Trees with vertex-edge roman domination number twice the domination number minus one

H. Naresh Kumar¹ orcid.org/0000-0002-1717-239X

Y. B. Venkatakrishnan² orcid.org/0000-0003-4560-2040

SASTRA University, Dept. of Mathematics, School of Arts, Science and Humanities, Tanjore, TN, India.

¹ nareshhari1403@gmail.com ; ² venkatakrish2@maths.sastra.edu

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Abstract:

A vertex-edge Roman dominating function (or just ve-RDF) of a graph $G = (V, E)$ is a function $f: V(G) \rightarrow \{0, 1, 2\}$ such that for each edge $e = uv$ either $\max\{f(u), f(v)\} \neq 0$ or there exists a vertex w such that either $wu \in E$ or $wv \in E$ and $f(w) = 2$. The weight of a ve-RDF is the sum of its function values over all vertices. The vertex-edge Roman domination number of a graph G , denoted by $\gamma_{veR}(G)$, is the minimum weight of a ve-RDF G . We characterize trees with vertexedge roman domination number equal to twice domination number minus one.

Keywords: Vertex-edge roman dominating set; Dominating set; Trees.

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1. Introduction

Let $G = (V, E)$ be a simple graph with order $n = |V|$. For every vertex $v \in V$, the set $\{u \in V \mid uv \in E\}$ denoted by $N(v)$ is the open neighborhood of v and the set $N(v) \cup \{v\}$ is the closed neighborhood of v and is denoted by $N[v]$. The *degree* of a vertex v is the cardinality of its open neighborhood, denoted $d_G(v) = |N(v)|$. A vertex of degree one is called a *leaf* and its neighbor is called a support vertex. A support vertex is *strong* (*weak*, respectively) if it is adjacent to at least two leaves (exactly one leaf, respectively). A star of order $n \geq 2$, denoted by $K_{1,n-1}$, is a tree with at least $n - 1$ leaves. A double star is a tree that contains exactly two vertices that are not leaves. A double star with respectively r and s leaves attached to each support vertex is denoted by $D_{r,s}$. The number of edges in a shortest path between vertices u and v is the *distance* between vertices u and v in a connected graph G . The longest distance between any pair of vertices is defined as the *diameter* of the graph G , and is denoted by $\text{diam}(G)$. The *edge subdivision* for an edge uv is the deletion of uv from G and addition of two edges uw and wv along with the new vertex w .

Domination theory is one of the widely studied topic in graph theory. Minimum number of servers in an adhoc network can be modelled as a dominating set of a network graph. A set S of vertices is a *dominating set* (abbreviated DS) of G if every vertex not in S is adjacent to some vertex in S . The minimum cardinality of a dominating set of G is called the *domination number*, $\gamma(G)$. In domination theory, a set of vertices dominate another set of vertices or a set of edges dominate another set of edges. Peters [7] in 1986 defined a variant of a dominating set in which a set of vertices dominate the edges of a graph G . A vertex v *ve-dominates* every edge incident to a vertex in $N[v]$. A set $S \subseteq V$ is a *vertex-edge dominating set* (or simply, a *ve-dominating set*) if for every edge $e \in E$, there exists a vertex $v \in S$ such that v *ve-dominates* e . The minimum cardinality of a *ve-dominating set* of G is called the *ve-domination number* $\gamma_{ve}(G)$. The concept of vertex-edge domination is studied further in [2, 5, 6].

A variant of domination, namely *Roman dominating function* (or just RDF) is a function $f : V(G) \rightarrow \{0, 1, 2\}$ in which every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The *weight* of an RDF f is $f(V(G)) = \sum_{u \in V(G)} f(u)$. The *Roman domination number* $\gamma_R(G)$ is the minimum weight of an RDF on G . For more information on Roman domination and its variants, see [1, 3, 4, 8, 10].

