Proyecciones Journal of Mathematics
Vol. 40, N^o 5, pp. 1147-1154, October 2021.
Universidad Católica del Norte
Antofagasta - Chile

10.22199/issn.0717-6279-3532

Vertex cover and edge-vertex domination in trees

B. Senthilkumar
SASTRA Deemed University, India
H. Naresh Kumar
SASTRA Deemed University, India
and
Y. B. Venkatakrishnan
SASTRA Deemed University, India
Received : May 2019. Accepted : March 2021

Abstract

Let $G = (V, E)$ be a simple graph. An edge $e \in E(G)$ edge-vertex dominates a vertex $v \in V(G)$ if e is incident with v or e is incident with a vertex adjacent to v. A subset $D \subseteq E(G)$ is an edge-vertex dominating set of a graph G if every vertex of G is edge-vertex dominated by an edge of D. A vertex cover of G is a set $C \subseteq V$ such that for each edge $uv \in E$ at least one of u and v is in C. We characterize trees with edge-vertex domination number equals vertex covering number.

AMS classification: *05C69, 05C70.*

Keywords: *Edge vertex dominating set, vertex cover, trees.*

1. Introduction

Let $G = (V, E)$ be a graph. By the neighborhood of a vertex v of G we mean the set $N_G(v) = \{u \in V(G)\colon uv \in E(G)\}$. The degree of a vertex v, denoted by $d_G(v)$, is the cardinality of its neighborhood. By a leaf, we mean a vertex of degree one, while a support vertex is a vertex adjacent to a leaf. We say that a support vertex is strong (weak, respectively) if it is adjacent to at least two leaves (exactly one leaf, respectively). The path on n vertices we denote by P_n. Let T be a tree, and let v be a vertex of T. We say that v is adjacent to a path P_n if there is a neighbor of v, say x, such that the subtree resulting from T by removing the edge vx and which contains the vertex x as a leaf, is a path P_n. By a star, we mean a connected graph in which exactly one vertex has degree greater than one.

A vertex cover, abbreviated VC, of G is a set $C \subseteq V$ such that for each edge $uv \in E$ at least one of u and v is in C. Let $\alpha(G)$ be the vertex covering number of G, the minimum number of vertices required to cover all edges of G.

An edge $e \in E(G)$ edge-vertex dominates a vertex $v \in V(G)$ if e is incident with v or e is incident with a vertex adjacent to v. A subset $D \subseteq E(G)$ is an edge-vertex dominating set, abbreviated EVDS, of a graph G if every vertex of G is edge-vertex dominated by an edge of D. The edge-vertex domination number of G, denoted by $\gamma_{ev}(G)$, is the minimum cardinality of an edge-vertex dominating set of G. An edge-vertex dominating set of G of minimum cardinality is called a $\gamma_{ev}(G)$-set. Edge-vertex domination in graphs was introduced in [5] and studied further in [3, 4, 6].

Relating vertex covering number with other dominating parameters is studied in [1, 2]. In this paper, we characterize trees with equal vertex covering number and edge-vertex domination number.

2. Results

We begin with the following proposition.

Proposition 1. *For any graph G, we have $\gamma_{ev}(G) \leq \alpha(G)$.*

Proof. Let D be any $\alpha(G)$-set. Let $x_1, x_2, \cdots, x_\alpha$ be the vertices in D. Consider an edge e_i incident with the vertex x_i. Define A as the set of all selected edges e_i. Clearly the set A is an edge vertex dominating set. Thus $\gamma_{ev}(G) \leq \alpha(G)$. □

We now characterize trees with equal edge-vertex domination number and vertex covering number. For this purpose, we introduce the family $\mathcal{T}$ of trees $T = T_k$ that can be obtained as follows. Let $T_1 \in \{P_2, P_3, P_5\}$. If k is a positive integer, then T_{k+1} can be constructed recursively from T_k by one of the following operations:

- Operation $\mathcal{O}_1$: Attach a vertex by joining it to a support vertex of T_k.
- Operation $\mathcal{O}_2$: Attach a path P_2 by joining one of its vertex to a vertex of T_k adjacent to path P_2.
- Operation $\mathcal{O}_3$: Attach a path P_3 by joining one of its leaf to a vertex of T_k adjacent to path P_3.
- Operation $\mathcal{O}_4$: Attach a path P_3 by joining one of its leaf to a support vertex of T_k.

