A note on modified third-order Jacobsthal numbers

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Abstract:

Modified third-order Jacobsthal sequence is defined in this study. Some properties involving this sequence, including the Binet-style formula and the generating function are also presented.

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1. Introduction

The Jacobsthal numbers have many interesting properties and applications in many fields of science (see, e. g., [1, 6, 7]). The Jacobsthal numbers J_n are defined by the recurrence relation

$$(1.1) \quad J_0 = 0, \quad J_1 = 1, \quad J_{n+2} = J_{n+1} + 2J_n, \quad n \geq 0.$$

Another important sequence is the Jacobsthal-Lucas sequence. This sequence is defined by the recurrence relation $j_{n+2} = j_{n+1} + 2j_n$, where $j_0 = 2$ and $j_1 = 1$ (see, [7]).

In [5] the Jacobsthal recurrence relation is extended to higher order recurrence relations and the basic list of identities provided by A. F. Horadam [7] is expanded and extended to several identities for some of the higher order cases. For example, the third-order Jacobsthal numbers, $\{J_n^{(3)}\}_{n \geq 0}$, and third-order Jacobsthal-Lucas numbers, $\{j_n^{(3)}\}_{n \geq 0}$, are defined by

$$(1.2) \quad J_{n+3}^{(3)} = J_{n+2}^{(3)} + J_{n+1}^{(3)} + 2J_n^{(3)}, \quad J_0^{(3)} = 0, \quad J_1^{(3)} = J_2^{(3)} = 1, \quad n \geq 0,$$

and

$$(1.3) \quad j_{n+3}^{(3)} = j_{n+2}^{(3)} + j_{n+1}^{(3)} + 2j_n^{(3)}, \quad j_0^{(3)} = 2, \quad j_1^{(3)} = 1, \quad j_2^{(3)} = 5, \quad n \geq 0,$$

respectively.

Some of the following properties given for third-order Jacobsthal numbers and third-order Jacobsthal-Lucas numbers are used in this paper (for more details, see [2, 3, 4, 5]). Note that Eqs. (1.8) and (1.12) have been corrected in this paper, since they have been wrongly described in [5].

$$(1.4) \quad 3J_n^{(3)} + j_n^{(3)} = 2^{n+1},$$

$$(1.5) \quad j_n^{(3)} - 3J_n^{(3)} = 2j_{n-3}^{(3)}, \quad n \geq 3,$$

$$(1.6) \quad J_{n+2}^{(3)} - 4J_n^{(3)} = \begin{cases} -2 & \text{if } n \equiv 1 \pmod{3} \\ 1 & \text{if } n \not\equiv 1 \pmod{3} \end{cases},$$

$$(1.7) \quad j_n^{(3)} - 4J_n^{(3)} = \begin{cases} 2 & \text{if } n \equiv 0 \pmod{3} \\ -3 & \text{if } n \equiv 1 \pmod{3} \\ 1 & \text{if } n \equiv 2 \pmod{3} \end{cases},$$

$$(1.8) \quad j_{n+1}^{(3)} + j_n^{(3)} = 3J_{n+2}^{(3)},$$

$$(1.9) \quad j_n^{(3)} - J_{n+2}^{(3)} = \begin{cases} 1 & \text{if } n \equiv 0 \pmod{3} \\ -1 & \text{if } n \equiv 1 \pmod{3} \\ 0 & \text{if } n \equiv 2 \pmod{3} \end{cases},$$

$$(1.10) \quad \left(j_{n-3}^{(3)}\right)^2 + 3J_n^{(3)}j_n^{(3)} = 4^n,$$

$$(1.11) \quad \sum_{k=0}^n J_k^{(3)} = \begin{cases} J_{n+1}^{(3)} & \text{if } n \not\equiv 0 \pmod{3} \\ J_{n+1}^{(3)} - 1 & \text{if } n \equiv 0 \pmod{3} \end{cases}$$

and

$$(1.12) \quad \left(j_n^{(3)}\right)^2 - 9\left(J_n^{(3)}\right)^2 = 2^{n+2}j_{n-3}^{(3)}, \quad n \geq 3.$$