In [9], a variation of Roman dominating function and also a variation

of vertex-edge dominating set, say, vertex-edge Roman dominating function is defined as a function $f : V(G) \rightarrow \{0, 1, 2\}$ such that each edge $e = vu$ is either incident with a vertex having function value at least one or uv is ve -dominated by some vertex w with $f(w) = 2$. The *vertex-edge Roman domination number* $\gamma_{veR}(G)$ equals the minimum weight of all ve -RDF on G .

In this paper we characterize all trees with vertex-edge roman domination number equals twice domination number minus one.

2. Main results

We begin the section with an observation.

Observation 1. *Let T' be a tree with $\gamma_{veR}(T') = 2\gamma(T') - 1$. The tree T is obtained from T' by attaching a vertex to a support vertex of T' . Then $\gamma_{veR}(T) = 2\gamma(T) - 1$ if and only if $\gamma_{veR}(T') = 2\gamma(T') - 1$.*

Proof. Let x be the support vertex in the tree T' . Let y be the leaf adjacent to x . The tree T is obtained from T' by adding a vertex z to the vertex x . Let D be a $\gamma(T)$ -set. To dominate the leaves y and z , the vertex $x \in D$. It is easy to see that D is a DS of the tree T' . Thus $\gamma(T') \leq \gamma(T)$. Let D' be a $\gamma(T')$ -set. If $x \notin D'$, then $y \in D'$ then obviously $(D' \setminus \{y\}) \cup \{x\}$ is a DS of T . Thus $\gamma(T) \leq \gamma(T')$. We get $\gamma(T) = \gamma(T')$.

Let f be a $\gamma_{veR}(T)$ -dominating function of the tree T . To dominate the edges xy and xz , the vertex x is assigned weight one or a vertex in $N(x)$ is assigned weight 2. If z is assigned weight 2, then the vertex y has weight zero. Replacing the weight of y and z , we get $f|_{V(T) \setminus \{y\}}$ and y is assigned weight two is a veR -dominating function of the tree T' . Thus $\gamma_{veR}(T') \leq \gamma_{veR}(T)$. Let f' be a $\gamma_{veR}(T')$ -dominating function of the tree T . To dominate the edge xy , the vertex x or y is assigned weight one. If y is assigned weight 1, then weight of x is zero. Replacing the weight of x and y gives f is a veR -dominating function of T . If weight of x and y are zero, then the vertex which dominates xy dominates xz and in all the cases of f is a veR -dominating function of T . Thus $\gamma_{veR}(T) \leq \gamma_{veR}(T')$. We get $\gamma_{veR}(T) = \gamma_{veR}(T')$. This implies that $\gamma_{veR}(T) = 2\gamma(T) - 1$ if and only if $\gamma_{veR}(T') = 2\gamma(T') - 1$. $\square$

To prove the required result, we define a family $\mathcal{T}$ containing stars and all trees obtained from a double star by subdividing the edge joining the center twice. We now prove that for every tree of family $\mathcal{T}$, the vertex-edge roman domination number equals the twice domination number minus one.

Lemma 2. If $T \in \mathcal{T}$, then $\gamma_{veR}(T) = 2\gamma(T) - 1$.

Proof. Let $T \in \mathcal{T}$. If T is star $K_{1,n}$ ($n \geq 1$), then $\gamma_{veR}(T) = 1 = 2(1) - 1 = 2\gamma(T) - 1$. If T is a tree obtained from a double star by subdividing the edge joining the center twice, then the support vertices is a minimum dominating set. Thus $\gamma(T) = 2$. If two support vertices are assigned weight one and two respectively, then we have $\gamma_{veR}(T) = 3$. We get $\gamma_{veR}(T) = 3 = 2(2) - 1 = 2\gamma(T) - 1$. $\square$

We now prove that if vertex-edge roman domination number of a tree is equal to twice domination number minus one, then the tree belongs to the family $\mathcal{T}$.

