Proyecciones (Antofagasta. On line) | vol. 39, n. 2 (2020) | pp. 435-449.

10.22199/issn.0717-6279-2020-02-0027

PROYECCIONES
Journal of Mathematics

ISSN 0717-6279 (On line)

Fuzzy δ^*-almost continuous and fuzzy δ^*-continuous functions in mixed fuzzy ideal topological spaces

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Received: April 2019 | Accepted: December 2019

Abstract:

In this paper we introduce two new classes of functions between mixed fuzzy topological spaces, namely fuzzy δ^*-almost continuous and fuzzy δ^*-continuous functions and investigate some of their properties. The description of these two types of functions facilitated by the introduction of generalized open sets, called fuzzy δ-preopen sets, fuzzy δ-precluster point, fuzzy preopen sets, fuzzy δ-pre-q-neighbourhoods.

Keywords: Fuzzy δ-preopen set; Fuzzy δ-regular open set; Fuzzy δ-pre neighbourhood; Fuzzy δ-regular neighbourhood.

MSC (2010): 54A40, 54D20, 54C08, 54C10, 54A05, 54C60, 54E55.

Cite this article as (IEEE citation style):

B. C. Tripathy and G. C. Ray "Fuzzy δ^*-almost continuous and fuzzy δ^*-continuous functions in mixed fuzzy ideal topological spaces", *Proyecciones (Antofagasta, On line)*, vol. 39, no. 2, pp. 435-449, Apr. 2020, doi: 10.22199/issn.0717-6279-2020-02-0027.

1. Introduction

Topological spaces has been expanded in different directions. The notion of mixed topology was investigated by Alexiewicz and Semadeni [1]. Mixed topology is a technique of combining two different topologies on a set to get a third topology on that set. The works on mixed topology is due to ([3, 4, 5, 6, 8, 11, 12, 13, 14, 15, 16]) and many others.

In 1965, Zadeh [17] introduced the concept of fuzzy set. The notion of fuzzy topology was introduced by Chang [2]. Different properties of fuzzy topological spaces have been investigated by Das and Baishya [6], Ganster et al. [7], Ganguly and Singha [8] and others.

Mixed fuzzy topological spaces has also been studied from different aspects by Das and Baishya [6], Tripathy and Ray [12] and others. This concept is not the generalization of the concept of crisp mixed topology due to Alexiewicz and Semadeni [1], Cooper ([4, 5]), Warren [15] and others. Tripathy and Ray [12] studied and investigated the concept of mixed fuzzy topological spaces in a slightly different way as introduced and investigated by Das and Baishya [6]. The concept of fuzzy continuity and δ-continuity in fuzzy setting in fuzzy topological spaces has been introduced and investigated by Ganguly and Saha [9] . Tripathy and Shravan [11] studied mixed multiset topological spaces. The concept of fuzzy δ-almost continuous and fuzzy δ^*-almost continuous functions between fuzzy topological spaces was introduced and studied by Chilana [3]. In this paper we introduce the concepts of fuzzy δ^*-almost continuous and fuzzy δ^*-continuous functions in mixed fuzzy topological spaces.

2. Preliminaries

Let X be a non-empty set and I, the unit interval [0, 1]. A fuzzy set A in X is characterized by a function $\mu_A : X \to I$, where μ_A is called the membership function of A, $\mu_A(x)$ representing the membership grade of x in A, one may refer to [16]. The empty fuzzy set is denoted by 0_X, such that $\mu_X(x) = 0$ for all $x \in X$. Also X can be regarded as a fuzzy set in itself denoted by 1_X such that $\mu_X(x) = 1$ for all $x \in X$. Further, an ordinary subset A of X can also be regarded as a fuzzy set in X if its membership function is taken as usual characteristic function of A that is $\mu_A(x) = 1$, for all $x \in A$ and $\mu_A(x) = 0$, for all $x \in X - A$. Two fuzzy set A and B are said to be equal if $\mu_A(x) = \mu_B(x)$, for all $x \in X$. A fuzzy set A is said to be contained in a fuzzy set B, written as $A \leq B$, if $\mu_A(x) \leq \mu_B(x)$, for all $x \in X$. The complement of a fuzzy set A in X is a fuzzy set A^c in X defined by $\mu_{A^c}(x) = 1 - \mu_A(x)$, for all $x \in X$. We write $A^c = coA$, if there is no confusion. The union and intersection of a collection $\{A_i : i \in \Delta\}$, (where Δ is an index set) of fuzzy sets in X, to be written as $\bigvee\limits_{i\in\Delta} A_i$ and $\bigwedge\limits_{i\in\Delta} A_i$ respectively are defined as follows:

