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H-supplemented modules with respect to images of fully invariant submodules

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Abstract:

Lifting modules plays important roles in module theory. H-supplemented modules are a nice generalization of lifting modules which have been studied extensively recently. In this article, we introduce a proper generalization of H-supplemented modules via images of fully invariant submodules. Let F be a fully invariant submodule of a right Rmodule M. We say that M is IF -H-supplemented in case for every endomorphism φ of M, there is a direct summand D of M such that $\varphi(F) + X = M$ if and only if $D + X = M$, for every submodule X of M. It is proved that M is I_F -H-supplemented if and only if F is a dual Rickart direct summand of M for a fully invariant noncosingular submodule F of M. It is shown that the direct sum of I_F –H supplemented modules is not in general I_F -H-supplemented. Some sufficient conditions such that the direct sum of I_F -H-supplemented modules is I_F -H-supplemented are given

Keywords: *H*-supplemented module; I_F -lifting module; I_F-H-supplemented module; Dual Rickart module; Endomorphisms ring.

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1. Introduction

All rings considered in this article are associative with an identity element and all modules are unitary right modules unless otherwise stated. Let R be a ring and M a right R-module. The ring of all R-endomorphisms of M is denoted by $S = End_R(M)$. We use the notation $N \ll M$ to indicate that N is small in M (i.e., for all $L \underset{\neq}{<} M$, $L + N \neq M$). A module M is called *hollow* if every proper submodule of M is small in M. $Rad(M)$ and $Soc(M)$ denote the radical and the socle of a module M, respectively. A submodule N of M is called a *fully invariant submodule* of M if for all $\phi \in End_R(M)$, $\phi(N) \subseteq N$.

Let $L \subseteq K \leq M$. We recall that K lies above L in M, if $K/L \ll M/L$. A module M is called *lifting* if every submodule A of M lies above a direct summand D of M. A submodule N of M is called *supplement* in M if there exists a submodule K of M such that $M = N + K$ and $N \cap K \ll N$. A module M is said to be *supplemented* if every submodule of M has a supplement [4].

Recall that a module M is called *H-supplemented* in case for every submodule N of M, there exists a direct summand D of M such that $M = N + X$ if and only if $M = D + X$ for every submodule X of M [9]. In [7], the authors presented some equivalent conditions for a module to be H-supplemented that shows that this class of modules is closely related to the concept of small submodules. In [3], the authors introduced a new generalization of H-supplemented modules that is Goldie*-supplemented modules via an equivalence relation namely β^*. Let X and Y be submodules of M. Then $X\beta^*Y$ in M provided $(X+Y)/X \ll M/X$ and $(X+Y)/Y \ll M/Y$. Here it is convenient to state that M is H-supplemented if and only if for each submodule X of M there exists a direct summand D of M such that $X\beta^*D$ in M.

Recall from [8], that a module M is *dual Rickart* in case, for every endomorphism φ of M, the image of φ is a direct summand of M. The author in [1] introduced a generalization of both lifting modules and dual Rickart modules as $\mathcal{I}$-lifting modules. The author showed that a projective $\mathcal{I}$-lifting module is a direct sum of cyclic modules. In [10], the authors studied H-supplemented modules via homomorphisms, which generalizes both H-supplemented modules and $\mathcal{I}$-lifting modules. They called a module M, *endomorphism H-supplemented* (E-H-supplemented, for short) provided for every φ in $End_R(M)$, there is a direct summand D of M such that $Im\varphi\beta^*D$.

The present authors, in [2] introduced a new generalization of $\mathcal{I}$-lifting modules via image of fully invariant submodules. Let M be a module and let F be a fully invariant submodule of M. Then M is called *$\mathcal{I}_F$-lifting* in case for every endomorphism φ of M, there is a direct summand D of M contained in $\varphi(F)$ such that $\varphi(F)/D \ll M/D$. Some properties of $\mathcal{I}_F$-lifting modules were investigated in [2].

Inspired by mentioned works on lifting modules and H-supplemented modules via a homological approach, we are interested to study on H-supple-mented modules via image of fully invariant submodules. In fact, in the definition of an E-H-supplemented module, one can replaced M by a fully invariant submodule of M. Let M be a module and let F be a fully invariant submodule of M. We say M is *$\mathcal{I}_F$-H-supplemented* provided for every endomorphism φ of M there is a direct summand D of M such that $\varphi(F) + X = M$ if and only if $D + X = M$, equivalently $\varphi(F)\beta^* D$. In what follows by F we mean a fully invariant submodule of M.