Lemma 3. For every tree T , $\gamma_{veR}(T) = 2\gamma(T) - 1$, then $T \in \mathcal{T}$

Proof. First assume that $\text{diam}(T) = 1$. The tree $T = K_2$. We have $\gamma_{veR}(T) = 1 = 2\gamma(T) - 1$. Thus $T \in \mathcal{T}$. Now assume that $\text{diam}(T) = 2$. The tree T is a star. We have $\gamma_{veR}(T) = 2\gamma(T) - 1$. Thus $T \in \mathcal{T}$. Now assume that $\text{diam}(T) \geq 3$. Thus the order n of the tree T is at least four. The result we obtain by the induction on the number n . Assume that the theorem is true for every tree T' of order $n' < n$.

We now root T at a vertex r of maximum eccentricity $\text{diam}(T)$. Let t be a leaf at maximum distance from r , v be the parent of t , u be the parent of v , w be the parent of u , d be the parent of w and e be the parent of d in the rooted tree. By T_x we denote the subtree induced by a vertex x and its descendants in the rooted tree T .

By Observation 1, we can assume that $d_T(v) = 2$. Assume that among the children of u there is a support vertex, say x , other than v . The leaf adjacent to x we denote by y . Let $T' = T - T_v$. Let f_1 be a $\gamma_{veR}(T')$ -dominating function of tree T' . The function f defined on $V(T)$ as

$$(2.1) \quad f(a) = \begin{cases} f_1(a) & \text{if } a \in V(T') \\ 1 & \text{if } a = v \\ 0 & \text{if otherwise} \end{cases}$$

is a veR -dominating function of T . Thus $\gamma_{veR}(T) \leq \gamma_{veR}(T') + 1$. Let D be a $\gamma(T)$ -set. To dominate the vertices t and y , the vertices $v, x \in D$. It is easy to observe that $D \setminus \{v\}$ is a DS of T' . Thus $\gamma(T') \leq \gamma(T) - 1$. We have $\gamma_{veR}(T) \leq \gamma_{veR}(T') + 1 \leq 2\gamma(T') - 1 + 1 \leq 2(\gamma(T) - 1) < 2\gamma(T) - 1$.

Now assume that among the children of u , say x , is a leaf. Let $T' = T - T_v$. Let f_1 be a $\gamma_{veR}(T')$ -dominating function of the tree T' . The function f defined on $V(T)$ as

$$(2.2) \quad f(a) = \begin{cases} f_1(a) & \text{if } a \in V(T') \\ 1 & \text{if } a = v \\ 0 & \text{if otherwise} \end{cases}$$

is a veR -dominating function of T . Thus $\gamma_{veR}(T) \leq \gamma_{veR}(T') + 1$. Let D be a $\gamma(T)$ -set. To dominate the vertices t and x , the vertices $v, u \in D$. It is easy to observe that $D \setminus \{v\}$ is a DS of T' . Thus $\gamma(T') \leq \gamma(T) - 1$. We have $\gamma_{veR}(T) \leq \gamma_{veR}(T') + 1 \leq 2\gamma(T') - 1 + 1 \leq 2(\gamma(T) - 1) < 2\gamma(T) - 1$.

Assume that $d_T(u) = 2$. Now assume that among the children of w , other than u , say x , such that the distance of w to the most distant vertex of T_x is three. It suffices to consider the case $T_x = P_3 = xyz$. Let $T' = T - T_u$. Let f_1 be a $\gamma_{veR}(T')$ -dominating function of the tree T' . To dominate the edges zy and xy , the vertex x is assigned weight two. The function f defined on $V(T)$ as

$$(2.3) \quad f(a) = \begin{cases} f_1(a) & \text{if } a \in V(T') \\ 1 & \text{if } a = v \\ 0 & \text{if otherwise} \end{cases}$$

is a veR -dominating function of T . Thus $\gamma_{veR}(T) \leq \gamma_{veR}(T') + 1$. Let D be a $\gamma(T)$ -set. To dominate the vertices t and z , the vertices $v, y \in D$. It is easy to observe that $D \setminus \{v\}$ is a DS of T' . Thus $\gamma(T') \leq \gamma(T) - 1$. We have $\gamma_{veR}(T) \leq \gamma_{veR}(T') + 1 \leq 2\gamma(T') - 1 + 1 \leq 2(\gamma(T) - 1) < 2\gamma(T) - 1$.