$$\mu_{\bigvee\limits_{i\in\Delta} A_i}(x) = sup\{\mu_{A_i}(x) : i \in \Delta\}, \text{ for all } x \in X,$$

$$\text{and } \mu_{\bigwedge\limits_{i\in\Delta} A_i}(x) = inf\{\mu_{A_i}(x) : i \in \Delta\}, \text{ for all } x \in X.$$

We procure the following definitions, those will be used in this article.

Definition 2.1. A collection $\mathcal{B}$ of open fuzzy sets in a fts X is said to be an open base for X if every open fuzzy set in X is a union of members of $\mathcal{B}$.

Definition 2.2. If A is a fuzzy set in X and B is a fuzzy set in Y then, $A \times B$ is a fuzzy set in $X \times Y$ with the membership $\mu_{A\times B}$, defined by $\mu_{A\times B}(x, y) = min\{\mu_A(x), \mu_B(y)\}$, for all $x \in X$ and for all $y \in Y$.

Definition 2.3. Let f be a function from X into Y. Then, for each fuzzy set B in Y, the inverse image of B under f, written as $f^{-1}(B)$, is a fuzzy set in X, whose membership is defined by $\mu_{f^{-1}(B)}(x) = \mu_B(f(x))$ for all $x \in X$.

Definition 2.4. A fuzzy set A in a fuzzy topological space (X, τ) is called a neighborhood of a point $x \in X$ if and only if there exists $B \in \tau$ such that $B \leq A$ and $\mu_A(x), \mu_B(x) > 0$.

Definition 2.5 (see[6]). A fuzzy point x_α is said to be quasi-coincident with A, denoted by $x_\alpha qA$, if and only if $\alpha + A(x) > 1$ or $\alpha > A(x))^c$.

Definition 2.6 (see[6]). A fuzzy set A is said to be quasi-coincident with B and is denoted by AqB, if and only if there exists a $x \in X$ such that $A(x) + B(x) > 1$.

It is clear that if A and B are quasi-coincident at x both $A(x)$ and $B(x)$

are not zero at x and hence A and B intersect at x.

Definition 2.7. A fuzzy set A in a fts (X, τ) is called a quasi-neighborhood of a fuzzy point x_λ if and only if $A_1 \in \tau$ such that $\overline{A_1} \subseteq A$ and $x_\lambda q A_1$. The family of all Q-neighborhood of the fuzzy point x_λ is called the system of Q-neighborhoods of x_λ. The intersection of two quasi-neighborhood of x_λ is a quasi-neighborhood of x_λ.

Definition 2.8 (see [12]). Let (X, τ_1) and (X, τ_2) be two fuzzy topological spaces and let $\tau_1(\tau_2) = \{A \in I^X :$ for every fuzzy set $B \in X$ with $AqB,$ there exists a $\tau_2 -$ open set $A_1,$ such that $A_1 qB$ and $\tau_1 -$ closure, $\overline{A_1} \leq B\}$.

It is proved that this family of fuzzy sets will form a topology on X and this topology we call a mixed fuzzy topology on X (One may refer to Tripathy and Ray [12]).

Definition 2.9. A fuzzy set A in a mixed fuzzy topological space $(X, \tau_1(\tau_2))$ is said to be regularly open set in X if and only if $int(clA) = A$ (closure with respect to τ_2 and interior with respect to τ_1).

