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Maps preserving fixed points of generalized product of operators

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Abstract:

Let $\mathcal{B}(X)$ be the algebra of all bounded linear operators in a complex Banach space X. For $A \in \mathcal{B}(X)$ let $F(A)$ be the subspace of fixed point of A. For an integer $k \geq 2$, let $(i_1, .., i_m)$ be a finite sequence with terms chosen from $\{1, \cdots, k\}$, and assume at least one of the terms in $(i_1, \cdots, i_m)$ appears exactly once. The generalized product of k operators $A_1, ..., A_k \in \mathcal{B}(X)$ is defined by

$$A_1 * A_2 * \cdots * A_k = A_{i_1} A_{i_2} \cdots A_{i_m},$$

and includes the usual product and the triple product. We characterize the form of maps from $\mathcal{B}(X)$ onto itself satisfying

$$F(\phi(A_1) * \cdots * \phi(A_k)) = F(A_1 * \cdots * A_k)$$

for all $A_1, \cdots, A_k \in \mathcal{B}(X)$.

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1. Introduction

Let X be a Banach space over $\mathbf{K}$, where $\mathbf{K}$ is the complex field $\mathbf{C}$ or the real field $\mathbf{R}$, and let $\mathcal{B}(X)$ be the algebra of all bounded linear operators on X. The dual space of X will be denoted by X^*. For a vector $x \in X$ and linear functional f in the dual space X^* of X, let $x \otimes f$ stands for the operator of rank at most one defined by

$$(x \otimes f)y := f(y)x, \quad (y \in X).$$

The problem of characterizing maps on matrices or operators that preserve certain functions, subsets and relations has attracted the attention of many mathematicians in the last decade; for example we can see [1, 2, 3, 4, 5, 6, 7] and their references. In recent years, a great activity has occurred in characterising maps preserving the subspace of fixed points of a matrix or operators. Recall that a vector $x \in X$ is said to be fixed point of an operator $A \in \mathcal{B}(X)$ if $Ax = x$, denote by $F(A)$ the set of all fixed points of an operator A. Note that if we consider the rank-one operator $x \otimes f$ for $x \in X$ and $f \in X^*$ then

$$(1.1) \qquad x \otimes f \;\; is \;\; idempotent \Longleftrightarrow F(x \otimes f) = \langle x \rangle$$

$$(1.2) \qquad x \otimes f \;\; is \;\; not \;\; idempotent \Longrightarrow F(x \otimes f) = \{0\}.$$

Clearly that $F(A) \in Lat(A)$, where $Lat(A)$ the lattice of A, is the set of all invariant subspaces of A. In [4], A. A. Jafarian and A.R. Sourour described linear maps preserving the lattice of an operator in Banach alge- bra. In particular they showed that a linear map $\phi : \mathcal{B}(X) \to \mathcal{B}(X)$ satisfied $Lat(\phi(A)) = Lat(A)$, if and only if $\phi(A) = \alpha A + \varphi(A)I$ for all $A \in \mathcal{B}(X)$ where I is the identity operator, α a nonzero scalar in $\mathbf{K}$ and $\varphi : \mathcal{B}(X) \to \mathbf{K}$ linear functional.

This result has been extended in [3], where G. Dolinar et al. characterised the form of maps preserving the lattice of sum of operators, they showed that maps (not necessarily linear) $\phi : \mathcal{B}(X) \to \mathcal{B}(X)$ satisfied $Lat(\phi(A) + \phi(B)) = Lat(A + B)$ for all $A, B \in \mathcal{B}(X)$, if and only if there is a non zero scalar α and a map $\varphi : \mathcal{B}(X) \to \mathbf{K}$ such that $\phi(A) = \alpha A + \varphi(A)I$ for all $A \in \mathcal{B}(X)$. They proved also, in the same paper, that a non necessarily linear maps $\phi : \mathcal{B}(X) \to \mathcal{B}(X)$ satisfied $Lat(\phi(A)\phi(B)) = Lat(AB)$ (resp. $Lat(\phi(A)\phi(B)\phi(A)) = Lat(ABA)$) for all $A, B \in \mathcal{B}(X)$, if and only if there is a map $\varphi : \mathcal{B}(X) \to \mathbf{K}$ such that $\varphi(A) = 0$ if $A = 0$ and $\phi(A) = \varphi(A)A$ for all $A \in \mathcal{B}(X)$.In [5], A. Taghavi and R. Hosseinzadeh proved that if X is a complex Banach space with $\dim X \geq 3$ and if a surjective map $\phi : \mathcal{B}(X) \to \mathcal{B}(X)$ satisfies