Assume that among the children of w , other than u , say x , such that the distance of w to the most distant vertex of T_x is two. It suffices to consider the case $T_x = P_2 = xy$. Let $T' = T - T_u$. Let f_1 be a $\gamma_{veR}(T')$ -dominating function of the tree T' . To dominate the edges xy and wx , the vertex w is assigned weight two. The function f defined on $V(T)$ as

$$(2.4) \quad f(a) = \begin{cases} f_1(a) & \text{if } a \in V(T') \\ 1 & \text{if } a = v \\ 0 & \text{if otherwise} \end{cases}$$

is a veR -dominating function of T . Thus $\gamma_{veR}(T) \leq \gamma_{veR}(T') + 1$. Let D be a $\gamma(T)$ -set. To dominate the vertices t and y , the vertices $v, x \in D$. It is easy to observe that $D \setminus \{v\}$ is a DS of T' . Thus $\gamma(T') \leq \gamma(T) - 1$. We have $\gamma_{veR}(T) \leq \gamma_{veR}(T') + 1 \leq 2\gamma(T') - 1 + 1 \leq 2(\gamma(T) - 1) < 2\gamma(T) - 1$.

Assume that among the children of w , say x is a leaf. Let $T' = T - T_u$. Let f_1 be a $\gamma_{veR}(T')$ -dominating function of the tree T' . The vertex which dominates the edge wx in tree T' dominates the edge wu in the tree T . The function f defined on $V(T)$ as

$$(2.5) \quad f(a) = \begin{cases} f_1(a) & \text{if } a \in V(T') \\ 1 & \text{if } a = v \\ 0 & \text{if otherwise} \end{cases}$$

is a veR -dominating function of T . Thus $\gamma_{veR}(T) \leq \gamma_{veR}(T') + 1$. Let D be a $\gamma(T)$ -set. To dominate the vertices t and x , the vertices $v, w \in D$. It is easy to observe that $D \setminus \{v\}$ is a DS of T' . Thus $\gamma(T') \leq \gamma(T) - 1$. We have $\gamma_{veR}(T) \leq \gamma_{veR}(T') + 1 \leq 2\gamma(T') - 1 + 1 \leq 2(\gamma(T) - 1) < 2\gamma(T) - 1$.

Assume that among the children of d , other than w , say x , such that the distance of d to the most distant vertex of T_x is three. It suffices to consider the case $T_x = P_3 = xyz$. Let $T' = T - T_w - T_y$. Let f_1 be a $\gamma_{veR}(T')$ -dominating function of the tree T' . The function f defined on $V(T)$ as

$$(2.6) \quad f(a) = \begin{cases} f_1(a) & \text{if } a \in V(T') \\ 1 & \text{if } a = y \\ 2 & \text{if } a = u \\ 0 & \text{if otherwise} \end{cases}$$

is a veR -dominating function of T . Thus $\gamma_{veR}(T) \leq \gamma_{veR}(T') + 3$. Let D be a $\gamma(T)$ -set. To dominate the vertices w, u, t, x and z , the vertices $d, v, y \in D$. It is easy to observe that $D \setminus \{v, y\}$ is a DS of T' . Thus $\gamma(T') \leq \gamma(T) - 2$. We have $\gamma_{veR}(T) \leq \gamma_{veR}(T') + 3 \leq 2\gamma(T') - 1 + 3 \leq 2(\gamma(T) - 1) < 2\gamma(T) - 1$.