Definition 2.10. A mapping $f : (X, \tau_1(\tau_2)) \to (Y, \tau_3(\tau_4))$ is said to be fuzzy δ-continuous mapping if for every regularly open q-neighbourhood U of a fuzzy point y_λ of Y, there exists a regularly open q-neighbourhood V of x_λ of X such that $f(V) < U$, where $f(x) = y$.

Definition 2.11. A mapping $f : (X, \tau_1(\tau_2)) \to (Y, \tau_3(\tau_4))$ is said to be fuzzy continuous mapping if for every open q-neighbourhood U of a fuzzy point y_λ of Y, there exists an open q-neighbourhood V of x_λ of X such that $f(V) < U$, where $f(x) = y$.

The following is an equivalence condition to definition 2.11.

A mapping $f : (X, \tau_1(\tau_2)) \to (Y, \tau_3(\tau_4))$ is said to be fuzzy continuous mapping if for any open fuzzy set U in Y, there exists an open fuzzy set $V \in X$ such that $f(V) < U$, where $f(x) = y$.

Definition 2.12. (see [9]). A fuzzy set A in an fuzzy topological space X is said to be δ-preopen if $A \leq int(\delta - clA)$. The complement of fuzzy

δ-preopen set is called fuzzy δ-preclosed. The set of all fuzzy δ-preopen sets in X will be denoted by $\delta - PO(X)$.

3. Fuzzy δ^*-almost continuous functions

In this section, we introduce the following definitions in mixed fuzzy topological spaces.

Definition 3.1. Let $(X, \tau_1(\tau_2))$ be a mixed fuzzy topological space. A fuzzy set $A \in X$ is said to be preopen set if $A \leq \tau_1 - int(\tau_2 - cl(A))$.

Definition 3.2. A fuzzy set A in a mixed fuzzy topological space $(X, \tau_1(\tau_2))$ is said to be δ-preopen set if $A \leq \tau_1 - int((\tau_2 - \delta - cl(A)$. The complement of δ-preopen set is said to be δ-preclosed. The set of all fuzzy δ-preopen sets in X is denoted by $\delta - PO(X)$.

Definition 3.3. A fuzzy set A in a mixed fuzzy topological space $(X, \tau_1(\tau_2))$ is called a fuzzy $\delta - pre - q$-neighbourhood of a fuzzy point x_λ in X if there exists a fuzzy δ-preopen set V in X such that $x_\lambda q V$ and $V \leq A$.

Definition 3.4. A fuzzy set A in a mixed fuzzy topological space $(X, \tau_1(\tau_2))$ is said to be a δ-pre-neighbourhood of a fuzzy point x_λ in X if there exists a fuzzy δ-preopen set $V \in X$ such that $x_\lambda \in V$ and $V \leq A$.

Definition 3.5. A fuzzy δ-interior of a fuzzy set A in a mixed fuzzy topological space $(X, \tau_1(\tau_2))$ is denoted by $\delta - int(A)$ and is define by $\delta - int(A) = 1 - \delta - clA)$. The union of all fuzzy δ-preopen sets in a mixed fuzzy topological space X, each contained in a fuzzy set A in X is called the fuzzy δ-pre-interior of A and is denoted by $\delta - pint(A)$.

Definition 3.6. A fuzzy point x_λ in a mixed fuzzy topological space $(X, \tau_1(\tau_2))$ is called a fuzzy δ-precluster point of a fuzzy set A in X if every fuzzy $\delta - pre - q - nbd$ of the fuzzy point x_λ is q-coincident with A. The union of all fuzzy δ-precluster point of A is called fuzzy preclosure of A and is denoted by $\delta - pcl(A)$.

Definition 3.7. A function $f : (X, \tau_1(\tau_2)) \to (Y, \tau_3(\tau_4))$ is said to be fuzzy δ^*-almost continuous function if $f^{-1}(V)$ is fuzzy δ-preopen set in X, for every fuzzy δ-preopen set $V \in Y$.

Now, we establish some equivalent conditions of fuzzy δ^*-almost continuous functions.