Using standard techniques for solving recurrence relations, the auxiliary equation, and its roots are given by

$$x^3 - x^2 - x - 2 = 0; \quad x = 2, \text{ and } x = \frac{-1 \pm i\sqrt{3}}{2}.$$

Note that the latter two are the complex conjugate cube roots of unity. Call them ω_1 and ω_2 , respectively. Thus the Binet formulas can be written as

$$(1.13) \quad J_n^{(3)} = \frac{2}{7}2^n - \frac{3 + 2i\sqrt{3}}{21}\omega_1^n - \frac{3 - 2i\sqrt{3}}{21}\omega_2^n$$

and

$$(1.14) \quad j_n^{(3)} = \frac{8}{7}2^n + \frac{3 + 2i\sqrt{3}}{7}\omega_1^n + \frac{3 - 2i\sqrt{3}}{7}\omega_2^n,$$

respectively. Now, we use the notation

$$(1.15) \quad V_n^{(2)} = \frac{A\omega_1^n - B\omega_2^n}{\omega_1 - \omega_2} = \begin{cases} 2 & \text{if } n \equiv 0 \pmod{3} \\ -3 & \text{if } n \equiv 1 \pmod{3} \\ 1 & \text{if } n \equiv 2 \pmod{3} \end{cases},$$

where $A = -3 - 2\omega_2$ and $B = -3 - 2\omega_1$. Furthermore, note that for all $n \geq 0$ we have

$$(1.16) \quad V_{n+2}^{(2)} = -V_{n+1}^{(2)} - V_n^{(2)}, \quad V_0^{(2)} = 2 \text{ and } V_1^{(2)} = -3.$$

From the Binet formulas (1.13), (1.14) and Eq. (1.15), we have

$$(1.17) \quad J_n^{(3)} = \frac{1}{7} (2^{n+1} - V_n^{(2)}) \text{ and } j_n^{(3)} = \frac{1}{7} (2^{n+3} + 3V_n^{(2)}).$$

Motivated essentially by the recent works [5], [2] and [4], in this paper we introduce the Modified third-order Jacobsthal sequences and we give some properties, including the Binet-style formula and the generating functions for these sequences. Some identities involving these sequences are also provided.

2. The Modified Third-order Jacobsthal sequence, Binet's formula and the generating function

The principal goals of this section will be to define the Modified third-order Jacobsthal sequence and to present some elementary results involving it.

First of all, we define the Modified third-order Jacobsthal sequence, denoted by $\{K_n^{(3)}\}_{n \geq 0}$, which first terms are $\{3, 1, 3, 10, 15, 31, 66, \dots\}$. This sequence is defined recursively by

$$(2.1) \quad K_{n+3}^{(3)} = K_{n+2}^{(3)} + K_{n+1}^{(3)} + 2K_n^{(3)},$$

with initial conditions $K_0^{(3)} = 3$, $K_1^{(3)} = 1$ and $K_2^{(3)} = 3$. In fact, this sequence appears when we study the third-order Jacobsthal numbers with indices from an arithmetic progression, for example,

$$J_{a(n+3)+r}^{(3)} = (2^a + \omega_1^a + \omega_2^a) J_{a(n+2)+r}^{(3)} - (2^a(\omega_1^a + \omega_2^a) + 1) J_{a(n+1)+r}^{(3)} + 2^a J_{an+r}^{(3)},$$

for fixed integers a, r with $0 \leq r < a$. Note that $K_n^{(3)} = J_n^{(3)} + 2J_{n-1}^{(3)} + 6J_{n-2}^{(3)}$, where $J_n^{(3)}$ is the n -th third-order Jacobsthal number.

In order to find the generating function for the Modified third-order Jacobsthal sequence, we shall write the sequence as a power series where each term of the sequence correspond to coefficients of the series. As a consequence of the definition of generating function of a sequence, the generating function associated to $\{K_n^{(3)}\}_{n \geq 0}$, denoted by $\{g_{K_n^{(3)}}(t)\}$, is defined by

$$g_{K_n^{(3)}}(t) = \sum_{n \geq 0} K_n^{(3)} t^n.$$

Consequently, we obtain the following result:

Theorem 2.1. *The generating function for the Modified third-order Jacobsthal numbers $\{K_n^{(3)}\}_{n \geq 0}$ is $g_{K_n^{(3)}}(t) = \frac{3-2t-t^2}{1-t-t^2-2t^3}$.*