Assume that among the children of d , other than w , say x , such that the distance of d to the most distant vertex of T_x is two. It suffices to consider the case $T_x = P_2 = xy$. Let $T' = T - T_w - y$. Let f_1 be a $\gamma_{veR}(T')$ -dominating function of the tree T' . The function f defined on $V(T)$ as

$$(2.7) \quad f(a) = \begin{cases} f_1(a) & \text{if } a \in V(T') \\ 1 & \text{if } a = x \\ 2 & \text{if } a = u \\ 0 & \text{if otherwise} \end{cases}$$

is a veR -dominating function of T . Thus $\gamma_{veR}(T) \leq \gamma_{veR}(T') + 3$. Let D be a $\gamma(T)$ -set. To dominate the vertices w, u, t and y , the vertices $d, v, x \in D$. It is easy to observe that $D \setminus \{v, x\}$ is a DS of T' . Thus $\gamma(T') \leq \gamma(T) - 2$. We have $\gamma_{veR}(T) \leq \gamma_{veR}(T') + 3 \leq 2\gamma(T') - 1 + 3 \leq 2(\gamma(T) - 1) < 2\gamma(T) - 1$.

Assume that among the children of d , other than w , say x , such that the distance of w to the most distant vertex of T_x is four. It suffices to consider the tree $T_x = P_4 = xyza$. We fix $d_T(d) = 3$. Assume that e is adjacent to

T_k , where T_k is isomorphic to T_d or T_k is P_5 or P_2 . Let $T' = T - T_d$. Let f_1 be a $\gamma_{veR}(T')$ -dominating function of the tree T' . The function f defined on $V(T)$ as

$$(2.8) \quad f(c) = \begin{cases} f_1(c) & \text{if } c \in V(T') \\ 2 & \text{if } c = d \\ 1 & \text{if } c = v, z \\ 0 & \text{if otherwise} \end{cases}$$

is a veR -dominating function of T . Thus $\gamma_{veR}(T) \leq \gamma_{veR}(T') + 4$. Let D be a $\gamma(T)$ -set. To dominate the vertices e, w, x, u, t, y and a , the vertices $d, v, z \in D$. Since $k \in D$, $D \setminus \{d, v, z\}$ is a DS of T' . Thus $\gamma(T') \leq \gamma(T) - 3$. We have $\gamma_{veR}(T) \leq \gamma_{veR}(T') + 4 \leq 2\gamma(T') - 1 + 4 \leq 2(\gamma(T) - 3) + 3 < 2\gamma(T) - 1$.

Now assume that T_k is P_4 or P_1 . Let $T' = T - T_d$. Let f_1 be a $\gamma_{veR}(T')$ -dominating function of the tree T' . The function f defined on $V(T)$ as

$$(2.9) \quad f(c) = \begin{cases} f_1(c) & \text{if } c \in V(T') \\ 2 & \text{if } c = d \\ 1 & \text{if } c = v, z \\ 0 & \text{if otherwise} \end{cases}$$

is a veR -dominating function of T . Thus $\gamma_{veR}(T) \leq \gamma_{veR}(T') + 4$. Let D be a $\gamma(T)$ -set. To dominate the vertices w, x, u, y, t and a , the vertices $d, v, y \in D$. Since $e \in D$, $D \setminus \{d, v, z\}$ is a DS of T' . Thus $\gamma(T') \leq \gamma(T) - 3$. We have $\gamma_{veR}(T) \leq \gamma_{veR}(T') + 4 \leq 2\gamma(T') - 1 + 4 \leq 2(\gamma(T) - 3) + 3 < 2\gamma(T) - 1$.