One preference of this generalization of the H-supplemented modules over other generalizations is that the fully invariant submodules form a complete modular sublattice of the lattice of submodules and they are well mannered with respect to endomorphisms. Many of the important submodules of a module are fully invariant submodules such as the Jacobson radical of a module, the socle of a module, the singular submodule, the cosingular submodule, etc.

In Section 2, we show that $\mathcal{I}_F$-H-supplemented modules are proper generalization of both $\mathcal{I}_F$-lifting modules and H-supplemented modules. We present some conditions under which the two concepts of $\mathcal{I}_F$-lifting and $\mathcal{I}_F$-H-supplemented coincide. We also study homomorphic images of $\mathcal{I}_F$-H-supplemented modules.

In Section 3, we give an example that shows a finite direct sum of $\mathcal{I}_F$-H-supplemented modules is not $\mathcal{I}_F$-H-supplemented, in general. We introduce the concept of relative $\mathcal{I}_F$-H-supplemented property and use this concept to investigate finite direct sums of $\mathcal{I}_F$-H-supplemented modules.

2. $\mathcal{I}_F$-H-supplemented modules

In this section we introduce $\mathcal{I}_F$-H-supplemented modules as a proper generalization of both $\mathcal{I}_F$-lifting modules and H-supplemented modules. Examples are provided to show that the concept of an $\mathcal{I}_F$-H-supplemented module is distinct from both an $\mathcal{I}_F$-lifting module and an H-supplemented module. As we state in the introduction, we provide conditions under which the two concepts of $\mathcal{I}_F$-lifting and $\mathcal{I}_F$-H-supplemented coincide.

Definition 2.1. Let M be a module and let F be a fully invariant submodule of M. We say M is $\mathcal{I}_F$-H-supplemented if for every $\varphi \in End_R(M)$, there exists a direct summand D of M such that $\varphi(F) + X = M$ if and only if $D + X = M$ for every submodule X of M.

It is clear that every H-supplemented module is $\mathcal{I}_F$-H-supplemented but the converse is not true (see Example 2.12). Obviously, the sentences "M is E-H-supplemented" and "M is $\mathcal{I}_M$-H-supplemented" are the same. Note that every module M is clearly $\mathcal{I}_0$-H-supplemented.

It is proved in [7] that a module M is H-supplemented if and only if for every submodule N of M there is a direct summand D of M such that $(N + D)/D \ll M/D$ and $(N + D)/N \ll M/N$, i.e., $N\beta^* D$. The same is true for $\mathcal{I}_F$-H-supplemented modules. One can easily check the following:

Proposition 2.2. *The following sentences are equivalent for a module M:*
(1) *M is $\mathcal{I}_F$-H-supplemented;*
(2) *For every $\varphi \in S$, there exists a direct summand D of M such that $\varphi(F)\beta^* D$;*
(3) *For every $\varphi \in S$, there exist a direct summand D and a submodule N of M with $\varphi(F) \subseteq N$ and $D \subseteq N$ such that $\frac{N}{D} \ll \frac{M}{D}$ and $\frac{N}{\varphi(F)} \ll \frac{M}{\varphi(F)}$.*

Below, we shall provide some examples of $\mathcal{I}_F$-H-supplemented modules.

Examples 2.3. (1) Every $\mathcal{I}_F$-lifting module is $\mathcal{I}_F$-H-supplemented. In particular every lifting module M is $\mathcal{I}_F$-H-supplemented for every fully invariant submodule F of M.
(2) Let p be a prime number. Then the $\mathbf{Z}$-module $M = \mathbf{Z}_{p^2}$ is not a dual Rickart module. Now, $Rad(M) = (p) \neq 0$. Since M is a hollow module, M is $\mathcal{I}_{Rad(M)}$-lifting and hence $\mathcal{I}_{Rad(M)}$-H-supplemented.

A module M is called *epi-retractable* provided every submodule of M is a homomorphic image of M [5]. By [5, Example 2.4], every finitely generated module over an PID is epi-retractable. Note that for an epi-retractable module the two concepts H-supplemented and E-H-supplemented coincide.

Now it is easy to verify the following proposition:

Proposition 2.4. *Let F be a fully invariant submodule of an epi-retractable module M. If M is E-H-supplemented, then M is $\mathcal{I}_F$-H-supplemented.*

We show that the class of $\mathcal{I}_F$-H-supplemented modules contains properly the class of $\mathcal{I}_F$-lifting modules.

