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Spectra and fine spectra for the upper triangular band matrix $U(a_0, a_1, a_2; b_0, b_1, b_2)$ over the sequence space c_0

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Abstract

The aim of this paper is to obtain the spectrum, fine spectrum, approximate point spectrum, defect spectrum and compression spectrum of the operator

$$U(a_0,a_1,a_2;b_0,b_1,b_2)=\begin{bmatrix} a_0 & b_0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & a_1 & b_1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & a_2 & b_2 & 0 & 0 & \cdots \\ 0 & 0 & 0 & a_0 & b_0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & a_1 & b_1 & \cdots \\ 0 & 0 & 0 & 0 & 0 & a_2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} (b_0, b_1, b_2 \neq 0)$$

on the sequence space c_0 where b_0, b_1 , b_2 are nonrzero and the nonzero diagonals are the entries of an oscillatory sequence.

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Keywords : *upper triangular band matrix, spectrum, fine spectrum, approximate point spectrum, defect spectrum, compression spectrum.*

1. Introduction

The spectral theory is one of the most useful tools in science. There are many applications in mathematics and physics which involves matrix theory, control theory, function theory, differential and integral equations, complex analysis, and quantum physics. For example, atomic energy levels are determined and therefore the frequency of a laser or the spectral signature of a star are obtained by it in quantum mechanics.

Let $L : X \to Y$ be a bounded linear operator where X and Y are Banach spaces. Denote the range of L, $R(L)$ and the set of all bounded linear operators on X into itself $B(X)$.

Assume that X be a Banach space and $L \in B(X)$. The adjoint operator $L^* \in B(X^*)$ of L is defined by $(L^* f)(x) = f(Lx)$ for all $f \in X^*$ and $x \in X$ where X^* is the dual space X.

Let X is a complex normed linear space and $D(L) \subset X$ be domain of L where $L : D(L) \to X$ be a linear operator. For $L \in B(X)$ we determine a complex number λ by the operator $(\lambda I - L)$ denoted by L_λ which has the same domain $D(L)$, such that I is the identity operator. Recall that the resolvent operator of L is $L_\lambda^{-1} := (\lambda I - L)^{-1}$.

Let $\lambda \in \mathbf{C}$. If L_λ^{-1} exists, is bounded and, is defined on a set which is dense in X then λ is called a regular value of L.

The set $\rho(L, X)$ of all regular values of L is called the resolvent set of L. $\sigma(L, X) := \mathbf{C} \backslash \rho(L, X)$ is called the spectrum of L where $\mathbf{C}$ is complex plane. Hence those values $\lambda \in \mathbf{C}$ for which L_λ is not invertible are contained in the spectrum $\sigma(L, X)$.

The spectrum $\sigma(L, X)$ is union of three disjoint sets as follows: The point spectrum $\sigma_p(L, X)$ is the set such that L_λ^{-1} does not exist. Further $\lambda \in \sigma_p(L, X)$ is called the eigen value of L. We say that $\lambda \in \mathbf{C}$ belongs to the continuous spectrum $\sigma_c(L, X)$ of L if the resolvent operator L_λ^{-1} is defined on a dense subspace of X and is unbounded. Furthermore, we say that $\lambda \in \mathbf{C}$ belongs to the residual spectrum $\sigma_r(L, X)$ of L if the resolvent operator L_λ^{-1} exists, but its domain of definition (i.e. the range $R(\lambda I - L)$) of $(\lambda I - L)$ is not dense in X; in this case L_λ^{-1} may be bounded or unbounded. Together with the point spectrum, these two subspectra form a disjoint subdivision

$$\sigma(L, X) = \sigma_p(L, X) \cup \sigma_c(L, X) \cup \sigma_r(L, X) \tag{1.1}$$

of the spectrum of L.

1.1. Goldberg's Classification of Spectrum

If $T \in B(X)$, then there are three possibilities for $R(T)$:
(I) $R(T) = X$,
(II) $\overline{R(T)} = X$, but $R(T) \neq X$,
(III) $\overline{R(T)} \neq X$
and three possibilities for T^{-1}:
(1) T^{-1} exists and continuous,
(2) T^{-1} exists but discontinuous,
(3) T^{-1} does not exist.

If these possibilities are combined in all possible ways, nine different states are created. These are labelled by: I_1, I_2, I_3, II_1, II_2, II_3, III_1, III_2, III_3. If an operator is in state III_2 for example, then $\overline{R(T)} \neq X$ and T^{-1} exists but is discontinuous (see [9]).