Theorem 3.1. Let $f : X \to Y$ be a function from a mixed fuzzy topological space $(X, \tau_1(\tau_2))$ into another mixed fuzzy topological space $(Y, \tau_3(\tau_4))$, the following conditions are equivalents.
(i) f is fuzzy δ^*-almost continuous function.
(ii) For each fuzzy point x_λ in X and fuzzy δ-prenbd V of $f(x_\lambda)$, $f^{-1}(V)$ is fuzzy δ-prenbd of x_λ.
(iii) For each fuzzy point x_λ in X and fuzzy δ-prenbd V of $f(x_\lambda)$, there is a fuzzy δ-prenbd U of x_λ such that $f(U) \leq V$.
(iv) For each fuzzy set B in Y, $f(\delta - pint(f{-}1(B)) \leq \delta - pint(B)$.
(v) For each fuzzy δ-preclosed set F in Y, $f^{-1}(F)$ is fuzzy δ-preclosed in X.

Proof.
$(i) \Rightarrow (ii)$ For any fuzzy point x_λ in X and a fuzzy δ-prenbd V of $f(x_\lambda)$, there exists a fuzzy δ-preopen set B such that $f(x_\lambda) \in B$ and $B \leq V$. Since f is δ^* almost continuous, therefore $f^{-1}(B)$ is fuzzy δ-preopen set in Y containing x_λ.
We have,
$f(x_\lambda) \in B \Rightarrow x_\lambda \in f^{-1}(B)$
and
$B \leq V \Rightarrow f^{-1}(B) \leq f^{-1}(V)$.
Hence, $f^{-1}(V)$ is fuzzy δ-prenbd of x_λ.
$(ii) \Rightarrow (iii)$ Straightforward.
$(iii) \Rightarrow (iv)$ Since $\delta - pint(B)$ is fuzzy δ-preopen set in Y, so $\delta - pint(B)$ is $\delta - prenbd$ of each of its fuzzy points. Therefore for each fuzzy point x_λ in X and each fuzzy $\delta - prenbd$, $\delta - pint(B)$ of $f(x_\lambda)$, by condition (iii), we have a fuzzy $\delta - prenbd$, $(\delta - pint(f^{-1}(B)))$ of x_λ such that $f(\delta - pint(f^{-1}(B))) \leq \delta - pint(B)$.
Hence, $f^{-1}(V)$ is fuzzy δ-prenbd of x_λ.
$(iv) \Rightarrow (v)$ Let F be a fuzzy δ-preclosed set in Y. Then $1 - F$ is a fuzzy δ-preopen set in Y.

This implies, $1 - F = \delta - int(1 - F)$.

By the condition (iv) we have $f^{-1}(1 - F) \leq \delta\text{-pint}(f^{-1}(1 - F))$.

$\Rightarrow 1-f^{-1}(F) \leq \delta - pint(f^{-1}(1-F))$. [Since $f^{-1}(1-F) = 1-f^{-1}(1-F)$]
$\Rightarrow 1-f^{-1}(F) \leq \delta - pint(1-f^{-1}(F))$.
$\Rightarrow 1-f^{-1}(F)$ is fuzzy δ-preopen in X.
$\Rightarrow 1-f^{-1}(F)$is fuzzy δ-preclosed in X.
Let x_λ be a fuzzy point in $f^{-1}(1-F)$ i.e. $x_\lambda \in f^{-1}(1-F)$
$\Rightarrow f(x_\lambda) \in 1-F$
$\Rightarrow f(x_\lambda) \notin F$
$\Rightarrow x_\lambda \notin f^{-1}(F)$
$\Rightarrow x_\lambda \in 1-f^{-1}(F)$.

Hence, $f^{-1}(1-F) \leq 1-f^{-1}(F)$.

Similarly, we can show that $f^{-1}(1-F) \geq 1-f^{-1}(F)$.
Thus we conclude that $f^{-1}(1-F) = 1-f^{-1}(F)$.
$(v) \Rightarrow (i)$ We have, for every fuzzy δ-preclosed set F in Y, $f^{-1}(F)$ is fuzzy δ-preclosed set in X.
$\Rightarrow 1-f^{-1}(F)$ is fuzzy δ-preopen set in X
$\Rightarrow f^{-1}(1-F)$ is fuzzy δ-preopen set in X.