Suppose e is adjacent to two P_3 , say klm and $k_1l_1m_1$. Let $T' = T - T_k$. It is easy to check that $\gamma_{veR}(T) < 2\gamma(T) - 1$. Now assume that e is adjacent to exactly one $P_3 = klm$. Let $T' = T - T_e$. Let f_1 be a $\gamma_{veR}(T')$ -dominating function of the tree T' . The function f defined on $V(T)$ as

$$(2.10) \quad f(c) = \begin{cases} f_1(c) & \text{if } c \in V(T') \\ 2 & \text{if } c = d \\ 1 & \text{if } c = v, z, l \\ 0 & \text{if otherwise} \end{cases}$$

is a veR -dominating function of T . Thus $\gamma_{veR}(T) \leq \gamma_{veR}(T') + 5$. Let D be a $\gamma(T)$ -set. To dominate the vertices e, w, x, u, y, t, a, k and m , the vertices $d, v, z, l \in D$. It is easy to observe that $D \setminus \{d, v, z, l\}$ is a DS of T' . Thus $\gamma(T') \leq \gamma(T) - 4$. We have $\gamma_{veR}(T) \leq \gamma_{veR}(T') + 5 \leq 2\gamma(T') - 1 + 5 \leq 2(\gamma(T) - 4) + 4 < 2\gamma(T) - 1$.

Now assume that $d_T(e) = 2$. Let $T' = T - T_e$. Let f_1 be a $\gamma_{veR}(T')$ -dominating function of the tree T' . The function f defined on $V(T)$ as

$$(2.11) \quad f(c) = \begin{cases} f_1(c) & \text{if } c \in V(T') \\ 2 & \text{if } c = d \\ 1 & \text{if } c = v, z \\ 0 & \text{if otherwise} \end{cases}$$

is a veR -dominating function of T . Thus $\gamma_{veR}(T) \leq \gamma_{veR}(T') + 4$. Let D be a $\gamma(T)$ -set. To dominate the vertices e, w, x, u, y, t and a , the vertices $d, v, z \in D$. It is easy to observe that $D \setminus \{d, v, z\}$ is a DS of T' . Thus $\gamma(T') \leq \gamma(T) - 3$. We have $\gamma_{veR}(T) \leq \gamma_{veR}(T') + 4 \leq 2\gamma(T') - 1 + 4 \leq 2(\gamma(T) - 3) + 3 < 2\gamma(T) - 1$.

Assume among the children of d , say h , is a leaf. We have $d_T(d) = 3$. Assume e is adjacent to e say T_k , where T_k is isomorphic to T_d or T_k is P_5 or P_2 . Let $T' = T - T_d$. Let f_1 be a $\gamma_{veR}(T')$ -dominating function of the tree T' . The function f defined on $V(T)$ as

$$(2.12) \quad f(a) = \begin{cases} f_1(a) & \text{if } a \in V(T') \\ 2 & \text{if } a = u \\ 1 & \text{if } a = d \\ 0 & \text{if otherwise} \end{cases}$$

is a veR -dominating function of T . Thus $\gamma_{veR}(T) \leq \gamma_{veR}(T') + 3$. Let D be a $\gamma(T)$ -set. To dominate the vertices h, w, u, t and e , the vertices $v, d \in D$. Since $k \in D$, $D \setminus \{d, v\}$ is a DS of T' . Thus $\gamma(T') \leq \gamma(T) - 2$. We have $\gamma_{veR}(T) \leq \gamma_{veR}(T') + 3 \leq 2\gamma(T') - 1 + 3 \leq 2(\gamma(T) - 1) < 2\gamma(T) - 1$.

Now assume that T_k is P_4 or P_1 . Let $T' = T - T_d$. Let f_1 be a $\gamma_{veR}(T')$ -dominating function of the tree T' . The function f defined on $V(T)$ as

$$(2.13) \quad f(a) = \begin{cases} f_1(a) & \text{if } a \in V(T') \\ 2 & \text{if } a = u \\ 1 & \text{if } a = d \\ 0 & \text{if otherwise} \end{cases}$$

is a veR -dominating function of T . Thus $\gamma_{veR}(T) \leq \gamma_{veR}(T') + 3$. Let D be a $\gamma(T)$ -set. To dominate the vertices h, w, u and t , the vertices $v, d \in D$. Since $e \in D$, $D \setminus \{d, v\}$ is a DS of T' . Thus $\gamma(T') \leq \gamma(T) - 2$. We have $\gamma_{veR}(T) \leq \gamma_{veR}(T') + 3 \leq 2\gamma(T') - 1 + 3 \leq 2(\gamma(T) - 1) < 2\gamma(T) - 1$.