If λ is a complex number such that $T = \lambda I - L \in I_1$ or $T = \lambda I - L \in II_1$, then $\lambda \in \rho(L, X)$. All scalar values of λ not in $\rho(L, X)$ comprise the spectrum of L. The further classification of $\sigma(L, X)$ gives rise to the fine spectrum of L. That is, $\sigma(L, X)$ can be divided into the subsets $I_2\sigma(L, X) = \emptyset$, $I_3\sigma(L, X)$, $II_2\sigma(L, X)$, $II_3\sigma(L, X)$, $III_1\sigma(L, X)$, $III_2\sigma(L, X)$, $III_3\sigma(L, X)$. For example, if $T = \lambda I - L$ is in a given state, III_2 (say), then we write $\lambda \in III_2\sigma(L, X)$.

Let w be the space of all real or complex valued sequences. The space of all bounded, convergent, null and bounded variation sequences are denoted by ℓ_∞, c, c_0 and bv, respectively. Also by ℓ_1, ℓ_p, bv_p we denote the spaces of all absolutely summable sequences, $p-$absolutely summable sequences and $p-$bounded variation sequences, respectively.

Many researchers have investigated the spectrum and the fine spectrum of linear operators defined by some determined limitation matrices over certain sequence spaces. There are a lot of studies about spectrum and fine spectrum. For instance, the fine spectrum of the Cesàro operator has been examined by Gonzalez [10] on the sequence space ℓ_p for $(1 < p < \infty)$. Also, Wenger [23] has studied the fine spectrum Hölder summability operator over c, and Rhoades [14] generalized this result to the weighted mean methods. Reade [13] has investigated the spectrum of the Cesàro operator on the sequence space c_0. The spectrum of the Rhaly operators on the sequence spaces c_0 and c has examined by Yildirim [21] and the fine spectrum of the Rhaly operators on the sequence space c_0 has examined by Yildirim [22].

In [19], Tripathy and Das determined the spectrum and fine spectrum of the upper triangular matrix $U(r, s)$ on the sequence space

$$cs = \left\{ x = (x_n) \in w : \lim_{n\to\infty} \sum_{i=0}^{n} x_i \text{ exists} \right\},$$

which is a Banach space with respect to the norm $\|x\|_{cs} = \sup_n |\sum_{i=0}^{n} x_i|$. Also they determined the subdivisions of the spectrum of the operator $U(r,s)$ on the same space. In [16], the norm and spectrum of the Cesàro matrix considered as a bounded operator on $\overline{bv_0} \cap \ell_\infty$ were studied by Tripathy and Saikia. In [17], Tripathy and Paul examined the spectra of the operator $D(r,0,0,s)$ on sequence spaces c_0 and c. In [11], Paul and Tripathy investigated the spectrum of the operator $D(r,0,0,s)$ over the sequence spaces ℓ_p and bv_p. In [18], the spectra of the Rhaly operator on the class of bounded statistically null bounded variation sequence space was determined by Tripathy and Das. In [12], Paul and Tripathy investigated the fine spectrum of the operator $D(r,0,0,s)$ over a sequence space bv_0. In [4], the spectrum and fine spectrum of the lower triangular matrix $B(r,s,t)$ on the sequence space cs were studied by Das and Tripathy. In [6], the fine spectrum of the lower triangular matrix $B(r,s)$ over the Hahn sequence space was investigated by Das.

2. Fine Spectrum

The upper triangular matrix $U(a_0,a_1,a_2;b_0,b_1,b_2)$ is an infinitite matrix with the non-zero diagonals are the entries of an oscillatory sequence of the form

$$U(a_0,a_1,a_2;b_0,b_1,b_2) = \begin{bmatrix} a_0 & b_0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & a_1 & b_1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & a_2 & b_2 & 0 & 0 & \cdots \\ 0 & 0 & 0 & a_0 & b_0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & a_1 & b_1 & \cdots \\ 0 & 0 & 0 & 0 & 0 & a_2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{2.1}$$

where b_0, b_1 , b_2 are nonrzero. The matrix $A = (a_{nk})$ gives rise to a bounded linear operator $T \in B(c_0)$ from c_0 to itself if and only if
(i) the rows of A in ℓ_1 and their ℓ_1 norms are bounded,
(ii) the columns of A are in c_0.

The operator norm of T is the supremum of ℓ_1 norm values of the rows.