Examples 2.5. (1) Let p be a prime number. Consider the $\mathbf{Z}$-module $M_1 = \mathbf{Z}_{p^3}$. Then by [7, Example 4.6], the $\mathbf{Z}$-module $M = M_1 \oplus \frac{M_1}{(p)} \oplus \frac{(p)}{(p^2)} \oplus \frac{(p^2)}{(0)}$ is H-supplemented. Since $\mathbf{Z}_p \oplus \mathbf{Z}_{p^3}$ is isomorphic to a direct summand of M, M is not lifting from [6, Corollary 2]. Being M a finitely generated $\mathbf{Z}$-module implies that M is epi-retractable by [5, Example 2.4]. Hence M is not $\mathcal{I}$-lifting which means that M is not $\mathcal{I}_M$-lifting. In other words, M is $\mathcal{I}_M$-H-supplemented as well as H-supplemented.
(2) (see [13, Example 2.3]) Let I and J be two ideals of a commutative local ring R with maximal ideal m such that $I \subset J \subseteq m$ and mJI (e.g., R is an DVR with maximal ideal m, $I = m^4$ and $J = m^2$). We consider the module $M = R/I \times R/J$. From [13, Proposition 2.1] it follows that M is H-supplemented and so M is $\mathcal{I}_M$-H-supplemented. In other words, from [13, Example 2.3], M is not lifting. Being M an epi-retractable module implies M is not $\mathcal{I}$-lifting ($\mathcal{I}_M$-lifting).

Recall from [12] that an R-module M is *noncosingular (cosingular)* provided $\overline{Z}(M) = M$ ($\overline{Z}(M) = 0$) where $\overline{Z}(M) = \cap\{Kerf \mid f : M \to U\}$ for all small R-modules U.

We present some conditions under which, the two concepts $\mathcal{I}_F$-lifting and $\mathcal{I}_F$-H-supplemented coincide.

Theorem 2.6. *Let F be a fully invariant submodule of a module M. If either F is noncosingular or $Rad(M) = 0$, then the following statements are equivalent:*
(1) *For every $\varphi \in End_R(M)$, the submodule $\varphi(F)$ is a direct summand of M;*
(2) *M is $\mathcal{I}_F$-lifting;*
(3) *M is $\mathcal{I}_F$-H-supplemented;*
(4) *F is a dual Rickart direct summand of M.*

Proof. We prove when F is noncosingular, the case $Rad(M) = 0$ is the same.
(1) $\Rightarrow$ (2) It is obvious.
(2) $\Rightarrow$ (3) It can be easily verified.
(3) $\Rightarrow$ (4) Let M be $\mathcal{I}_F$-H-supplemented and let φ be an endomorphism of M. Then there is a direct summand D of M such that $\varphi(F)\beta^* D$ which means that $(\varphi(F) + D)/D \ll M/D$ and $(\varphi(F) + D)/\varphi(F) \ll M/\varphi(F)$.

Note that $\varphi(F)$ is noncosingular as well as F. It follows that $(\varphi(F) + D)/D \cong \varphi(F)/(\varphi(F) \cap D)$ is a noncosingular submodule of M/D. Hence, $\varphi(F) + D = D$ implies that $\varphi(F) \subseteq D$. Now, $D/\varphi(F) \ll M/\varphi(F)$. Set $D \oplus D' = M$. Then $D/\varphi(F) + (D' + \varphi(F))/\varphi(F) = M/\varphi(F)$. Therefore, $D' + \varphi(F) = M$. Being $\varphi(F)$ a submodule of D combining with modularity implies $\varphi(F) = D$. Now, let ψ be an endomorphism of F. Then $h = jo\psi o\pi_F$ is an endomorphism of M where j is the inclusion map and π_F is the canonical projection. Then $\psi(F) = h(F)$ is a direct summand of M. Hence $\psi(F)$ is a direct summand of F showing that F is dual Rickart.
(4) $\Rightarrow$ (1) Let φ be an arbitrary endomorphism of M. Then $g = \pi_F o\varphi oj$ is an endomorphism of F. As F is a dual Rickart module, $g(F)$ is a direct summand of F and also a direct summand of M. It follows that $\varphi(F) = g(F)$ is a direct summand of M. □

Example 2.7. [2, Example 2.8] (1) Let K be a field and $R = \prod_{i=1}^{\infty} K_i$ where $K_i = K$ for each $i \in \mathbf{N}$.

Let L be an V-ring and let K be a field. Then $S = K \times L$ is an V-ring as well. Consider the central idempotent $e = (1, 0)$ of S. Then $Se = eS \cong K$ as both left and right S-module. Let R be the ring $M_n(S)$ (the ring of all $n \times n$ matrices with entries from S). As R is Morita-equivalent to S, it should be also an V-ring. Now, R has a central idempotent, $f = eI$ where I is the identity matrix of R. Then $fR = Rf$ is isomorphic to $M_n(Se)$ so that $fR = Rf \cong M_n(K)$. Note that $F = Rf$ is a two-sided ideal of R and also is a direct summand of R. Being K a field implies that $M_n(K)$ and hence F is semisimple (dual Rickart). It follows from Theorem 2.6 that R is an $\mathcal{I}_F$-H-supplemented module.