Now assume that T_k is $P_3 = xyz$. Let $T' = T - T_x$. Let f_1 be a $\gamma_{veR}(T')$ -dominating function of the tree T' . To dominate the edges dw, wv, vu, ut

the vertex v is assigned a weight 2 and to dominate the edges hd, de vertex e is assigned weight 2. The function f defined on $V(T)$ as

$$(2.14) \quad f(a) = \begin{cases} f_1(a) & \text{if } a \in V(T') \\ 2 & \text{if } a = y \\ 0 & \text{if otherwise} \end{cases}$$

is a veR -dominating function of T . Thus $\gamma_{veR}(T) \leq \gamma_{veR}(T') + 1$. Let D be a $\gamma(T)$ -set. To dominate the vertices x and z , the vertex $y \in D$. It is easy to observe that $D \setminus \{y\}$ is a DS of T' . Thus $\gamma(T') \leq \gamma(T) - 1$. We have $\gamma_{veR}(T) \leq \gamma_{veR}(T') + 1 \leq 2\gamma(T') - 1 + 1 \leq 2(\gamma(T) - 1) < 2\gamma(T) - 1$.

Now assume that $d_T(e) = 2$. Let $T' = T - T_e$. Let f_1 be a $\gamma_{veR}(T')$ -dominating function of the tree T' . The function f defined on $V(T)$ as

$$(2.15) \quad f(a) = \begin{cases} f_1(a) & \text{if } a \in V(T') \\ 2 & \text{if } a = d \\ 1 & \text{if } a = v \\ 0 & \text{if otherwise} \end{cases}$$

is a veR -dominating function of T . Thus $\gamma_{veR}(T) \leq \gamma_{veR}(T') + 3$. Let D be a $\gamma(T)$ -set. To dominate the vertices e, h, w, u, v and t , the vertices $v, d \in D$. It is easy to observe that $D \setminus \{d, v\}$ is a DS of T' . Thus $\gamma(T') \leq \gamma(T) - 2$. We have $\gamma_{veR}(T) \leq \gamma_{veR}(T') + 3 \leq 2\gamma(T') - 1 + 3 \leq 2(\gamma(T) - 1) < 2\gamma(T) - 1$.

Now assume $d_T(d) = 2$. Assume that some child of e , say x , other than d such that the distance of e to the most distant vertex of T_x is five or two. It suffices to consider the case $T_x = P_5 = xyzab$ or $T_x = P_2 = xy$. Let $T' = T - T_d$. Let f_1 be a $\gamma_{veR}(T')$ -dominating function of the tree T' . The function f defined on $V(T)$ as

$$(2.16) \quad f(a) = \begin{cases} f_1(a) & \text{if } a \in V(T') \\ 2 & \text{if } a = u \\ 1 & \text{if } a = d \\ 0 & \text{if otherwise} \end{cases}$$

is a veR -dominating function of T . Thus $\gamma_{veR}(T) \leq \gamma_{veR}(T') + 3$. Let D be a $\gamma(T)$ -set. To dominate the vertices w, u and t , the vertices $d, v \in D$. The vertex e is adjacent to P_5 or P_2 , the child of e other than d is in D . It is easy to observe that $D \setminus \{v, d\}$ is a DS of T' . Thus $\gamma(T') \leq \gamma(T) - 2$. We have $\gamma_{veR}(T) \leq \gamma_{veR}(T') + 3 \leq 2\gamma(T') - 1 + 3 \leq 2(\gamma(T) - 1) < 2\gamma(T) - 1$.