Proof. Using the definition of generating function, we have $g_{K_n^{(3)}}(t) = K_0^{(3)} + K_1^{(3)}t + K_2^{(3)}t^2 + \dots + K_n^{(3)}t^n + \dots$. Multiplying both sides of this identity by $-t$, $-t^2$ and by $-2t^3$, and then from Eq. (2.1), we have $(1 - t - t^2 - 2t^3)g_{K_n^{(3)}}(t) = K_0^{(3)} + (K_1^{(3)} - K_0^{(3)})t + (K_2^{(3)} - K_1^{(3)} - K_0^{(3)})t^2$ and the result follows.

The following result gives the Binet-style formula for $K_n^{(3)}$.

Theorem 2.2. *For $n \geq 0$, we have $K_n^{(3)} = 2^n + \omega_1^n + \omega_2^n = 2^n + M_n^{(2)}$, where*

$$(2.2) \quad M_n^{(2)} = \begin{cases} 2 & \text{if } n \equiv 0 \pmod{3} \\ -1 & \text{if } n \equiv 1, 2 \pmod{3} \end{cases}$$

and ω_1, ω_2 are the roots of the characteristic equation associated with the respective recurrence relations $x^2 + x + 1 = 0$.

Proof. Since the characteristic equation has three distinct roots, the sequence $K_n^{(3)} = a2^n + b\omega_1^n + c\omega_2^n$ is the solution of the Eq. (2.1). Considering $n = 0, 1, 2$ in this identity and solving this system of linear equations, we obtain a unique value for a, b and c , which are, in this case, $a = b = c = 1$.

So, using these values in the expression of $K_n^{(3)}$ stated before, we get the required result.

Using the fact that $\omega_1 + \omega_2 = -\omega_1\omega_2 = -1$, we have

$$(2.3) \quad M_n^{(2)} = -\frac{1}{7} \left(4V_{n+1}^{(2)} - V_n^{(2)} \right)$$

and $V_n^{(2)}$ as in Eq. (1.15). Furthermore, $M_n^{(2)} = -M_{n-1}^{(2)} - M_{n-2}^{(2)}$, $M_0^{(2)} = 2$ and $M_1^{(2)} = -1$. Furthermore, we easily obtain the identities stated in the following result:

Proposition 2.3. *For a natural number n and m , if $J_n^{(3)}$, $j_n^{(3)}$, and $K_n^{(3)}$ are, respectively, the n -th third-order Jacobsthal, third-order Jacobsthal-Lucas and Modified third-order Jacobsthal numbers, then the following identities are true:*

$$(2.4) \quad 147J_{n+2}^{(3)} = 13K_{n+2}^{(3)} + 48K_{n+1}^{(3)} + 20K_n^{(3)},$$

$$(2.5) \quad 6K_{n+2}^{(3)} = 5j_{n+2}^{(3)} + 3j_{n+1}^{(3)} - 5j_n^{(3)},$$

$$(2.6) \quad 49j_{n+2}^{(3)} = 43K_{n+2}^{(3)} + 8K_{n+1}^{(3)} + 36K_n^{(3)},$$

$$(2.7) \quad K_n^{(3)}K_m^{(3)} + K_{n+1}^{(3)}K_{m+1}^{(3)} + K_{n+2}^{(3)}K_{m+2}^{(3)} = \left\{ \begin{array}{c} 21 \cdot 2^{n+m} \\ +2^n \left(M_{m+1}^{(2)} + 3M_{m+2}^{(2)} \right) \\ +2^m \left(M_{n+1}^{(2)} + 3M_{n+2}^{(2)} \right) \\ +3(\omega_1^n \omega_2^m + \omega_1^m \omega_2^n) \end{array} \right\},$$

$$(2.8) \quad \left(K_n^{(3)} \right)^2 + \left(K_{n+1}^{(3)} \right)^2 + \left(K_{n+2}^{(3)} \right)^2 = 21 \cdot 2^{2n} + 2^{n+1} \left(M_{n+1}^{(2)} + 3M_{n+2}^{(2)} \right) + 6,$$

and $M_n^{(2)}$ as in Eq. (2.3).