Recall from [11] that a module M is *weak duo* in case every direct summand of M is a fully invariant submodule of M. We recall that L is a *cosmall submodule of* K *in* M (denoted by $L \overset{cs}{\hookrightarrow} K$ in M) if K lies above L in M. Recall that a submodule L of M is called *coclosed* in M, if L has no proper cosmall submodule. It is clear that every direct summand of M is a coclosed submodule of M. A module M is said to have cosmall intersection property or $CSIP$ if for any $A, B, C, D \leq M$, $A \overset{cs}{\hookrightarrow} B$ in M and $C \overset{cs}{\hookrightarrow} D$ in M imply that $A \cap C \overset{cs}{\hookrightarrow} B \cap D$ in M.

Proposition 2.8. *Let F be a fully invariant submodule of a module M. If either*
(1) *M is noncosingular or*

(2) *M is a weak duo module or*
(3) *M is projective or*
(4) *M has $CSIP$,*
then the two concepts $\mathcal{I}_F$-H-supplemented and $\mathcal{I}_F$-lifting coincide.

Proof. (1) Let M be $\mathcal{I}_F$-H-supplemented and let $\varphi \in End_R(M)$ be arbitrary. Then by assumption, there is a direct summand D of M such that $(\varphi(F)+D)/D \ll M/D$ and $(\varphi(F)+D)/\varphi(F) \ll M/\varphi(F)$. Since M is noncosingular, D is noncosingular. It follows that $(\varphi(F)+D)/\varphi(F) = 0$ which implies that $\varphi(F) + D = \varphi(F)$. So, D is contained in $\varphi(F)$. Therefore, $\varphi(F)/D \ll M/D$. The converse is straightforward.
(2) Let M be an $\mathcal{I}_F$-H-supplemented weak duo module and $\varphi \in End_R(M)$. Then there exists a direct summand D of M such that $\varphi(F) + X = M$ if and only if $D + X = M$ for every $X \leq M$. Set $D \oplus D' = M$. Then $\varphi(F)+D' = M$. As F is fully invariant we have $F = (F\cap D)\oplus(F\cap D')$. It follows that $M = \varphi((F\cap D)+(F\cap D'))+D' = \varphi(F\cap D)\oplus D'$ (note that D' is also fully invariant so that $\varphi(F\cap D') \subseteq D'$). Hence $\varphi(F\cap D) = D$ implies that D is contained in $\varphi(F)$. Suppose that $\varphi(F)/D + L/D = M/D$ for a submodule L of M containing D. Then $\varphi(F)+L = M$. Hence $D+L = M$ which implies $L = M$ as required. Therefore, $\varphi(F)/D \ll M/D$.
(3) Similar to the proof of [10, Theorem 2.16].
(4) Let $\phi \in End_R(M)$. Then there exists a direct summand D of M such that $\phi(F) \overset{cs}{\hookrightarrow} (\phi(F) + D)$ in M and $D \overset{cs}{\hookrightarrow} (\phi(F) + D)$ in M. By $CSIP$, $\phi(F) \cap D \overset{cs}{\hookrightarrow} (\phi(F) + D)$ in M. But $\phi(F) \cap D \subseteq D \subseteq \phi(F) + D$, so $\phi(F) \cap D \overset{cs}{\hookrightarrow} D$ in M. As D is coclosed in M, $\phi(F) \cap D = D$, hence $D \leq \phi(F)$ and $D \overset{cs}{\hookrightarrow} \phi(F)$ in M. Therefore M is $\mathcal{I}_F$-lifting. □

A characterization of indecomposable $\mathcal{I}_F$-H-supplemented modules is presented in the following.

Proposition 2.9. *Let $M \neq 0$ be an indecomposable module and let $F < M$ be fully invariant. Then M is $\mathcal{I}_F$-H-supplemented if and only if $F \ll M$. In case $F = M$, then M is $\mathcal{I}_M$-H-supplemented if and only if every nonzero endomorphism φ of M is epimorphism or $Im\varphi \ll M$.*

Proof. Let M be $\mathcal{I}_F$-H-supplemented and let $\imath \in End_R(M)$ be the identity endomorphism. Then there exists a direct summand D of M such that $F + X = \imath(F) + X = M$ if and only if $D + X = M$ for every submodule X of M. By assumption, either $D = 0$ or $D = M$. Second case will not happen as $F < M$. On the other hand, $D = 0$ implies that $F \ll M$.