We now prove that for every tree T of the family $\mathcal{T}$, the tree T has equal edge-vertex domination number and vertex covering number.

Lemma 2. *If $T \in \mathcal{T}$, then $\gamma_{ev}(T) = \alpha(T)$.*

Proof. We use induction on the number k of operations performed to construct the tree T. Suppose $T = P_2$ or P_3, then $\gamma_{ev}(T) = 1 = \alpha(T)$. If $T = P_5$, then $\gamma_{ev}(T) = 2 = \alpha(T)$. Let k be a positive integer. Assume that the result is true for every tree $T' = T_k$ of the family $\mathcal{T}$ constructed by $k-1$ operations. Let $T = T_{k+1}$ be a tree of the family $\mathcal{T}$ constructed by k operations.

Assume that T is obtained from T' by operation $\mathcal{O}_1$. Let x be the support vertex to which a leaf y is attached. Let z be a leaf different from y adjacent to x. Let D be a $\alpha(T)$-set. To cover the edges xz, xy the vertex $x \in D$. It is easy to see that D is a vertex cover of tree T'. Thus $\alpha(T') \leq \alpha(T)$. Let D' be a $\alpha(T')$-set. To cover the edge xz the vertex $x \in D$. The vertex x also covers the edge xy in tree T. Thus $\alpha(T') \leq \alpha(T)$. This implies that $\alpha(T) = \alpha(T')$. Let S be a $\gamma_{ev}(T)$-set. Suppose $xy \in S$. Then $xz \notin S$. It is easy to see that $D \setminus \{xy\} \cup \{xz\}$ is an EVDS of tree T'. If $xy \notin S$ then obviously S is an EVDS of tree T'. Thus $\gamma_{ev}(T') \leq \gamma_{ev}(T)$. Let S' be a $\gamma_{ev}(T')$-set. The edge which dominates the vertex z dominates the vertex y in tree T. Thus, $\gamma_{ev}(T) \leq \gamma_{ev}(T')$. We have $\gamma(T) = \gamma(T') = \alpha(T') = \alpha(T)$.

Assume that T is obtained from T' by operation $\mathcal{O}_2$. Let x be the vertex to which a path $P_2 : uv$ is attached. Let u be adjacent to x. Let yz be a path different from uv adjacent to x. Let y be adjacent to x. Let D' be a $\gamma_{ev}(T')$-set. It is easy to see that $D' \cup \{xu\}$ is an EVDS of tree T. Thus $\gamma_{ev}(T) \leq \gamma_{ev}(T') + 1$. Let D be a $\gamma_{ev}(T)$-set. To dominate the vertices z and v the edges $xy, xu \in D$. It is obvious that $D \setminus \{xu\}$ is an EVDS of tree T'. Thus, $\gamma_{ev}(T') \leq \gamma_{ev}(T) - 1$. This implies that $\gamma_{ev}(T) \leq \gamma_{ev}(T') + 1$. Let S' be a $\alpha(T')$-set. It is clear that $S' \cup \{u\}$ is a vertex cover of tree T. Thus $\alpha(T) \leq \alpha(T') + 1$. Let S be a $\alpha(T)$-set. To cover the edges yz, xy, xu and uv the vertices $y, u \in S$. It is clear that $S \setminus \{u\}$ is a vertex cover of tree T'. Thus, $\alpha(T') \leq \alpha(T) - 1$. This implies that $\alpha(T) = \alpha(T') + 1$. We have $\gamma_{ev}(T) = \gamma_{ev}(T') + 1 = \alpha(T') + 1 = \alpha(T)$.

Assume that T is obtained from T' by operation $\mathcal{O}_3$. Let x be the vertex to which a path $P_3 : wyz$ is attached. Let w be adjacent to x. Let abc be a path different from wyz adjacent to x. Let a be adjacent to x. Let D' be a $\gamma_{ev}(T')$-set. It is easy to see that $D' \cup \{wy\}$ is an EVDS of tree T. Thus $\gamma_{ev}(T) \leq \gamma_{ev}(T') + 1$. Let D be a $\gamma_{ev}(T)$-set. To dominate the vertices x, w, y, z, a, b and c the edges $wy, ab \in D'$. It is clear that $D \setminus \{xy\}$ is an EVDS of tree T'. Thus, $\gamma_{ev}(T') \leq \gamma_{ev}(T) - 1$. This implies that $\gamma_{ev}(T) \leq \gamma_{ev}(T') + 1$. Let S' be a $\alpha(T')$-set. To cover the egdes bc and ab the vertex $b \in S'$. To cover the edge ax the vertex $b \in S'$. The vertex x covers the edge xw in tree T. Thus $S' \cup \{y\}$ is a vertex cover of tree T. Thus $\alpha(T) \leq \alpha(T') + 1$. Let S be a $\alpha(T)$-set. To cover the edges xw and xa the vertices $y, b \in S$. It is clear that $S \setminus \{y\}$ is a vertex cover of tree T'. Thus $\alpha(T') \leq \alpha(T) - 1$. This imples that $\alpha(T) = \alpha(T') + 1$. We have $\gamma_{ev}(T) = \gamma_{ev}(T') + 1 = \alpha(T') + 1 = \alpha(T)$.