Hence for each fuzzy δ-preopen set $1-F$ in Y, $f^{-1}(1-F)$ is fuzzy δ-preopen set in X.
Therefore f is fuzzy δ^*-almost continuous function.

Theorem 3.2. Let U be a fuzzy set in a mixed fuzzy topological space $(X, \tau_1(\tau_2))$, then $\delta - pcl(U)$ is the intersection of all fuzzy δ-preclosed sets containing U.

Proof. Let $V = \wedge\{A \in I^X : A \text{ is } \delta - \text{ preclosed and } U \leq A\}$.

We have to show that $V = \delta - pcl(U)$.

Let $x_\lambda \in V$ be any fuzzy point in X.

Suppose, $x_\lambda \notin \delta - pcl(U)$. Then x_λ is not a fuzzy δ-precluster point of U, so there exists a fuzzy $\delta - pre - q - nbdB$ of x_λ such that UqB (U is not quasi-coincident with B).

Therefore, there exists a fuzzy δ-preopen set C in X such that $x_\lambda \in C$

and $C \leq B$.
But $UqB \Rightarrow UqC \Rightarrow \leq 1 - C$.

Since, C is fuzzy δ-preopen, so $1 - C$ is fuzzy δ-preclosed set and $V \leq 1 - C$ as V is the smallest fuzzy δ-preclosed set containing U.

Hence, $x_\lambda \notin V$. This leads to a contradiction.

Thus, $x_\lambda \in \delta - pcl(U)$ and so $V \leq \delta - pcl(U)$.

Conversely, we shall show that $V \geq \delta - pcl(U)$.

Suppose $x_\lambda \in \delta - pcl(U)$, but $x_\lambda \notin V$.

If $x_\lambda \notin V$, then there exists a fuzzy δ-preclosed set F in X containing U such that $x_\lambda \notin F$.

$\Rightarrow x_\lambda \in 1-F$ and $Uq(1-F)$. This contradicts to x_λ is fuzzy δ-precluster point of U.

Therefore, $x_\lambda \in V$ and so we get $V \geq \delta - pcl(U)$.

Thus $V = \delta - pcl(U)$ i.e. for any fuzzy set U in a mixed fuzzy topological space $(X, \tau_1(\tau_2))$, $\delta - pcl(U)$ is the intersection of all fuzzy δ-preclosed sets containing U.

Theorem 3.3. Let A and B be two fuzzy subsets in a mixed fuzzy topological space $(X, \tau_1(\tau_2))$, the following results hold:

(a) $A \leq B \Rightarrow \delta - pcl(A) \leq \delta - pcl(B)$.
(b) A is fuzzy δ-preclosed if and only if $A = \delta - pcl(A)$.
(c) $\delta - pcl(A)$ is fuzzy δ-preclosed in X.
(d) $\delta - pcl(\delta - pcl(A)) = \delta - pcl(A)$.

Proof.
(a) Let A and B be two fuzzy subsets of a mixed fuzzy topological space X and $A \leq B$.
Let $x_\lambda \in \delta - pcl(A)$. Then there exists a fuzzy $\delta - pre - q - nbd(U)$ of x_λ such that AqU and $U \leq A$.

We have $A \leq B$. Therefore, UqA and $A \leq B \Rightarrow UqB$.
Hence, we get a fuzzy $\delta - pre - q - nbdU$ of x_λ such that BqU and $U \leq B$ i.e. x_λ is a fuzzy δ-precluster point of B.
Therefore, $x_\lambda \in \delta - pc(B)$ and consequently we get $\delta - pcl(A) \leq \delta - pcl(B)$.
(b) Let A be a fuzzy δ-preclosed set in X.
Clearly, $\delta - pcl(A) \leq A$.
Also, A is fuzzy δ-preclosed set $\Rightarrow A$ contains all its δ-precluster points $\Rightarrow A \leq \delta - pcl(A)$.
Hence, $A = \delta - pcl(A)$.
Converse part is straightforward from definition of δ-preclosed set.
(c) This result directly follows from Theorem 3.2.
(d) Follows from (b) and (c).