The converse is obvious as for every $\varphi \in End_R(M)$, the condition $F \ll M$ implies $\varphi(F) + X = M$ if and only if $0 + X = M$, for every $X \leq M$. The latter follows from [10, Proposition 2.3]. □

Following presents a characterization of an $\mathcal{I}_F$-H-supplemented module M when F is a direct summand of M.

Theorem 2.10. *Let F be a fully invariant direct summand of a module M. If F is E-H-supplemented, then M is $\mathcal{I}_F$-H-supplemented. The converse holds, in case M is a weak duo module.*

Proof. ($\Rightarrow$) Let F be E-H-supplemented and let φ be an endomorphism of M. Consider $q = \pi_F o \varphi o j : F \to F$, which is an endomorphism of F, where $j : F \to M$ is the inclusion and $\pi_F : M \to F$ is the projection map on F. Being F a fully invariant submodule of M implies that $q(F) = \varphi(F)$. As F is E-H-supplemented, there is a direct summand D of F (so that of M) such that $q(F) + Y = F$ if and only if $D + Y = F$ for every submodule Y of F. Now, suppose that $\varphi(F) + X = M$ for a submodule X of M. Then $\varphi(F) + X \cap F = F$. Then, $D + (X \cap F) = F$. It follows that $D + X = F + X = M$. For the converse, let $D + X = M$ where $X \leq M$. Then modularity implies $D + (X \cap F) = F$. Since F is E-H-supplemented, $\varphi(F) + (X \cap F) = F$. Hence $\varphi(F) + X = F + X = M$ as required. Therefore, M is $\mathcal{I}_F$-H-supplemented.
($\Leftarrow$) Let $g : F \to F$ be an endomorphism of F and $F \oplus F' = M$ for a submodule F' of M. Then $h = jog o \pi_F : M \to M$ is an endomorphism of M where $j : F \to M$ is the inclusion and $\pi_F : M \to F$ is the projection on F. It is straightforward to check $h(F) = g(F)$. As M is $\mathcal{I}_F$-H-supplemented, there exists a direct summand D of M such that $g(F) + X = M$ if and only if $D + X = M$ for every submodule X of M. We shall verify $g(F) + Y = F$ if and only if $(F \cap D) + Y = F$. Now, let $g(F) + Y = F$ for $Y \leq F$. Then $g(F) + Y + F' = M$. By assumption we have $D + Y + F' = M$. By modular law, we conclude that $Y + (D + F') \cap F = F$. Since M is weak duo, D is a fully invariant submodule of M so that $D = (D \cap F) \oplus (D \cap F')$. Therefore, $Y + [(D \cap F) + F'] \cap F = F$. Hence $Y + (D \cap F) = F$. The other implication can be verified similarly. □

Corollary 2.11. (1) *Let M be a module such that $\overline{Z}(M)$ is a direct summand of M. If $\overline{Z}(M)$ is E-H-supplemented, then $\varphi(\overline{Z}(M))$ is a direct summand of M for every $\varphi \in End_R(M)$.*

(2) *Let M be a module such that $Soc(M)$ is a direct summand of M. Then M is $\mathcal{I}_{Soc(M)}$-H-supplemented.*

Proof. (1) Let $\overline{Z}(M)$ be an E-H-supplemented direct summand of M. Then by Theorem 2.10, M is $\mathcal{I}_{\overline{Z}(M)}$-$H$-supplemented. Note that since $\overline{Z}(M)$ is a direct summand of M, it is noncosinsgular. Therefore, $\varphi(\overline{Z}(M))$ is a direct summand of M for every $\varphi \in End_R(M)$ by Theorem 2.6. (2) It is clear as $Soc(M)$ is semisimple. □

The following example introduces an $\mathcal{I}_F$-H-supplemented module which is not H-supplemented.

Example 2.12. Let K be a field and $R = K \times K[[x]]$. Then $J(R) = 0 \times (x)$. It follows that $R/J(R) \cong K \times (\frac{K[[x]]}{(x)})$ is semisimple. Hence R is a commutative semilocal ring with $Soc(R) = K \times 0$. Let $M = R_R^{(\mathbf{N})}$. Then $Rad(R_R^{(\mathbf{N})})$ is not small in $R_R^{(\mathbf{N})}$ by [14, 43.9]. Hence, by [14, 42.5], $M = R_R^{(\mathbf{N})}$ is not supplemented. So M is not H-supplemented. By [2, Proposition 2.13], M is not $\mathcal{I}_{Rad(M)}$-lifting and it is not $\mathcal{I}_{Rad(M)}$-H-supplemented, by Proposition 2.8(3).