Assume that T is obtained from T' by operation $\mathcal{O}_4$. Let x be a support vertex to which a path $P_3 : uvw$ is attached. Let u be adjacent to x. The leaf adjacent to x is denoted by y. Let S' be a $\alpha(T')$-set. To cover the edge xy the vertex $x \in S'$. The vertex x covers the edge xu in tree T. It is easy to see that $S' \setminus \{v\}$ is a vertex cover of tree T. Thus $\alpha(T) \leq \alpha(T') + 1$. Let S be a $\alpha(T)$-set. To cover the edges vw, uv, xu and xy the vertices $v, x \in S$. It is obvious that $S \setminus \{v\}$ is a vertex cover of tree T'. Thus $\alpha(T') \leq \alpha(T) - 1$. This imples that $\alpha(T) = \alpha(T') + 1$. Let D' be a $\gamma_{ev}(T')$-set. It is easy to see that $D' \cup \{uv\}$ is an EVDS of tree T. Thus $\gamma_{ev}(T) \leq \gamma_{ev}(T') + 1$. Let D be a $\gamma_{ev}(T)$-set. To dominate the vertex y the edge incident with x other than xy and xu is in D. It is easy to see that $xy, xu \in D$. It is easy to see that $D \setminus \{uv\}$ is an EVDS of tree T'. Thus, $\gamma_{ev}(T') \leq \gamma_{ev}(T) - 1$. This implies that $\gamma_{ev}(T) \leq \gamma_{ev}(T') + 1$. We have

$\gamma_{ev}(T) = \gamma_{ev}(T') + 1 = \alpha(T') + 1 = \alpha(T)$. $\square$

We now prove that if T has equal edge-vertex domination number and vertex covering number, then the tree belongs to the family $\mathcal{T}$.

Lemma 3. *Let T be a tree. If $\gamma_{ev}(T) = \alpha(T)$, then $T \in \mathcal{T}$.*

Proof. If $diam(T) = 1$, then T is P_2. We have $\gamma_{ev}(T) = 1 = \alpha(T)$, thus $T \in \mathcal{T}$. If $diam(T) = 2$, then T is star. If T is P_3, then $T \in \mathcal{T}$. If T is a star other than P_3, then T is obtained from P_3 by applying operation $\mathcal{O}_1$ appropriately.

Now assume that $diam(T) \geq 3$. Thus the order n of the tree is at least four. We obtain the result by induction on the number n. Assume that the theorem is true for every T' of order $n' < n$.

First assume that some support vertex of T, say x, is strong. Let y and z be the two leaves adjacent to x. Let $T' = T - y$. Let D' be a $\gamma_{ev}(T')$-set. It is obvious that D' is an EVDS of tree T. Thus $\gamma_{ev}(T) \leq \gamma_{ev}(T')$. Let S be a $\alpha(T)$-set. To cover the edges xy, yz the vertex $x \in S$. It is easy to see that S is a vertex cover of tree T'. Thus $\alpha(T') \leq \alpha(T)$. We have $\gamma_{ev}(T) \leq \gamma_{ev}(T') \leq \alpha(T') \leq \alpha(T)$. If $\gamma_{ev}(T) = \alpha(T)$ then $\gamma_{ev}(T') = \alpha(T')$. By inductive hypothesis $T' \in \mathcal{T}$. The tree T is obtained from T' by operation $\mathcal{O}_1$. Thus $T \in \mathcal{T}$.