$$\dim F(\phi(A)\phi(B)) = \dim F(AB)$$

for all A, $B \in \mathcal{B}(X)$, then there exists an invertible operator $S \in \mathcal{B}(X)$ such that $\phi(A) = \pm SAS^{-1}$ for all $A \in \mathcal{B}(X)$. In [6] A. Taghavi et al. studied the surjective maps $\phi : \mathcal{B}(X) \to \mathcal{B}(X)$ which satisfy $F(\phi(A)+\phi(B))$ $= F(A+B)$ for all $A, B \in \mathcal{B}(X)$, they conclude that $\phi(A) = UA + R$ for all $A \in \mathcal{B}(X)$ where $U = I - 2\phi(0)$ and $R = \phi(0)$. In [7] A. Taghavi et al. proved that if X is a complex Banach space with $\dim X \geq 3$ and $\phi : \mathcal{B}(X)$ $\to \mathcal{B}(X)$ is sur- jective maps satifying $F(\phi(A)\phi(B)\phi(A)) = F(ABA)$ for all $A, B \in \mathcal{B}(X)$, then $\phi(A) = \alpha A$ for all $A \in \mathcal{B}(X)$ where $\alpha \in \mathbf{C}$ with $\alpha^3 = 1$. The aim of this note is to prove the last result for the generalized product.

For an integer $k \geq 2$, let $(i_1, \cdots, i_m)$ be a finite sequence such that $\{i_1, \cdots, i_m\} = \{1 \cdots k\}$ and let at least one of the terms in $(i_1, \cdots, i_m)$ appears exactly once. The generalized product of width m of k operators $A_1 \cdots A_k \in \mathcal{B}(X)$ is defined by

$$A_1 * A_2 * \cdots * A_k = A_{i_1} A_{i_2} \cdots A_{i_m}.$$

Evidently, the generalized product includes the usual product and the triple product. The following theorem is the main result of this paper. Its proof use ideas from [1, 7].

Theorem 1.1. *Consider the generalized product of width m, $T_1 * \cdots * T_k$. Let $\phi : \mathcal{B}(X) \to \mathcal{B}(X)$ be a surjective map. Then, the following statements are equivalent.*

1. $F(\phi(A_1) * \cdots * \phi(A_k)) = F(A_1 * \cdots * A_k)$ *for all* $A_1, \cdots, A_k \in \mathcal{B}(X)$.
2. *There exists* $\alpha \in \mathbf{C}$ *with* $\alpha^m = 1$ *such that* $\phi(A) = \alpha A$ *for all* $A \in \mathcal{B}(X)$.

2. Preliminaries

In this section, we collect and prove some lemmas that will be used in the sequel. The first and the second are quoted from [7]. Denote $C^* = \mathbf{C} \backslash \{0, 1\}$. **Lemma 2.1.** *Let $A \in \mathcal{B}(X)$, then $A \in \mathbf{C}^* I$ if and only if $F(PAP) = \{0\}$, for every rank one idempotent operators $P \in \mathcal{B}(X)$.*

Proof. See [7, Lemma 2.1]. □

Lemma 2.2. *Let A and B be non-scalar operators. If $F(PAP) = F(PBP)$, for every rank one idempotent operators $P \in \mathcal{B}(X)$, then there exists a scalar $\lambda \in \mathbf{C} \setminus \{1\}$ such that $B = \lambda I + (1-\lambda)A$.*

Proof. See [7, Lemma 2.2]. □

In the following we give a conditions in term of dimension of fixed points of generalized product for two operators to be the same.