Since $Soc(M) = K^{(\mathbf{N})}$ is a direct summand of M, by Corollary 2.11, M is $\mathcal{I}_{K^{(\mathbf{N})}}$-$H$-supplemented. Therefore, by Proposition 2.8, M is also $\mathcal{I}_{K^{(\mathbf{N})}}$-lifting.

Proposition 2.13. *Let F be a fully invariant submodule of a module M and let K be a fully invariant direct summand of M contained in F. If M is $\mathcal{I}_F$-H-supplemented, then M/K is $\mathcal{I}_{F/K}$-H-supplemented.*

Proof. Let $g : M/K \to M/K$ be an endomorphism of M/K and $M = K \oplus K'$. Then $f = johog o\pi : M \to M$ is an endomorphism of M. Note that $\pi : M \to M/K$ is the canonical projection, $h : M/K \to K'$ is the isomorphism induced by the decomposition $M = K \oplus K'$ and $j : K' \to M$ is the inclusion. By assuming $g(F/K) = T/K$, one can $f(F) = T \cap K'$. Since M is $\mathcal{I}_F$-H-supplemented, there is a direct summand D of M such that $(T \cap K') + X = M$ if and only if $D + X = M$ for every $X \leq M$. Set $M = D \oplus D'$. Then $M/K = (D + K)/K + (D' + K)/K$. Since K is a fully invariant submodule of M, we have $K = (K \cap D) \oplus (K \cap D')$. Hence $(D + K) \cap (D' + K) = K$ which implies that $(D + K)/K$ is a direct summand of M/K. We shall show that $T/K + Y/K = M/K$ if and only if $(D + K)/K + Y/K = M/K$ for every submodule Y of M containing K. In first step, let $T/K+Y/K = M/K$. Then $T+Y = M$. As T contains K, we have $[K + (T \cap K')] + Y = M = (T \cap K') + Y = M$. Then by assumption, $D + Y = M$. It follows that $(D + K)/K + Y/K = M/K$. In other words, suppose that $(D+K)/K+Y/K = M/K$. Then $D+Y = M$ which implies that $T + Y = M$. Therefore, $T/K + Y/K = M/K$ as required. □

3. Direct sums of $\mathcal{I}_F$-H-supplemented modules

Let $F = \oplus_{i \in I} F_i$ where F_i $(i \in I)$ is a fully invariant submodule of M. The following example shows that a finite direct sum of $\mathcal{I}_{F_i}$-H-supplemented modules need not be $\mathcal{I}_F$-H-supplemented, in general.

Example 3.1. Let R be a discrete valuation ring and let $I_1, \ldots, I_n$ be some ideals of R. Consider the R-module $M \cong R/I_1 \times \cdots \times R/I_n$. Since R is commutative, each R/I_i is H-supplemented and so each R/I_i is $\mathcal{I}_{R/I_i}$-H-supplemented. If $I_1 \subseteq \cdots \subseteq I_n \subset R$, then M is H-supplemented by [13, Proposition 2.1]. Therefore, M is $\mathcal{I}_M$-H-supplemented. Otherwise, i.e., the condition $I_1 \subseteq \cdots \subseteq I_n \subset R$ does not hold, M is not H-supplemented. Note also that M is an epi-retractable R-module by [5, Example 2.4(3)]. It means that in this case M is not $\mathcal{I}_M$-H-supplemented.

Now we define relative $\mathcal{I}_F$-H-supplemented modules and we apply this concept to study finite direct sums of $\mathcal{I}_F$-H-supplemented modules.

Definition 3.2. Let M and N be R-modules and let F be a fully invariant submodule of M. We say M is N-$\mathcal{I}_F$-H-supplemented if for every homomorphism $\phi : M \to N$, there exists a direct summand D of N such that $\phi(F) + X = N$ if and only if $D + X = N$ for every submodule X of N.

It is clear that a module M is $\mathcal{I}_F$-H-supplemented if and only if M is M-$\mathcal{I}_F$-H-supplemented.

Theorem 3.3. *Let M and N be right R-modules and let F be a fully invariant submodule of M. Then M is N-$\mathcal{I}_F$-H-supplemented if and only if for every direct summand M' of M and every fully invariant direct summand N' of N, M' is N'-$\mathcal{I}_{F \cap M'}$-H-supplemented.*

Proof. Let $M' = eM$ for some $e^2 = e \in End_R(M)$, and let N' be a fully invariant direct summand of N. Then $N = N' \oplus T$ for some $T \leq N$. Suppose that $\psi \in \mathrm{Hom}\,(M', N')$. We want to show that for any submodule

X of N', there exists a direct summand D of N' such that $\psi(F \cap M') + X = N'$ if and only if $D + X = N'$.