We now root T at a vertex r of maximum eccentricity $diam(T)$. Let t be a leaf at maximum distance from r, v be the parent of t, u be the parent of v in the rooted tree. If $diam(T) \geq 4$, then let w be the parent of u. If $diam(T) \geq 5$, then let d be the parent of w. If $diam(T) \geq 6$, then let e be the parent of d. By T_x we denote the subtree induced by a vertex x and its descendants in the rooted tree T.

Assume that some child of u there is a support vertex, say x, other than v. Let y be the leaf adjacent to x. Let $T' = T - T_x$. Let D' be a $\gamma_{ev}(T')$-set. It is obvious that $D' \cup \{ux\}$ is an EVDS of tree T. Thus $\gamma_{ev}(T) \leq \gamma_{ev}(T') + 1$. Let S be a $\alpha(T)$-set. To cover the edges vt, uv the vertex $v \in S$. To cover the edges xy, ux the vertex $x \in S$. It is easy to see that $S \setminus \{x\}$ is a vertex cover of tree T'. Thus $\alpha(T') \leq \alpha(T) - 1$. We have $\gamma_{ev}(T) \leq \gamma_{ev}(T') + 1 \leq \alpha(T') + 1 \leq \alpha(T)$. If $\gamma_{ev}(T) = \alpha(T)$ then $\gamma_{ev}(T') = \alpha(T')$. By inductive hypothesis $T' \in \mathcal{T}$. The tree T is obtained from T' by operation $\mathcal{O}_2$. Thus $T \in \mathcal{T}$.

Assume that some child of u, say x is a leaf. Let $T' = T - T_u$. Let D' be a $\gamma_{ev}(T')$-set. It is easy to see that $D' \cup \{uv\}$ is an EVDS of tree T. Thus $\gamma_{ev}(T) \leq \gamma_{ev}(T') + 1$. Let S be a $\alpha(T)$-set. To cover the edges wu, ux, uv and vt, the vertices $u, v \in S$. It is obvious that $S \setminus \{u, v\}$ is a vertex cover of tree T'. Thus $\alpha(T') \leq \alpha(T) - 2$. We now get $\gamma_{ev}(T) \leq \gamma_{ev}(T') + 1 \leq \alpha(T') + 1 \leq \alpha(T) - 1 < \alpha(T)$.

Assume $d_T(u) = 2$. Assume that some child of w, other than u, say x such that the distance of w to the most distant vertex of T_x is three. It suffices to consider the case T_x is a path $P_3 = xyz$. Let $T' = T - T_w$. Let D' be a $\gamma_{ev}(T')$-set. To dominate the vertices w, x, y an z the edge $xy \in D'$. It is clear that $D' \setminus \{uv\}$ is an EVDS of tree T. Thus $\gamma_{ev}(T) \leq \gamma_{ev}(T') + 1$. Let S be a $\alpha(T)$-set. To cover the edges yz, xy, wx, wu, uv and vt the vertices $y, w, v \in S$. It is clear that $S \setminus \{v\}$ is a vertex cover of tree T'. Thus $\alpha(T') \leq \alpha(T) - 1$. We have $\gamma_{ev}(T) \leq \gamma_{ev}(T') + 1 \leq \alpha(T') + 1 \leq \alpha(T)$. If $\gamma_{ev}(T) = \alpha(T)$ then $\gamma_{ev}(T') = \alpha(T')$. By inductive hypothesis $T' \in \mathcal{T}$. The tree T is obtained from T' by operation $\mathcal{O}_3$. Thus $T \in \mathcal{T}$.

Assume that some child of w, other than u, say x such that the distance of w to the most distant vertex of T_x is two. It suffices to consider the case T_x is a path $P_2 = xy$. By operation $\mathcal{O}_2$, we can assume that w is adjacent to exactly one $P_2 = xy$. Thus, $d_T(w) = 3$. Let $T' = T - T_w$. Let D' be a $\gamma_{ev}(T')$-set. It is easy to observe that $D' \setminus \{wx, uv\}$ is an EVDS of tree T. Thus $\gamma_{ev}(T) \leq \gamma_{ev}(T') + 2$. Let S be a $\alpha(T)$-set. To cover the edges wx, wu, wd, uv and vt the vertices $x, w, v \in S$. It is easy to see that $S \setminus \{x, w, v\}$ is a vertex cover of tree T'. Thus $\alpha(T') \leq \alpha(T) - 3$. We have $\gamma_{ev}(T) \leq \gamma_{ev}(T') + 2 \leq \alpha(T') + 2 \leq \alpha(T) - 3 + 2 < \alpha(T)$.