Proof. First, we will just prove Eqs. (2.4) and (2.7) since Eqs. (2.5), (2.6) and (2.8) can be dealt with in the same manner.

(2.4): To prove Eq. (2.4), we use induction on n . Let $n = 0$, we get

$$\begin{aligned} 147J_2^{(3)} &= 147 = 13 \cdot 3 + 48 \cdot 1 + 20 \cdot 3 \\ &= 13K_2^{(3)} + 48K_1^{(3)} + 20K_0^{(3)}. \end{aligned}$$

In the same way for the case $n = 1, 2$. Let us assume that $147J_{m+2}^{(3)} = 13K_{m+2}^{(3)} + 48K_{m+1}^{(3)} + 20K_m^{(3)}$ is true for all values $m \geq 0$ less than or equal $n \geq 2$. Then,

$$\begin{aligned} 147J_{(n+1)+2}^{(3)} &= 147J_{n+3}^{(3)} \\ &= 147 \left(J_{n+2}^{(3)} + J_{n+1}^{(3)} + 2J_n^{(3)} \right) \\ &= 13K_{n+2}^{(3)} + 48K_{n+1}^{(3)} + 20K_n^{(3)} + 13K_{n+1}^{(3)} + 48K_n^{(3)} + 20K_{n-1}^{(3)} \\ &\quad + 26K_n^{(3)} + 96K_{n-1}^{(3)} + 40K_{n-2}^{(3)} \\ &= 13 \left(K_{n+2}^{(3)} + K_{n+1}^{(3)} + 2K_n^{(3)} \right) + 48 \left(K_{n+1}^{(3)} + K_n^{(3)} + 2K_{n-1}^{(3)} \right) \\ &\quad + 20 \left(K_n^{(3)} + K_{n-1}^{(3)} + 2K_{n-2}^{(3)} \right) \\ &= 13K_{n+3}^{(3)} + 48K_{n+2}^{(3)} + 20K_{n+1}^{(3)}. \end{aligned}$$

(2.7): Using the the Binet formula of $K_n^{(3)}$ in Theorem 2.2, we have

$$\begin{aligned} K_n^{(3)} K_m^{(3)} &+ K_{n+1}^{(3)} K_{m+1}^{(3)} + K_{n+2}^{(3)} K_{m+2}^{(3)} \\ &= \left\{ \begin{aligned} &\left(2^n + M_n^{(2)} \right) \left(2^m + M_m^{(2)} \right) \\ &+ \left(2^{n+1} + M_{n+1}^{(2)} \right) \left(2^{m+1} + M_{m+1}^{(2)} \right) \\ &+ \left(2^{n+2} + M_{n+2}^{(2)} \right) \left(2^{m+2} + M_{m+2}^{(2)} \right) \end{aligned} \right\}. \end{aligned}$$

lll Then, we obtain

$$\begin{aligned} K_n^{(3)} K_m^{(3)} &+ K_{n+1}^{(3)} K_{m+1}^{(3)} + K_{n+2}^{(3)} K_{m+2}^{(3)} \\ &= \left\{ \begin{aligned} &21 \cdot 2^{n+m} + 2^n \left(M_m^{(2)} + 2M_{m+1}^{(2)} + 4M_{m+2}^{(2)} \right) \\ &+ 2^m \left(M_n^{(2)} + 2M_{n+1}^{(2)} + 4M_{n+2}^{(2)} \right) \\ &+ M_n^{(2)} M_m^{(2)} + M_{n+1}^{(2)} M_{m+1}^{(2)} + M_{n+2}^{(2)} M_{m+2}^{(2)} \end{aligned} \right\} \\ &= \left\{ \begin{aligned} &21 \cdot 2^{n+m} + 2^n \left(M_{m+1}^{(2)} + 3M_{m+2}^{(2)} \right) \\ &+ 2^m \left(M_{n+1}^{(2)} + 3M_{n+2}^{(2)} \right) \\ &+ 3(\omega_1^n \omega_2^m + \omega_1^m \omega_2^n) \end{aligned} \right\}. \end{aligned}$$

Then, we obtain the Eq. (2.8) if $m = n$ in Eq. (2.7).

3. Some identities involving this type of sequence

In this section, we state some identities related with these type of third-order sequence. As a consequence of the Binet formula of Theorem 2.2, we get for this sequence the following interesting identities.