Lemma 2.3. *Let A and B in $\mathcal{B}(X)\setminus\{0\}$, and r and s two positive integers such that $r+s \geq 1$. The following statements are equivalent.*

1. *$A = B$.*
2. *$\dim F(T^r AT^s) = \dim F(T^r BT^s)$ for all operators $T \in \mathcal{B}(X)$.*
3. *$\dim F(R^r AR^s) = \dim F(R^r BR^s)$ for all rank one operators $R \in \mathcal{B}(X)$.*

Proof. The implications $(i) \Rightarrow (ii) \Rightarrow (iii)$ can be easily obtained. It remains to show that $(iii) \Rightarrow (i)$. So, by Lemma 2.1, it is clear that $A \in \mathbf{C}^* I$ if and only if $B \in \mathbf{C}^* I$. Let $A = \alpha I$ and $B = \beta I$, for some $\alpha, \beta \in \mathbf{C}^*$. By assumption, we have

$$\dim F(f(x)^{r+s-2} f(Ax)x \otimes f) = \dim F(f(x)^{r+s-2} f(Bx)x \otimes f)$$

which implies that $f(x)^{r+s} = \alpha^{-1}$ if only if $f(x)^{r+s} = \beta^{-1}$ and so $\alpha = \beta$. Now let A be a non-scalar operator. Since B is a non-scalar operator too, we can apply Lemma 2.2. Thus, there exists a scalar $\lambda \in \mathbf{C} \setminus \{1\}$ such that $B = \lambda I + (1-\lambda)A$. It is enough to prove that $\lambda = 0$. Assume on the contrary that $\lambda = 0$ and let $x \in X$ and $f \in X^*$ such that $f(x) =, 1, -1$ and $f(Ax) = \frac{1}{f(x)^{r+s-1}}$. It is clear that

$$\dim((x \otimes f)^r A(x \otimes f)^s) = 1.$$

Also we obtain

$$(x \otimes f)^r (\lambda I + (\lambda)A)(x \otimes f)^s = f(x)^{r+s-1}(\lambda f(x) + (1-\lambda) f(Ax)) = 1$$

which implies that

$$\dim F(x \otimes f)^r (\lambda I + (\lambda)A)(x \otimes f)^s = \dim F((x \otimes f)^r B(x \otimes f)^s) = 0.$$

This is a contradiction. It follows that $A = B$ and thus the lemma is established. □

Finally, we close this section with the following lemma that gives a characterization of rank-one operators by the dimension of fixed points of generalized product of operators.

Lemma 2.4. *Let r and s be positive integers such that $r + s \geq 1$. For a nonzero operator $R \in \mathcal{B}(X)$, the following statements are equivalent.*

1. *R is a rank one operator.*
2. *$\dim F(T^r RT^s)) \leq 1$ for all $T \in \mathcal{B}(X)$.*

Proof. If R is a rank one operator and $T \in \mathcal{B}(X)$ is an arbitrary operator, then $T^r RT^s$ has rank at most one. Therefore, by (1.1) and (1.2), we have $\dim F(T^r RT^s) \leq 1$, and the implication $(i) \Rightarrow (ii)$ is established. Conversely, assume that R has rank at least two. Let us show that there exists $T \in \mathcal{B}(X)$ such that $\dim F(T^r RT^s) \geq 2$. Since $rank(R) \geq 2$, let y_1, y_2 be two linearly independent vectors in the range of R, and x_1, x_2 in X such that $Rx_1 = y_1$ and $Rx_2 = y_2$. Assume that $s \neq 0$ and $r \neq 0$. Since X has infinite dimension, we can find linearly independent vectors

$$z_0, z_1, ..., z_s, z_{s+1}, ..., z_{s+r-1}, w_0, w_1, ..., w_s, ..., w_{s+r-1}$$

with $z_s := y_1$ and $w_s := y_2$. Now, we can find a finite rank operator $T \in \mathcal{B}(X)$ such that