Assume that some child of w other than u, say x is a leaf. Let $T' = T - T_u$. Let D' be a $\gamma_{ev}(T')$-set. It is easy to observe that $D' \setminus \{uv\}$ is an EVDS of tree T. Thus $\gamma_{ev}(T) \leq \gamma_{ev}(T') + 1$. Let S be a $\alpha(T)$-set. To cover the edges vt, uv and wx the vertices $v, w \in S$. It is clear that $S \setminus \{v\}$ is a vertex cover of tree T'. Thus $\alpha(T') \leq \alpha(T) - 1$. We have $\gamma(T) \leq \gamma(T') + 1 \leq \alpha(T') + 1 \leq \alpha(T)$. If $\gamma_{ev}(T) = \alpha(T)$ then $\gamma_{ev}(T') = \alpha(T')$. By inductive hypothesis $T' \in \mathcal{T}$. The tree T is obtained from T' by operation $\mathcal{O}_4$. Thus $T \in \mathcal{T}$.

Assume $d_T(w) = 2$. If $d_T(d) = 1$, then $T = P_5$. We have $\gamma_{ev}(T) = 2 = \alpha(T)$, thus $T \in \mathcal{T}$. Assume $d_T(d) \geq 3$. Let $T' = T - T_w$. Let D' be a $\gamma_{ev}(T')$-set. It is easy to see that $D' \cup \{uv\}$ is an EVDS of tree T. Thus, $\gamma_{ev}(T) \leq \gamma_{ev}(T') + 1$. Let S be a $\alpha(T)$-set. To cover the edges dw, wu, uv and vt the vertices $w, v \in S$. It is obvious that $S \setminus \{w, v\}$ is a vertex cover of tree T'. Thus, $\alpha(T') \leq \alpha(T) - 2$. We have $\gamma_{ev}(T) \leq \gamma_{ev}(T') + 1 \leq \alpha(T') + 1 \leq \alpha(T) - 2 + 1 < \alpha(T)$ □

As an immediate consequence of Lemma 2 and 3, we have the following characterization of trees with edge-vertex domination number equals vertex covering number.

Theorem 4. *Let T be a tree. Then $\gamma_{ev}(T) = \alpha(T)$ if and only if $T \in \mathcal{T}$.*

Acknowledgement: The authors thank TATA-Realty and Infrastructure Limited for its support. The third author thank DST-SERB(MATRICS), India-grant MTR/2018/000234 for its support.

References

[1] R. Dutton and W. F. Klostermeyer, "Edge dominating sets and Vertex Covers", *Discussiones Mathematicae Graph Theory*, vol. 33, pp. 437-456, 2013.

[2] W. F. Klostermeyer, M. E. Messinger and A. Yeo, "Dominating Vertex Covers: The Vertex-Edge Domination Problem", *Discussiones Mathematicae Graph Theory*, vol. 41, pp. 123-132, doi: 10.7151/dmgt.2175

[3] B. Krishnakumari, Y. B. Venkatakrishnan and M. Krzywkowski, "On trees with total domination number equal to edge-vertex domination number plus one", *Proceedings - Mathematical Sciences*, vol. 126, pp. 153-157, 2016.

[4] J. R. Lewis, "Vertex-edge and edge-vertex parameters in graphs", Ph. D. Thesis, Clemson University, 2007.

[5] K. W. Peters, "Theoretical and Algorithmic Results on Domination and Connectivity", Ph.D. Thesis, Clemson University, 1986.

[6] Y. B. Venkatakrishnan and B. Krishnakumari, "An improved upper bound of edge-vertex domination number of a tree", *Information Processing Letters*, vol. 134, pp. 14-17, 2018.

B. Senthilkumar
Department of Mathematics,
School of Arts,
Science and Humanities
SASTRA Deemed University,
Tanjore, Tamilnadu 613 401,
India
e-mail: senthilsubramanyan@gmail.com

H. Naresh Kumar
Department of Mathematics,
School of Arts,
Science and Humanities
SASTRA Deemed University,
Tanjore, Tamilnadu 613 401,
India
e-mail: nareshhari1403@gmail.com

and

Y. B. Venkatakrishnan
Department of Mathematics,
School of Arts,
Science and Humanities
SASTRA Deemed University,
Tanjore, Tamilnadu 613 401,
India
e-mail: venkatakrish2@maths.sastra.edu
Corresponding author