 3 \pmod 4$.

Corollary - 2.2 : $W_n \oplus K_{1,t}$ is graceful, $\forall\ t, n \in N$.

Theorem - 3 : If G is a universal graceful graph, then its one vertex super graph $G \oplus K_2$ is a graceful graph.

Proof : Let $v \in V(G)$ be any fixed vertex. Since, G is a universal graceful graph, there is a graceful labeling $f : V(G) \longrightarrow \{0, 1, 2, \ldots, q\}$ such that $f(v) = 0$.

Let $H = G \oplus K_2$, the ring sum of G with K_2 by considering vertex v and one pendant vertex of K_2 as a common vertex.

It is obvious that $|V(H)| = |V(G)| + 1$ and $|E(H)| = |E(G)| + 1$. Let $V(H) = V(G) \cup \{w\}$. Then we see that $E(H) = E(G) \cup \{vw\}$, as v and w are adjacent vertices of K_2.

Define $h : V(H) \longrightarrow \{0, 1, \ldots, |E(H)|\}$ as follows.

h(w)=q+1 and
h(u)=f(u), $\forall u \in V(G)$, where $q = |E(G)|$.

It is observed that h is an injective map as f is injective. Moreover
$h^\star(uw) = h(w) - h(v)$
$= q + 1 - f(v)$
$= q + 1 - 0$
$= q + 1$ and for any $u_1u_2 \in V(G)$
$h^\star(u_1u_2) = |h(u_1) - h(u_2)|$
$= |f(u_1) - f(u_2)|$
$= f^\star(u_1u_2)$.

Therefore, $h^\star : E(H) \longrightarrow \{1, 2, \ldots, |E(H)|\}$ is bijective and so, h becomes a graceful labeling for H. Thus, $G \oplus K_2$ is a graceful graph.

Theorem - 4 : Let G_1 be a graceful graph and G_2 be an α-graceful graph. Then ring sum $G_1 \oplus G_2$ by considering graceful center of G_1 and the graceful center of G_2 as a common vertex is a graceful graph.

Proof : Let $f_1 : V(G_1) \longrightarrow \{0, 1, \ldots, q_1\}$ be a graceful labeling and $f_1(w_1) = 0$, for some $w_1 \in V(G_1)$, where $q_1 = |E(G_1)|$. Since, G_2 is an α-graceful graph, $\exists\ f_2 : V(G_2) \longrightarrow \{0, 1, \ldots, q_2\}$ a graceful labeling for G_2 and an integer $k (0 \leq k < q_2)$ such that for each $uv \in E(G_2)$, $\min\{f_2(u), f_2(v)\} \leq k < \max\{f_2(u), f_2(v)\}$, where $q_2 = |E(G_2)|$. Let $f_2(w_2) = 0$, where $w_2 \in V(G_2)$. Take $H = G_1 \oplus G_2$ by considering w_1 and w_2 as a common vertex. It is obvious that $E(H) = E(G_1) \cup E(G_2)$, $|E(G)| = q_1 + q_2$.

Define $g : V(H) \longrightarrow \{0, 1, \ldots, q_1 + q_2\}$ as follows:

g=k-f_2 on V_1,
$g = q_1 + q_2 + k + 1 - f_2$ on V_2 and
$g = k + f_1$ on $V(G_1)$, where
$V_1 = \{w \in V(G_2)/f_2(w) \leq k\}$ and $V_2 = V(G_2) - V_1$.

Since, range of g on $V_1 \subseteq \{0, 1, 2, \ldots, k\}$, range of g on $V_2 \subseteq \{q_1 + k + 1, q_1 + k + 2, \ldots, q_1 + q_2\}$ and range of g on $V(G) \subseteq \{k+1, k+2, \ldots, k+q_1\}$, g is a one-one map.

Moreover

$$g^\star = f_1^\star \text{ on } E(G_1) \text{ and } g^\star = q_1 + f_2^\star \text{ on } E(G_2).$$

Thus, range of $g^\star = \{1, 2, \ldots, q_1, q_1 + 1, \ldots, q_1 + q_2\}$ and so, it is a bijective map. Therefore, g is a a graceful labeling for H and so, $H = G \oplus G_2$ is a graceful graph.