Assume that some child of e , say x , other than d such that the distance of e to the most distant vertex of T_x is four or one. It suffices to consider the case $T_x = P_4 = xyza$ or $T_x = P_1 = x$. Let $T' = T - T_w$. Let f_1 be

a $\gamma_{veR}(T')$ -dominating function of the tree T' . The function f defined on $V(T)$ as

$$(2.17) \quad f(a) = \begin{cases} f_1(a) & \text{if } a \in V(T') \\ 2 & \text{if } a = u \\ 0 & \text{if otherwise} \end{cases}$$

is a veR -dominating function of T . Thus $\gamma_{veR}(T) \leq \gamma_{veR}(T') + 2$. Let D be a $\gamma(T)$ -set. To dominate the vertices w, u and t , the vertices $d, v \in D$. The vertex e is adjacent to P_4 or P_1 . To dominate the children of e other than d , the vertex e is in D . It is easy to observe that $D \setminus \{v, d\}$ is a DS of T' . Thus $\gamma(T') \leq \gamma(T) - 2$. We have $\gamma_{veR}(T) \leq \gamma_{veR}(T') + 2 \leq 2\gamma(T') - 1 + 2 < 2\gamma(T) - 1$.

Assume that some child of e , say x , other than d such that the distance of e to the most distant vertex of T_x is three. It suffices to consider the case $T_x = P_3 = xyz$. Let $T' = T - T_x$. Let f_1 be a $\gamma_{veR}(T')$ -dominating function of the tree T' . To dominate the edges vt, uv the vertex u is assigned weight two. To dominate the edge ed , we assign a weight one to the vertex e or two to the vertex f . This vertex dominates the edge ex in the tree T . The function f defined on $V(T)$ as

$$(2.18) \quad f(a) = \begin{cases} f_1(a) & \text{if } a \in V(T') \\ 1 & \text{if } a = y \\ 0 & \text{if otherwise} \end{cases}$$

is a veR -dominating function of T . Thus $\gamma_{veR}(T) \leq \gamma_{veR}(T') + 1$. Let D be a $\gamma(T)$ -set. To dominate the vertex z , the vertex $y \in D$. It is clear that $D \setminus \{y\}$ is a DS of T' . Thus $\gamma(T') \leq \gamma(T) - 1$. We have $\gamma_{veR}(T) \leq \gamma_{veR}(T') + 1 \leq 2\gamma(T') < 2\gamma(T) - 1$.

Assume that $d_T(e) = 1$. We have $T = P_6$. Then $\gamma_{veR}(P_6) = 3 = 2(2) - 1 = 2\gamma(P_6) - 1$, $T \in \mathcal{T}$. Now assume that $d_T(e) = 2$. Let $T' = T - T_e$. Let f_1 be a $\gamma_{veR}(T')$ -dominating function of the tree T' . To dominate the edges vt, uv the vertex u is assigned weight two. To dominate the edge ed , we assign a weight one to the vertex e . The function f defined on $V(T)$ as

$$(2.19) \quad f(a) = \begin{cases} f_1(a) & \text{if } a \in V(T') \\ 1 & \text{if } a = e \\ 2 & \text{if } a = u \\ 0 & \text{if otherwise} \end{cases}$$

is a veR -dominating function of T . Thus $\gamma_{veR}(T) \leq \gamma_{veR}(T') + 3$. Let D be a $\gamma(T)$ -set. To dominate the vertices t and w , the vertices $v, d \in D$. It

is obvious that $D \setminus \{v, d\}$ is a DS of T' . Thus $\gamma(T') \leq \gamma(T) - 2$. We have $\gamma_{veR}(T) \leq \gamma_{veR}(T') + 3 \leq 2\gamma(T') - 1 + 3 \leq 2\gamma(T') + 2 < 2\gamma(T) - 1$. $\square$

As a immediate consequence of Lemmas 2 and 3, we have the following characterization of trees with vertex-edge roman domination number equal to twice domination number minus one.

Theorem 4. *Let T be a tree. Then $\gamma_{veR}(T) = 2\gamma(T) - 1$ if and only if $T \in \mathcal{T}$.*

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