Proposition 3.1 (Catalan-like identity). *For a natural numbers n, s , with $n \geq s$, if $K_n^{(3)}$ is the n -th Modified third-order Jacobsthal numbers, then the following identity*

$$K_{n+s}^{(3)} K_{n-s}^{(3)} - \left(K_n^{(3)}\right)^2 = \begin{pmatrix} 2^n (2^{-s} - 2^s) U_s^{(2)} M_{n+1}^{(2)} \\ -2^n \left(2^s U_s^{(2)} + 2 + (2^{-s} + 2^s) U_{s-1}^{(2)}\right) M_n^{(2)} \\ -3 \left(U_s^{(2)}\right)^2 \end{pmatrix}$$

is true, where $M_n^{(2)}$ as in Eq. (2.3), $U_n^{(2)} = \frac{\omega_1^n - \omega_2^n}{\omega_1 - \omega_2}$ and ω_1, ω_2 are the roots of the characteristic equation associated with the recurrence relation $x^2 + x + 1 = 0$.

Proof. Using the Eq. (2.3) and the Binet formula of $K_n^{(3)}$ in Theorem 2.2, we have

$$\begin{aligned} K_{n+s}^{(3)} K_{n-s}^{(3)} - \left(K_n^{(3)}\right)^2 &= \begin{pmatrix} \left(2^{n+s} + M_{n+s}^{(2)}\right) \left(2^{n-s} + M_{n-s}^{(2)}\right) \\ - \left(2^n + M_n^{(2)}\right)^2 \end{pmatrix} \\ &= \begin{pmatrix} 2^n \left(2^s M_{n-s}^{(2)} - 2 M_n^{(2)} + 2^{-s} M_{n+s}^{(2)}\right) \\ + M_{n+s}^{(2)} M_{n-s}^{(2)} - \left(M_n^{(2)}\right)^2 \end{pmatrix}. \end{aligned}$$

Using the following identity for the sequence $M_n^{(2)}$:

$$\begin{aligned} M_{n+s}^{(2)} &= \frac{1}{\omega_1 - \omega_2} \left(\omega_1^{n+s+1} - \omega_1^{n+s-1} + \omega_2^{n+s-1} - \omega_2^{n+s+1} \right) \\ &= \left(\frac{\omega_1^s - \omega_2^s}{\omega_1 - \omega_2} \right) (\omega_1^{n+1} + \omega_2^{n+1}) - \left(\frac{\omega_1^{s-1} - \omega_2^{s-1}}{\omega_1 - \omega_2} \right) (\omega_1^n + \omega_2^n) \\ &= U_s^{(2)} M_{n+1}^{(2)} - U_{s-1}^{(2)} M_n^{(2)}, \end{aligned}$$

where $U_s^{(2)} = \frac{\omega_1^s - \omega_2^s}{\omega_1 - \omega_2}$ and $U_{-s}^{(2)} = -U_s^{(2)}$. Then, we obtain

$$K_{n+s}^{(3)} K_{n-s}^{(3)} - \left(K_n^{(3)}\right)^2 = \begin{pmatrix} 2^n (2^{-s} - 2^s) U_s^{(2)} M_{n+1}^{(2)} \\ -2^n \left(2^s U_s^{(2)} + 2 + (2^{-s} + 2^s) U_{s-1}^{(2)}\right) M_n^{(2)} \\ -3 \left(U_s^{(2)}\right)^2 \end{pmatrix}.$$

Hence the result.

Note that for $s = 1$ in the Catalan-like identity obtained, we get the Cassini-like identity for the Modified third-order Jacobsthal sequence. Furthermore, for $s = 1$, the identity stated in Proposition 3.1, yields

$$K_{n+1}^{(3)} K_{n-1}^{(3)} - \left(K_n^{(3)}\right)^2 = \left\{ \begin{array}{c} 2^n (2^{-1} - 2^1) U_1^{(2)} M_{n+1}^{(2)} \\ -2^n \left(2^1 U_1^{(2)} + 2 + (2^{-1} + 2^1) U_0^{(2)}\right) M_n^{(2)} - 3 \left(U_1^{(2)}\right)^2 \end{array} \right\}$$

and using $U_0^{(2)} = 0$ and $U_1^{(2)} = 1$ in Proposition 3.1, we obtain the following result.