Theorem - 5 : If G_1 and G_2 be two α-graceful graphs, then the ring sum $G_1 \oplus G_2$ considering two graceful centers of G_1 and G_2 as a common vertex is an α-graceful graph.

Proof : Since, G_1 and G_2 both are α-graceful graphs, $\exists\ f_i : V(G_i) \longrightarrow \{0, 1, 2, \ldots, q_i\}$ graceful labeling for G_i and non-negative integer k_i $(0 \leq k_i < q_i)$ such that for each $uv \in E(G_i)$, $\min\{f_i(u), f_i(v)\} \leq k_i < \max\{f_i(u), f_i(v)\}$, where $q_i = |E(G_i)|$ and $i = 1, 2$.

Let $f_1(w_1) = 0$, $f_2(w_2) = 0$, where $w_i \in V(G_i)$, $i = 1, 2$. Take $H = G_1 \oplus G_2$ by considering w_1 and w_2 as a common vertex. It is obvious that $E(H) = E(G_1) \cup E(G_2)$ and $|E(H)| = q_1 + q_2$.

Take $V_1 = \{w \in V(G_1)/f_1(w) \leq k_1\}, V_2 = V(G_1) - V_1$, $V_3 = \{w \in V(G_2)/f_2(w) \leq k_2\}$ and $V_4 = V(G_2) - V_3$.

Define $g : V(H) \longrightarrow \{0, 1, 2, \ldots, q_1 + q_2\}$ as follows.

g=k$_2 - f_2$ on V_3,

$= q_1 + q_2 + k_2 + 1 - f_2$ on V_4 and

$= k_2 + f_1$ on $V(G_1)$.

Since, range of g on $V_3 \subseteq \{0, 1, 2, \ldots, k_2\}$ range of g on $V_4 \subseteq \{q_1 + k_2 + 1, q_1 + k_2 + 2, \ldots, q_1 + q_2\}$ and range of g on $V(G_1) \subseteq \{k_2 + 1, k_2 + 2, \ldots, k_2 + q_1\}$, g is a one-one map. Moreover, $g^\star = f_1^\star$ on $E(G_1)$ and $g^\star = g_1 + f_2^\star$ on $E(G_2)$ gives range of $g^\star = \{1, 2, \ldots, q_1, q_2 + 1, \ldots, q_1 + q_2\}$. Therefore, $g^\star$ is a bijective map and so, it is a graceful labeling for $H = G_1 \oplus G_2$.

Take $k = k_1 + k_2$. Let $uv \in E(H)$ be any edge.

$\Rightarrow$ Either $uv \in E(G_1)$ or $uv \in E(G_2)$.

Case−I : $\imath uv \in E(G_1)$.

Without loss of generality we assume here $u \in V_1$ and $v \in V_2$. Now $g(u) = k_2 + f_1(u) \leq k_2 + k_1 = k$ and $g(v) = k_2 + f_1(v) > k_2 + k_1 = k$.

Case−II : $\imath uv \in E(G_2)$.

Without loss of generality we assume here $u \in V_3$ and $v \in V_4$. Now $g(u) = k_2 - f_2(u) \leq k_2$ and $g(v) = q_1 + q_2 + k_2 + 1 - f_2(v) = q_1 + k_2 + 1 + (q_2 - f_2(v)) < k$, as $q_2 - f_2(v) \geq 0$ and $k_1 < q_1$.

Thus, for any case we get $\min\{g(u), g(v)\} \leq k < \max\{g(u), g(v)\}$, $\forall$ $uv \in E(H)$.

Hence, h is an α-graceful labeling for H and so, $H = G_1 \oplus G_2$ is an α-graceful graph.