Corollary 1. *$U(a_0, a_1, a_2; b_0, b_1, b_2) : c_0 \to c_0$ is a bounded linear operator and $\|U(a_0, a_1, a_2; b_0, b_1, b_2)\|_{(c_0:c_0)} = \max\{|a_0| + |b_0|, |a_1| + |b_1|, |a_2| + |b_2|\}$.*

Lemma 1 (Golberg [9, p.59]). *T has a dense range if and only if T^* is 1-1.*

Lemma 2 (Golberg [9, p.60]). *T has a bounded inverse if and only if T^* is onto.*

Theorem 1. $\sigma_p(U(a_0, a_1, a_2; b_0, b_1, b_2), c_0)$
$= \{\lambda \in \mathbf{C} : |\lambda - a_0| |\lambda - a_1| |\lambda - a_2| < |b_0| |b_1| |b_2|\}$.

Proof. If λ be an eigenvalue of the operator $U(a_0, a_1, a_2; b_0, b_1, b_2)$, then there exists $x \neq \theta = (0, 0, 0, ...)$ in c_0 such that $U(a_0, a_1, a_2; b_0, b_1, b_2)x = \lambda x$. Then we have

$$\begin{array}{rcl} a_0x_0 + b_0x_1 & = & \lambda x_0 \\ a_1x_1 + b_1x_2 & = & \lambda x_1 \\ a_2x_2 + b_2x_3 & = & \lambda x_2 \\ a_0x_3 + b_0x_4 & = & \lambda x_3 \\ a_1x_4 + b_1x_5 & = & \lambda x_4 \\ & \vdots & \end{array}.$$

From here, we get

$$\begin{cases} x_{3n} & = & (\frac{(\lambda-a_2)(\lambda-a_1)(\lambda-a_0)}{b_2b_1b_0})^n x_0, \\ x_{3n+1} & = & \frac{(\lambda-a_0)}{b_0}(\frac{(\lambda-a_2)(\lambda-a_1)(\lambda-a_0)}{b_2b_1b_0})^n x_0, \qquad n \geq 0. \\ x_{3n+2} & = & \frac{(\lambda-a_0)(\lambda-a_1)}{b_0b_1}(\frac{(\lambda-a_2)(\lambda-a_1)(\lambda-a_0)}{b_2b_1b_0})^n x_0. \end{cases}$$

The subsequences (x_{3n}), (x_{3n+1}) and (x_{3n+2}) of $x = (x_n)$ are in c_0 if and only if $|\lambda - a_0| |\lambda - a_1| |\lambda - a_2| < |b_0| |b_1| |b_2|$ and so, $x = (x_n) \in c_0$ if and only if $|\lambda - a_0| |\lambda - a_1| |\lambda - a_2| < |b_0| |b_1| |b_2|$. Thus,
$\sigma_p(U(a_0, a_1, a_2; b_0, b_1, b_2), c_0) = \{\lambda \in \mathbf{C} : |\lambda - a_0| |\lambda - a_1| |\lambda - a_2| < |b_0| |b_1| |b_2|\}$.
□

We will use the following Lemma to find the adjoint of a linear transform on the sequence space c_0.

Lemma 3. *[23, p.266] Let $T : c_0 \longmapsto c_0$ be a linear map and define $T^* : \ell_1 \longmapsto \ell_1$, by $T^*g = g \circ T$, $g \in c_0^* \cong \ell_1$, then T must be given with the matrix A, moreover, T^* must be given with the matrix A^t.*

Theorem 2. $\sigma_p(U(a_0,a_1,a_2;b_0,b_1,b_2)^*, c_0^*\cong\ell_1) = \emptyset$.

Proof. From Lemma 3, the matrix of $U(a_0,a_1,a_2;b_0,b_1,b_2)^*$ is transpose of matrix of $U(a_0,a_1,a_2;b_0,b_1,b_2)$. If λ be an eigenvalue of the operator $U(a_0,a_1,a_2;b_0,b_1,b_2)^*$, then there exists $x \neq \theta = (0,0,0,...)$ in ℓ_1 with $U(a_0,a_1,a_2;b_0,b_1,b_2)^*x = \lambda x$.

Then, we get

$$\begin{array}{rcl} a_0x_0 & = & \lambda x_0 \\ b_0x_0 + a_1x_1 & = & \lambda x_1 \\ b_1x_1 + a_2x_2 & = & \lambda x_2 \\ b_2x_2 + a_0x_3 & = & \lambda x_3 \\ b_0x_3 + a_1x_4 & = & \lambda x_4 \\ & \vdots & \end{array}.$$