Theorem 3.4. The union of any collection of fuzzy δ-preopen sets in a mixed fuzzy topological space is fuzzy δ-preopen.

Proof. Let $(X, \tau_1(\tau_2))$ be a mixed fuzzy topological space.

Consider the collection $\{A_\alpha : \alpha \in \Delta\}$ of fuzzy δ-preopen sets in X, where Δ is the index set.

We show that $\bigvee_{\alpha\in\Delta} A_\alpha$ is also fuzzy δ-preopen set.

Since each A_α is fuzzy a δ-preopen set, so by definition of fuzzy δ-preopen set $A_\alpha \leq \tau_1 - int(\tau_2 - \delta - clA_\alpha)$ for each $\alpha \in \Delta$.

We have,

$$\begin{aligned}
\bigvee_{\alpha\in\Delta} A_\alpha &\leq \bigvee_{\alpha\in\Delta} (\tau_1 - int(\tau_2 - \delta - clA_\alpha) \\
&\leq \tau_1 - int(\bigvee_{\alpha\in\Delta} (\tau_2 - \delta - clA_\alpha) \\
&\leq \tau_1 - int(\tau_2 - \delta - cl(\bigvee_{\alpha\in\Delta} (A_\alpha))
\end{aligned}$$

Hence, $\bigvee_{\alpha\in\Delta} A_\alpha$ is a fuzzy δ-preopen set in X. Thus arbitrary union of fuzzy δ-preopen set is fuzzy δ-preopen.

Theorem 3.5. Finite intersection of fuzzy δ-preopen set is fuzzy δ-preopen set in mixed fuzzy topological spaces.

Proof: Suppose A_n be a fuzzy δ-preopen set in a mixed fuzzy topological space $(X, \tau_1(\tau_2))$.

Then by definition of fuzzy δ-preopen set we have $A_n \leq \tau_1 - int(\tau_2 - \delta - clA_n)$. To show that $\bigwedge_{n=1}^{m} A_n \leq \tau_1 - int(\tau_2 - \delta - cl(\bigwedge_{n=1}^{m} A_n))$.

We have,

$\bigwedge_{n=1}^{m} (\tau_1 - int(\tau_2 - \delta - cl(A_n))) = \tau_1 - int(\bigwedge_{n=1}^{m} (\tau_2 - \delta - cl(A_n)))$
(since $intA \bigwedge intB = int(A \bigwedge B)$, for any two fuzzy sets A and B in a fuzzy topological space).

$$(3.1) \Rightarrow \bigwedge_{n=1}^{m} (\tau_1 - int(\tau_2 - \delta - cl(A_n))) \leq \tau_1 - int(\tau_2 - \delta - cl(\bigwedge_{n=1}^{m} (A_n)))$$

(since $\overline{A \bigwedge B} \leq \overline{A} \bigwedge \overline{B}$ and $int(A) \leq int(B)$).

Also,

$$\bigwedge_{n=1}^{m} A_n \leq \bigwedge_{n=1}^{m} (\tau_1 - int(\tau_2 - \delta - cl(A_n))), (\text{since } A \leq B \Rightarrow int(A) \leq int(B))$$

(3.2)

From (3.1) and (3.2) it can be concluded that $\bigwedge_{n=1}^{m} A_n \leq (\tau_1 - int(\tau_2 - \delta - cl(\bigwedge_{n=1}^{m} A_n)))$.

Hence finite intersection of fuzzy δ-preopen set is fuzzy is fuzzy δ-preopen set.

4. Fuzzy δ^*-continuous functions

In this section, we introduce the fuzzy δ^*-continuous functions between two mixed fuzzy topological spaces.

Definition 4.1. A fuzzy set A in a mixed fuzzy topological space $(X, \tau_1(\tau_2))$ is said to be fuzzy δ-regular open set (in short $FR-\delta$-open) if $\tau_1 - int(\tau_2 - \delta - cl(A)) = A$ and its complement is said to be fuzzy δ-regular closed set.