Proposition 3.2 (Cassini-like identity). *For a natural numbers n , if $K_n^{(3)}$ is the n -th Modified third-order Jacobsthal numbers, then the identity*

$$K_{n+1}^{(3)} K_{n-1}^{(3)} - \left(K_n^{(3)}\right)^2 = 2^{n-1} \left(3M_{n+2}^{(2)} - 5M_n^{(2)}\right) - 3$$

is true.

The d'Ocagne-like identity can also be obtained using the Binet formula and in this case we obtain

Proposition 3.3 (d'Ocagne-like identity). *For a natural numbers m , n , with $m \geq n$ and $K_n^{(3)}$ is the n -th Modified third-order Jacobsthal number, then the following identity*

$$K_{m+1}^{(3)} K_n^{(3)} - K_m^{(3)} K_{n+1}^{(3)} = \left\{ \begin{array}{c} 2^m \left(2M_n^{(2)} - M_{n+1}^{(2)}\right) \\ +2^n \left(M_{m+1}^{(2)} - 2M_m^{(2)}\right) - 3U_{m-n}^{(2)} \end{array} \right\}$$

is true.

Proof. Using the Eq. (2.2) of Theorem 2.2, we get the required result.

In addition, some formulae involving sums of terms of the Modified third-order Jacobsthal sequence will be provided in the following proposition.

Proposition 3.4. For a natural numbers m, n , with $n \geq m$, if $j_n^{(3)}$ and $K_n^{(3)}$ are, respectively, the n -th third-order Jacobsthal-Lucas and Modified third-order Jacobsthal numbers, then the following identities are true:

$$(3.1) \quad \sum_{s=m}^n K_s^{(3)} = \frac{1}{3} \left(K_{n+2}^{(3)} + 2K_n^{(3)} + K_m^{(3)} - K_{m+2}^{(3)} \right),$$

$$\sum_{s=0}^n K_s^{(3)} = \begin{cases} K_{n+1}^{(3)} + 2 & \text{if } n \equiv 0 \pmod{3} \\ K_{n+1}^{(3)} + 1 & \text{if } n \equiv 1 \pmod{3} \\ K_{n+1}^{(3)} - 3 & \text{if } n \equiv 2 \pmod{3} \end{cases}, \quad (3.2)$$

$$(3.3) \quad \sum_{s=0}^n j_s^{(3)} = \frac{1}{49} \left(16K_{n+3}^{(3)} - 5K_{n+2}^{(3)} + 2K_{n+1}^{(3)} \right) - 1.$$

Proof. (3.1): Using Eq. (2.1), we obtain

$$\begin{aligned} \sum_{s=m}^n K_s^{(3)} &= K_m^{(3)} + K_{m+1}^{(3)} + K_{m+2}^{(3)} + \sum_{s=m+3}^n K_s^{(3)} \\ &= K_m^{(3)} + K_{m+1}^{(3)} + K_{m+2}^{(3)} \\ &\quad + \sum_{s=m+2}^{n-1} K_s^{(3)} + \sum_{s=m+1}^{n-2} K_s^{(3)} + 2 \sum_{s=m}^{n-3} K_s^{(3)} \\ &= 4 \sum_{s=m}^n K_s^{(3)} + K_{m+2}^{(3)} - K_m^{(3)} - 4K_n^{(3)} - 3K_{n-1}^{(3)} - 2K_{n-2}^{(3)} \\ &= 4 \sum_{s=m}^n K_s^{(3)} + K_{m+2}^{(3)} - K_m^{(3)} - 2K_n^{(3)} - K_{n+2}^{(3)}. \end{aligned}$$

Then, the result in Eq. (3.1) is completed.