First note that $e(F \cap M') = eF$ and $\psi e(F) = \psi(F \cap M')$. Let $\psi(F \cap M') + X = N'$ for a submodule X of N'. Then $\psi e(F) + X + T = N$. Since $\psi eM \subseteq N' \subseteq N$ and M is N-$\mathcal{I}_F$-H-supplemented, we conclude that $A + X + T = N$ for some direct summands A of N. So $X + (A \cap N') = N'$. Note that $A \cap N'$ is a direct summand of N' as N' is fully invariant.

Conversely, assume that $N' = A \cap N' + X$ where $A \cap N'$ is a direct summand of N' and $X \leq N'$. Then $N = N' + T = A \cap N' + X + T$. Since $A \cap N'$ is a direct summand of N and M is N-$\mathcal{I}_F$-H-supplemented, $N = \psi e(F) + X + T$. Hence $N' = \psi e(F) + X$, and so $N' = \psi(F \cap M') + X$. Therefore M' is N'-$\mathcal{I}_{F\cap M'}$-H-supplemented. The other side of this theorem is clear. □

Corollary 3.4. *Let M be R-module and let F be a fully invariant submodule of M. Then the following condition are equivalent:* (1) *M is $\mathcal{I}_F$-H-supplemented;* (2) *For any fully invariant direct summand N of M, every direct summand L of M is N-$\mathcal{I}_{F\cap L}$-H-supplemented.*

Corollary 3.5. *Let F be a fully invariant submodule of a module M and let L be a direct summand of M. If M is $\mathcal{I}_F$-H-supplemented, then L is $\mathcal{I}_{F\cap L}$-H-supplemented.*

Recall that a module M is said to have the *summand sum property* (SSP) if the sum of any two direct summands is a direct summand of M.

Theorem 3.6. *Let $M = \oplus_{i=1}^{n} M_i$ and N be right R-modules and let F be a fully invariant submodule of M. If N has the SSP, then $M = \oplus_{i=1}^{n} M_i$ is N-$\mathcal{I}_F$-H-supplemented if and only if M_i is N-$\mathcal{I}_{F\cap M_i}$-H-supplemented for all $i \in \{1, 2, \dots, n\}$.*

Proof. Assume that M is N-$\mathcal{I}_F$-H-supplemented. By Theorem 3.3, M_i is N-$\mathcal{I}_{F\cap M_i}$-H-supplemented for all $i \in \{1, 2, \dots, n\}$. Conversely, let ϕ be a homomorphism from M to N. Consider $\phi = (\phi_i)_{i=1}^{n}$ where $\phi_i \in Hom_R(M_i, N)$ and $i \in \{1, 2, \dots, n\}$. Since M_i is N-$\mathcal{I}_{F\cap M_i}$-H-supplemented, there exists a direct summand D_i of N such that $\phi_i(F \cap M_i)\beta^* D_i$ for all $i \in \{1, 2, \dots, n\}$. Using [3, Proposition 2.11] and this fact that $\phi(F) = \sum_{i=1}^{n} \phi_i(F \cap M_i)$, we conclude that $\phi(F)\beta^* \sum_{i=1}^{n} D_i$. As N has the SSP, $\sum_{i=1}^{n} D_i$ is a direct summand of N. Hence M is N-$\mathcal{I}_F$-H-supplemented. □

Proposition 3.7. *Let $M = M_1 \oplus M_2$ be a module, where M_1 and M_2 are fully invariant submodules of M. If M is $\mathcal{I}_{M_i}$-H-supplemented for $i = 1, 2$, and M has SSP, then M is $\mathcal{I}_M$-H-supplemented.*

Proof. Let $\phi \in End_R(M)$.Then, by assumption, there are direct summands D_1 and D_2 of M such that $\phi(M_1)\beta^* D_1$ and $\phi(M_2)\beta^* D_2$. Note that, since M has SSP, then $D_1 + D_2$ is a direct summand of M. By [3, Proposition 2.11], $\phi(M)\beta^* D_1 + D_2$. Therefore M is $\mathcal{I}_M$-H-supplemented. □

Theorem 3.8. *Let $M = M_1 \oplus M_2$ be a duo module and let F be a submodule of M. Then M is $\mathcal{I}_F$-H-supplemented if and only if M_i is $\mathcal{I}_{F\cap M_i}$-H-supplemented for $i = 1, 2$.*