Definition 4.2. A fuzzy set U in a mixed fuzzy topological space $(X, \tau_1(\tau_2))$ is said to be fuzzy δ-pre-neighbourhood (in short fuzzy δ-pre-nbd.) of a fuzzy point x_λ if there exists a fuzzy δ-regular open set V in X such that $x_\lambda \in V$ and $U \leq V$.

Definition 4.3. A function $f : (X, \tau_1(\tau_2)) \to (Y, \tau_3(\tau_4))$ is said to be fuzzy δ^*-continuous function if $f^{-1}(V)$ is fuzzy δ-regular open set in X, for every fuzzy δ-preopen set V in Y.

From the above definition it is clear that a fuzzy δ-regular open set is always fuzzy δ-preopen set. But the converse may not be true in general.

Example 4.1. Let us consider a non-empty set $X = \{x, y\}$ and consider the following fuzzy sets on X.

$A = \{(x, 0.7), (y, 0.3)\}$ and $B = \{(x, 0.3), (y, 0.7)\}$. Then the collection of fuzzy sets

$\tau_1 = \{0_X, 1_X, B\}$ and $\tau_2 = \{0_X, 1_X, A\}$ are two fuzzy topologies on X.

Then we construct the mixed fuzzy topology on X from these two fuzzy topologies τ_1 and τ_2 and we get $\tau_1(\tau_2) = \{0_X, 1_X, A\}$. If we consider the fuzzy set C in X defined by $C(x) = 0.3$ and $C(y) = 0.7$, then $\tau_2 - \delta - cl(C) = \wedge\{F : F \text{ is } \delta - \text{closed and } C \leq F\} = 1_X$.

Also, $\tau_1 - int(1_X) = 1_X$. Hence $C \leq \tau_1 - int(\tau_2 - \delta - cl(C)) \Rightarrow C$ is fuzzy δ-preopen set. But C is not fuzzy δ-regular open set.

Theorem 4.1. For a function $f : X \to Y$ from a mixed fuzzy topological space $(X, \tau_1(\tau_2))$ into another mixed fuzzy topological space $(Y, \tau_3(\tau_4))$, the following conditions are equivalent.
(i) f is fuzzy δ^*-continuous function.
(ii) For each fuzzy point x_λ in X and fuzzy δ-pre-nbd V of $f(x_\lambda)$, $f^{-1}(V)$ is fuzzy δ-regular nbd of x_λ.
(iii) For each fuzzy point x_λ in X and fuzzy δ-prenbd V of $f(x_\lambda)$, there is a fuzzy δ-regular-nbh U of x_λ such that $f(U) \leq V$.

(*iv*) For each fuzzy δ-preclosed set F in Y, $f^{-1}(F)$ is fuzzy δ-regular closed set in X.

Proof. (*i*)$\Rightarrow$ (*ii*) Suppose f be a fuzzy δ^*-continuous function.
Let x_λ be a fuzzy point in X and V be the fuzzy δ-prenbd of $f(x_\lambda)$. Then by definition of fuzzy δ-prenbd of $f(x_\lambda)$, there exists a fuzzy δ-preopen set A in Y such that $f(x_\lambda) \in A$ and $A \leq V$. Since f is fuzzy δ^*-continuous function, therefore by definition $f^{-1}(A)$ is fuzzy δ-regular open set in X.
We have $f(x_\lambda) \in A \Rightarrow x_\lambda \in f^{-1}(A)$
and $A \leq V \Rightarrow f^{-1}(A) \leq f^{-1}(V)$.
Hence, $f^{-1}(V)$ is fuzzy δ-regular open set in X.
(*ii*) $\Rightarrow$ (*iii*) Straightforward.
(*iii*) $\Rightarrow$ (*iv*) Let F be a fuzzy δ-preclosed set in Y. Then $1 - F$ is fuzzy δ-preopen set in Y.
Let x_λ be a fuzzy point in X such that $f(x_\lambda) \in 1 - F$.
Then, $1 - F$ itself a fuzzy δ-prenbd of $f(x_\lambda)$. Hence by the given condition (*iii*), we have a fuzzy δ-regular-nbd V of x_λ such that $f(V) \leq 1 - F$. This implies, $V \leq f^{-1}(1 - F)$.
$\Rightarrow V \leq 1 - f^{-1}(A) \Rightarrow f^{-1}(A) \leq A$.
Hence, $f^{-1}(A)$ is fuzzy δ-regular open set in X.
(*iv*) $\Rightarrow$ (*i*) Straightforward.