$$Tz_{s-1} = x_1,\ Tz_{s+r-1} = z_0,\ Tw_{s-1} = x_2,\ Tw_{s+r-1} = w_0$$

and for $i \in \{0, 1, ..., r + s - 1\} \setminus \{s - 1, r + s - 1\}$

$$Tz_i = z_{i+1} \text{ and } Tw_i = w_{i+1}.$$

Note that $T^r RT^s z_0 = z_0$ and $T^r RT^s w_0 = w_0$. Then $z_0, w_0 \in F(T^r RT^s)$, hence $\dim F(T^r RT^s) \geq 2$. This shows that the implication $(ii) \Rightarrow (i)$ always holds in case both r and s are positive integers.

To finish, we may and will assume that $s = 0$ as the case when $r = 0$ is similar. Then choose linearly independent vectors $z_0, z_1, ..., z_{r-1}, w_0, w_1, ..., w_{r-1}$ in X with $z_0 = y_1, w_0 = y_2$. Thus, there exists a finite rank operator $T \in \mathcal{B}(X)$ such that $Tz_i = z_{i+1}$ for $i \in \{0, r - 2\}$ and $Tz_{r-1} = x_1$ and $Tw_i = w_{i+1}$ for $i \in \{0, r - 2\}$ and $Tw_{r-1} = x_2$. We have $T^r Rx_1 = x_1$ and $T^r Rx_2 = x_2$. Hence, $x_1, x_2 \in F(T^r R)$ and $\dim F(T^r R)) \geq 2$. This establishes the implication in this case too and the proof is then complete. □

3. Main Results

Since all the necessary ingredients are collected in the preliminary section we will state and prove the promised main result. Let $A, B \in \mathcal{B}(X)$, set $A_{i_p} = B$ and $A_{i_j} = A$ for $j \neq p$ with i_p is the term which appears exactly once in $(i_1, \cdots, i_m)$. Note that $A_1 * A_2 * \cdots * A_k = A^r B A^s$ for some positive integers r and s such that $r + s = m - 1$. Then, the Theorem 1.1 is a consequence of the following one.

Theorem 3.1.
Let r and s be two positive integers with $r + s \geq 1$ and $\phi : \mathcal{B}(X) \to \mathcal{B}(X)$ be a surjective map. Then the following statements are equivalent.

1. *For all $A, B \in \mathcal{B}(X)$,*

$$F(\phi(A)^r \phi(B) \phi(A)^s) = F(A^r B A^s). \tag{3.1}$$

2. *There exists $\alpha \in \mathbf{C}$ with $\alpha^{r+s+1} = 1$ such that $\phi(A) = \alpha A$ for all $A \in \mathcal{B}(X)$.*

Proof. The 'if' part is easily verified, so we need only to prove the 'only if' part. Indeed, assume that ϕ is a surjective map from $\mathcal{B}(X)$ into itself such that for all $A, B \in \mathcal{B}(X)$,

$$F(\phi(A)^r \phi(B) \phi(A)^s) = F(A^r B A^s).$$

We divide the proof into several steps.

Step 1. $\phi(0) = 0$ and ϕ is injective.
By assumption, ϕ is surjective, so there exist $A \in \mathcal{B}(X)$ such that $\phi(A) = x \otimes f$. Then by hypothesis we have

$$\begin{aligned} \{0\} &= F(A^r 0 A^s) \\ &= F(\phi(A)^r \phi(0) \phi(A)^s) \\ &= F((x \otimes f)^r \phi(0) (x \otimes f)^s) \\ &= F(f(x)^{r+s-2} f(\phi(0)x) x \otimes f) \end{aligned}$$

which implies that

$$f(x)^{r+s-1} f(\phi(0)x) \neq 1. \tag{3.2}$$

So x and $\phi(0)x$ are linearly dependant for every $x \in X$, because otherwise, we can find a linear functional f such that $f(x) \neq 0$ and $f(\phi(0)x) =$

$\frac{1}{f(x)^{r+s-1}}$. Thus, there exists a complex number γ such that $\phi(0) = \gamma I$. Replace in 3.2, we get $\gamma f(x)^{r+s-1} \neq 1$. Hence $\gamma = 0$ and $\phi(0) = 0$.