Proof. ($\Rightarrow$) By Corollary 3.5. ($\Leftarrow$) Assume M_1 is $\mathcal{I}_{F\cap M_1}$-H-supplemented and M_2 is $\mathcal{I}_{F\cap M_2}$-H-supplemented. Let π_i be the projection of M on M_i and let j_i be the inclusion map from M_i to M for $i = 1, 2$. Assume that f is an endomorphism of M. Since M_1 is $\mathcal{I}_{F\cap M_1}$-H-supplemented, there exists a direct summand N_1 of M_1 such that $M_1 = N_1 + X$ if and only if $M_1 = \pi_1 f j_1(F \cap M_1) + X$ for any submodule X of M_1 and since M_2 is $\mathcal{I}_{F\cap M_2}$-H-supplemented, there exists a direct summand N_2 of M_2 such that $M_2 = N_2 + Y$ if and only if $M_2 = \pi_2 f j_2(F \cap M_2) + Y$ for any submodule Y of M_2. We claim that $M = N_1 \oplus N_2 + Z$ if and only if $M = f(F) + Z$ for any submodule Z of M. Note that $f(F) = \pi_1 f(F \cap M_1) + \pi_2 f(F \cap M_2)$, because, $F = (F \cap M_1) \oplus (F \cap M_2)$ implies $f(F) = f(F \cap M_1) \oplus f(F \cap M_2) = \pi_1 f(F \cap M_1) + \pi_2 f(F \cap M_2)$. Let $M = N_1 \oplus N_2 + Z$. Then

$$M_1 = N_1 + (M_1 \cap (N_2 + Z)) = \pi_1 f j_1(F \cap M_1) + (M_1 \cap (N_2 + Z))$$
$$= M_1 \cap [\pi_1 f j_1(F \cap M_1) + (N_2 + Z)].$$

Thus $M_1 \le \pi_1 f j_1(F \cap M_1) + (N_2 + Z)$. Then $M_1 \le \pi_1 f(F \cap M_1) + Z$, because, let $m_1 = \pi_1 f(y) + n_2 + z_0$, where $m_1 \in M_1, y \in F \cap M_1$, $n_2 \in N_2$ and $z_0 \in Z$. Since $Z = (Z \cap M_1) \oplus (Z \cap M_2)$, $z_0 = z_1 + z_2$ where $z_1 \in Z \cap M_1$ and $z_2 \in Z \cap M_2$. Then $m_1 = \pi_1 f(y) + z_1$ and so $M_1 \le \pi_1 f(F \cap M_1) + Z$. Similarly, $M_2 \le \pi_2 f(F \cap M_2) + Z$. Thus $M = \pi_1 f(F \cap M_1) + \pi_2 f(F \cap M_2) + Z$. Therefore $M = f(F) + Z$.

Conversely, assume that $M = f(F) + Z$. Then $M = \pi_1 f(F \cap M_1) + \pi_2 f(F \cap M_2) + Z$. By modularity,

$$M_1 = \pi_1 f(F \cap M_1) + [M_1 \cap (\pi_2 f(F \cap M_2) + Z)]$$

and so

$$M_1 = N_1 + [M_1 \cap (\pi_2 f(F \cap M_2) + Z)] = M_1 \cap (N_1 + \pi_2 f(F \cap M_2) + Z).$$

Thus $M_1 \leq N_1 + \pi_2 f(F \cap M_2) + Z$. Therefore $M_1 \leq N_1 + Z$. Similarly, $M_2 \leq N_2 + Z$. Hence $M = (N_1 \oplus N_2) + Z$. □

Proposition 3.9. *Let F be a fully invariant submodule of a module M. Assume $\phi(F)$ has a supplement that is a direct summand of M for every $\phi \in End_R(M)$ such that whenever $M = M_1 \oplus M_2$ then M_1 and M_2 are relatively projective. Then M is an $\mathcal{I}_F$-H-supplemented module.*

Proof. Let $\phi \in End_R(M)$. By hypothesis, there exists a decomposition $M = M_1 \oplus M_2$ such that $M = \phi(F) + M_2$ and $\phi(F) \cap M_2 \ll M_2$ for some submodules M_1 and M_2 of M. Since M_1 is M_2-projective, by [9, Lemma 4.47], we get $M = N \oplus M_2$ for some submodule N of M such that $N \leq \phi(F)$. Then $\phi(F) = N \oplus (M_2 \cap \phi(F))$. Let $X \leq M$ with $M = \phi(F) + X$. Then $M = N + (M_2 \cap \phi(F)) + X$. As $M_2 \cap \phi(F) \ll M_2$, $M = N + X$. Therefore $M = N + X$ if and only if $M = \phi(F) + X$. Hence M is $\mathcal{I}_F$-H-supplemented. □

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