This completes the proof of the theorem.

From the above definitions of fuzzy δ^*-almost continuity and fuzzy δ^*-continuity it is clear that fuzzy δ^*-continuous functions between two mixed fuzzy topological spaces is always fuzzy δ^*-almost continuous function. But every fuzzy δ^*-almost continuous function may not be fuzzy δ^*-almost continuous. This follows from the following example.

Example 4.2. Consider a non-empty set $X = \{x, y\}$ and the following fuzzy sets on X.

$A = \{(x, 0.7), (y, 0.3)\}$ and $B = \{(x, 0.3), (y, 0.7)\}$. Then the collection of fuzzy sets $\tau_1 = \{0_X, 1_X, B\}$ and $\tau_2 = \{0_X, 1_X, A\}$ are two fuzzy topologies on X and from these two topologies we get the mixed fuzzy topology $\tau_1(\tau_2) = \{0_X, 1_X, A\}$ on X.

Consider another mixed fuzzy topology on X defined as follows:

Let the fuzzy sets on X be defined by

$A_1 = \{(x, 0.2), (y, 0.8)\}$, $A_2 = \{(x, 0.2), (y, 0.2)\}$, $A_3 = \{(x, 0.8), (y, 0.2)\}$, $A_4 = \{(x, 0.8), (y, 0.8)\}$.

Then the collection $\tau_3 = \{0_X, 1_X, A_1, A_2, A_3, A_4\}$ will form a fuzzy topology on X.

Also, consider the following fuzzy sets in X as follows:
$B_1 = \{(x, 0.3), (y, 0.7)\}$, $B_2 = \{(x, 0.7), (y, 0.3)\}$, $B_3 = \{(x, 0.3), (y, 0.3)\}$, $B_4 = \{(x, 0.7), (y, 0.7)\}$

Then the collection of fuzzy sets $\tau_4 = \{0_X, 1_X, B_1, B_2, B_3, B_4\}$ will form a fuzzy topology on X and from these two topologies we get the mixed fuzzy topology $\tau_3(\tau_4) = \{0_X, 1_X\}$ on X.

Consider the identity function $f_i : (X, \tau_1(\tau_2)) \to (X, \tau_3(\tau_4))$, then f_i is fuzzy δ^*-almost continuous function but not fuzzy δ^*-continuous. Consider the fuzzy set $C = \{(x, 0.7), (y, 0.5)\}$, then C is fuzzy δ-preopen set in X and $f_i^{-1}(C) = \{(x, 0.7), (y, 0.5)\}$, the inverse image under f_i, is not fuzzy δ-regular open. Hence f_i is not fuzzy δ^*-continuous function.

Theorem 4.2. If $f : X \to Y$ is fuzzy δ^*-almost continuous function and $g : Y \to Z$ is fuzzy δ^*-continuous function, then $g \circ f$ is fuzzy δ^*-almost continuous.

Proof. Since g is fuzzy δ^*-continuous so $g^{-1}(V)$ is fuzzy δ-regular open set in Y, for any fuzzy δ-preopen set V in Z.

We have $g^{-1}(V)$ is fuzzy δ-regular open set in Y.
$\Rightarrow g^{-1}(V)$ is fuzzy δ-preopen set in Y.
$\Rightarrow f^{-1}(g^{-1}(V))$ is fuzzy δ-preopen set in X. (Since f is fuzzy δ^*-almost continuous)
$\Rightarrow (g \circ f)^{-1}(V)$ is fuzzy δ-preopen set in X.
Hence, $g \circ f$ is fuzzy δ^*-almost continuous.

Acknowledgement. The authors thank the unanimous reviewer for the comments on the first draft of the article.

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