Let $A, B \in \mathcal{B}(X)$ such that $\phi(A) = \phi(B)$. Then, by hypothesis, for every $R \in \mathcal{B}(X)$ we have

$$\begin{aligned} F(R^r A R^s) &= F(\phi(R)^r \phi(A) \phi(R)^s) \\ &= F(\phi(R)^r \phi(B) \phi(R)^s) \\ &= F(R^r B R^s). \end{aligned}$$

It follows, by lemma 2.3 that $A = B$, and so ϕ is injective. Thus ϕ is bijective since it is assumed surjective, moreover ϕ^{-1} satisfied the equation (3.1).

Step 2. ϕ preserves rank one operators in both directions.

This is obvious by Lemma 2.4 and the assumption that ϕ is surjective.

Step 3. $\phi(P) = \alpha P$ for every rank-one idempotent P and some scalar $\alpha \in \mathbf{C}$ with $\alpha^{r+s+1} = 1$.
Let $P \in \mathcal{P}_1(X)$ be a rank-one idempotent operator. Note $P = x \otimes f$ where $x \in X$ and $f \in X^*$ with $f(x) = 1$. By the previous step ϕ preserves rank-one operators in both directions. Then there exist $y \in X$ and $g \in X^*$ such that $\phi(P) = y \otimes g$. From 3.1 and hypothesis, we have

$$\begin{aligned} \langle x \rangle = F(P) &= F(P^r P P^s) \\ &= F(\phi(P)^r \phi(P) \phi(P)^s) \\ &= F(f(y)^{r+s} y \otimes g). \end{aligned} \tag{3.3}$$

Then $\langle x \rangle = \langle y \rangle$ and so x and y are linearly dependant. Without loss of generality we can assume that $x = y$ and so $\phi(x \otimes f) = x \otimes g$. By (3.3) we obtain

$$g(x)^{r+s+1} = 1 \tag{3.4}$$

and so $x \notin ker(g)$. Thus we have

$$ker(f) \cup \langle x \rangle = X, \; ker(g) \cup \langle x \rangle = X$$

and its clear that

$$ker(f) \cap \langle x \rangle = \{0\}, \; ker(g) \cap \langle x \rangle = \{0\}$$

which imply that $ker(f) = ker(g)$ and so f and g are linearly dependant. This and (3.4) yield that there exists a complex number α with $\alpha^{r+s+1} = 1$ such that $\phi(P) = \alpha P$.

Step 4. $\phi(A) = \alpha A$ for every $A \in \mathcal{B}(X)$ and some scalar $\alpha \in \mathbf{C}$ with $\alpha^{r+s+1} = 1$.
If A is a rank one operator, by Lemma 2.2 there exists a $\lambda \in \mathbf{C} \setminus \{1\}$ such that

$$\phi(A) = \alpha(\lambda I + (1 - \lambda)A),$$

because by Lemma 2.1, A and $\phi(A)$ are non-scalar operators and by Step 3, $\phi(P) = \alpha P$ for every rank one idempotent P. Since $\phi(A)$ is a rank one operator, we obtain $\lambda = 0$ and so $\phi(A) = \alpha A$ for every rank one operator A. Now if A is an arbitrary operator, the assertion follows from Lemma 2.4, then ϕ takes the desired form. $\square$

By taking $r = s = 1$ the main theorem of [7] becomes corollary of Theorem 1.1.

Corollary 3.2.

Let $\phi : \mathcal{B}(X) \to \mathcal{B}(X)$ be a surjective map. Then the following statements are equivalent.

1. $F(\phi(A)\phi(B)\phi(A)) = F(ABA)$ *for all* $A, B \in \mathcal{B}(X)$.
2. *There exists* $\alpha \in \mathbf{C}$ *with* $\alpha^3 = 1$ *such that* $\phi(A) = \alpha A$ *for all* $A \in \mathcal{B}(X)$.

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