(3.2): As a consequence of the Eq. (2.2) of Theorem 2.2 and

$$\sum_{s=0}^n M_s^{(2)} = \sum_{s=0}^n (\omega_1^s + \omega_2^s) = \frac{\omega_1^{n+1} - 1}{\omega_1 - 1} + \frac{\omega_2^{n+1} - 1}{\omega_2 - 1} = \frac{1}{3} \left(M_n^{(2)} - M_{n+1}^{(2)} \right) + 1,$$

we have

$$\begin{aligned} \sum_{s=0}^n K_s^{(3)} &= \sum_{s=0}^n 2^s + \sum_{s=0}^n M_s^{(2)} \\ &= 2^{n+1} - 1 + \frac{1}{3} \left(M_n^{(2)} - M_{n+1}^{(2)} \right) + 1 \\ &= K_{n+1}^{(3)} + \frac{1}{3} \left(M_n^{(2)} - 4M_{n+1}^{(2)} \right) \\ &= \begin{cases} K_{n+1}^{(3)} + 2 & \text{if } n \equiv 0 \pmod{3} \\ K_{n+1}^{(3)} + 1 & \text{if } n \equiv 1 \pmod{3} \\ K_{n+1}^{(3)} - 3 & \text{if } n \equiv 2 \pmod{3} \end{cases}. \end{aligned}$$

Hence we obtain the result.

(3.3): Using the Eqs. (2.6) and (2.1), we have $49j_n^{(3)} = 43K_n^{(3)} + 8K_{n-1}^{(3)} + 36K_{n-2}^{(3)} = 18K_{n+1}^{(3)} + 25K_n^{(3)} - 10K_{n-1}^{(3)}$. Then, from Eq. (3.1), we obtain

$$\begin{aligned} 49 \sum_{s=0}^n j_s^{(3)} &= 98 + 18 \sum_{s=1}^n K_{s+1}^{(3)} + 25 \sum_{s=1}^n K_s^{(3)} - 10 \sum_{s=1}^n K_{s-1}^{(3)} \\ &= 98 + 18 \sum_{s=2}^{n+1} K_s^{(3)} + 25 \sum_{s=1}^n K_s^{(3)} - 10 \sum_{s=0}^{n-1} K_s^{(3)} \\ &= 98 + 6 \left(K_{n+3}^{(3)} + 2K_{n+1}^{(3)} + K_2^{(3)} - K_4^{(3)} \right) \\ &\quad + \frac{25}{3} \left(K_{n+2}^{(3)} + 2K_n^{(3)} + K_1^{(3)} - K_3^{(3)} \right) \\ &\quad - \frac{10}{3} \left(K_{n+1}^{(3)} + 2K_{n-1}^{(3)} + K_0^{(3)} - K_2^{(3)} \right) \\ &= 16K_{n+3}^{(3)} - 5K_{n+2}^{(3)} + 2K_{n+1}^{(3)} - 49. \end{aligned}$$

So, the proof is completed.

For negative subscripts terms of the sequence of Modified third-order Jacobsthal we can establish the following result:

Proposition 3.5. *For a natural number n the following identities are true:*

$$(3.4) \quad K_{-n}^{(3)} = K_n^{(3)} + 2^{-n} - 2^n,$$

$$(3.5) \quad \sum_{s=0}^n K_{-s}^{(3)} = \frac{1}{3} \left(K_{n+2}^{(3)} + 2K_n^{(3)} \right) - 2^{n+1} - 2^{-n} + 3.$$

Proof. (3.4): Since $M_{-n}^{(2)} = M_n^{(2)}$, using the Binet formula stated in Theorem 2.2 and the fact that $\omega_1\omega_2 = 1$, all the results of this Proposition follow. In fact,

$$\begin{aligned} K_{-n}^{(3)} &= 2^{-n} + M_{-n}^{(2)} \\ &= 2^{-n} + M_n^{(2)} \\ &= 2^{-n} + 2^n + M_n^{(2)} - 2^n \\ &= K_n^{(3)} + 2^{-n} - 2^n. \end{aligned}$$

So, the proof is completed.

(3.5): The proof is similar to the proof of Eq. (3.1) using the Eq. (3.4).

Conclusion

Sequences of numbers have been studied over several years, including the well known Tribonacci sequence and, consequently, on the Tribonacci-Lucas

sequence. In this paper we have also contributed for the study of Modified third-order Jacobsthal sequence, deducing some formulae for the sums of such numbers, presenting the generating functions and their Binet-style formula. It is our intention to continue the study of this type of sequences, exploring some their applications in the science domain. For example, a new type of sequences in the quaternion algebra with the use of these numbers and their combinatorial properties.

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