Here four graceful centers of $G_1 \oplus G_2$ are w_3, w_4, w_5, w_6, where $f_1(w_3) = k_1$, $f_1(w_4) = k_1 + 1$, $f_2(w_5) = k_2$ and $f_2(w_6) = k_2 + 1$, $w_3, w_4 \in V(G_1)$, $w_5, w_6 \in V(G_2)$. Because

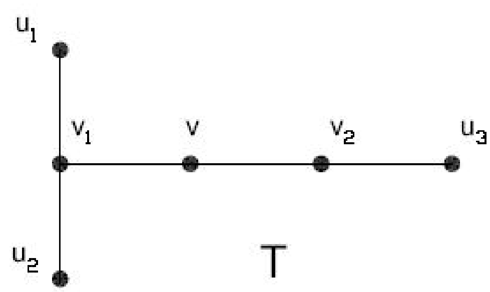
g(w$_3$) $= k_2 + f_1(w_3) = k_2 + k_1 = k$,

$g(w_4) = k_2 + f_1(w_4) = k_2 + k_1 + 1 = k + 1$,
$g(w_5) = k_2 - f_2(w_5) = k_2 - k_2 = 0$ and
$g(w_6) = q_1 + q_2 + k_2 + 1 - f_2(w_6) = q_1 + q_2$.

Now we give a counter example which is α-graceful but not universal graceful graph; Namely a special type of caterpillar.

A caterpillar is a tree with the property that the removal of its pendant vertices leaves a path. This path is known as spine of the caterpillar. It is denoted by $S(n_1, n_2, \ldots, n_k)$, where P_k is the spine of the given caterpillar and $n_1, n_2, \ldots, n_k$ are number of pendant vertices, which are adjacent with the spine of $S(n_1, n_2, \ldots, n_k)$.

Theorem - 6 : Let T be a caterpillar $S(2, 0, 1)$. Then T be an α-graceful graph, but it is not a universal graceful graph, as the vertex v can not be a graceful center for T with respect to any graceful labeling for T.



Proof : As above tree T is a caterpillar $S(2, 0, 1)$, it is an α-graceful graph.

Suppose T admits a graceful center v with respect to a graceful labeling f on T if possible. Here $f(v) = 0$ and v is adjacent to one vertex whose vertex label is $q = 5$. i.e. there are two cases either $f(v_1) = 5$ or $f(v_2) = 5$. In there both cases remaining four vertices have following $24 - 24$ possibilities are given in following table-1 and table-2.

From these table f creates one edge label twice and so, in any case f can not be a graceful labeling for T.

Therefore, T is not a universal graceful tree.

$f(v_2)$	$f(u_1)$	$f(u_2)$	$f(u_3)$	Possible four edge labels
1	2	3	4	1,3,3,2
1	2	4	3	1,2,3,1
1	3	2	4	1,3,3,2
1	3	4	2	1,1,2,1
1	4	2	3	1,2,1,3
1	4	3	2	1,1,1,2
2	1	3	4	2,2,4,2
2	1	4	3	2,1,4,1
2	3	1	4	2,2,2,4
2	3	4	1	2,1,2,1
2	4	1	3	2,1,1,4
2	4	3	1	2,1,1,2
3	1	2	4	3,1,4,3
3	1	4	2	3,1,4,1
3	2	1	4	3,1,3,4
3	2	4	1	3,2,3,1
3	4	1	2	3,1,1,4
3	4	2	1	3,2,1,3
4	1	2	3	4,1,4,3
4	1	3	2	4,2,4,2
4	2	1	3	4,1,3,4
4	2	3	1	4,3,3,2
4	3	1	2	4,2,2,4
4	3	2	1	4,3,2,3

Table-1**:** If $f(v_1) = 5$

$f(v_2)$	$f(u_1)$	$f(u_2)$	$f(u_3)$	Possible four edge labels
1	2	3	4	1,2,1,1
1	2	4	3	1,3,1,2
1	3	2	4	2,1,1,1
1	3	4	2	2,3,1,3
1	4	2	3	3,1,1,2
1	4	3	2	3,2,1,3
2	1	3	4	1,1,2,1
2	1	4	3	1,2,2,2
2	3	1	4	1,1,2,1
2	3	4	1	1,2,2,4
2	4	1	3	2,1,2,2
2	4	3	1	2,1,2,4
3	1	2	4	2,1,3,1
3	1	4	2	2,1,3,3
3	2	1	4	1,2,3,1
3	2	4	1	1,1,3,4
3	4	1	2	1,2,3,3
3	4	2	1	1,1,3,4
4	1	2	3	3,2,4,2
4	1	3	2	3,1,4,3
4	2	1	3	2,3,4,2
4	2	3	1	2,1,4,4
4	3	1	2	1,3,4,3
4	3	2	1	1,2,4,4

Table-2: If $f(v_2) = 5$

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