If $n = 3k$, then $b_0x_n + a_1x_{n+1} = \lambda x_{n+1}$, $k \geq 0$. If $n = 3k+1$, then $b_1x_n + a_2x_{n+1} = \lambda x_{n+1}$, $k \geq 0$. If $n = 3k+2$, then $b_2x_n + a_0x_{n+1} = \lambda x_{n+1}$, $k \geq 0$. Let x_k be the first non-zero of the sequence (x_n). If $k = 3n+2$, then we get $a_0 = \lambda$ since $b_2x_{k-1} + a_0x_k = \lambda x_k$. Then from the relation $b_2x_{k+1} + a_0x_{k+2} = \lambda x_{k+2}$, we have $x_{k+1} = 0$. But from $b_2x_n + a_0x_{n+1} = \lambda x_{n+1}$, we have $b_2x_n = 0$ which implies $x_k = 0$ as $b_2 \neq 0$, a contradiction. Similarly, if $k = 3n+1$ and $k = 3n$ we get a contradiction.
Thus, $\sigma_p(U(a_0,a_1,a_2;b_0,b_1,b_2)^*, c_0^*\cong\ell_1) = \emptyset$. □

Theorem 3. $\sigma_r(U(a_0,a_1,a_2;b_0,b_1,b_2), c_0) = \emptyset$.

Proof. Since, $\sigma_r(A) = \sigma_p(A^*,\ell_1)\backslash\sigma_p(A,c_0)$, Theorems 1 and 2 give us required result. □

Lemma 4. $\sum\limits_{n=0}^{\infty}\left(\sum\limits_{k=0}^{3n} a_kb_{nk}\right) = \sum\limits_{k=0}^{\infty} a_k\left(\sum\limits_{n=k}^{\infty} b_{nk}\right)$

where

(a_k) and (b_{nk}) are nonnegative real numbers.

Proof.

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{3n} a_kb_{nk}\right) = \sum_{k=0}^{0} a_kb_{0k} + \sum_{k=0}^{3} a_kb_{1k} + \sum_{k=0}^{6} a_kb_{2k} + \sum_{k=0}^{9} a_kb_{3k} + \cdots$$

$$
\begin{aligned}
&= a_0 b_{00} + (a_0 b_{10} + a_1 b_{11} + a_2 b_{12} + a_3 b_{13}) \\
&\quad + (a_0 b_{20} + a_1 b_{21} + a_2 b_{22} + \cdots + a_6 b_{26}) + \cdots \\
&= a_0 (b_{00} + b_{10} + b_{20} + \cdots) + a_1 (b_{11} + b_{21} + b_{31} + \cdots) \\
&\quad + a_2 (b_{12} + b_{22} + b_{32} + \cdots) + \cdots \\
&= a_0 \sum_{n=0}^{\infty} b_{n0} + a_1 \sum_{n=1}^{\infty} b_{n1} + a_2 \sum_{n=2}^{\infty} b_{n2} + \cdots \\
&= \sum_{k=0}^{\infty} a_k \left(\sum_{n=k}^{\infty} b_{nk} \right).
\end{aligned}
$$

□

Lemma 5. *Let $a_x = a_y$, $b_x = b_y$ for $x \equiv y(mod 3)$. Then*

$$
\prod_{m=0}^{3n-k-1} \frac{b_{3n-m}}{a_{3n-m} - \lambda} = B \left[\frac{b_2 b_1 b_0}{(a_2 - \lambda)(a_1 - \lambda)(a_0 - \lambda)} \right]^{n-j-1}
$$

where

$$
B = \begin{cases} \frac{b_2 b_1 b_0}{(a_2-\lambda)(a_1-\lambda)(a_0-\lambda)} & , \quad k = 3j \\ \frac{b_2 b_1}{(a_2-\lambda)(a_1-\lambda)} & , \quad k = 3j-1 \\ \frac{b_2}{(a_2-\lambda)} & , \quad k = 3j-2 \end{cases} .
$$

Proof. The required result is obtained by calculating finite product. □

Theorem 4. $\sigma_c(U(a_0, a_1, a_2; b_0, b_1, b_2), c_0) = \{\lambda \in \mathbf{C} : |\lambda - a_0| \, |\lambda - a_1| \, |\lambda - a_2| = |b_0| \, |b_1| \, |b_2|\}$ *and* $\sigma(U(a_0, a_1, a_2; b_0, b_1, b_2), c_0) = \{\lambda \in \mathbf{C} : |\lambda - a_0| \, |\lambda - a_1| \, |\lambda - a_2| \leq |b_0| \, |b_1| \, |b_2|\}$.

Proof. Let $y = (y_n) \in \ell_1$ be such that $(U(a_0, a_1, a_2; b_0, b_1, b_2) - \lambda I)^* x = y$ for some $x = (x_n)$.

Then we get system of linear equations:

$$\begin{array}{rcl} (a_0-\lambda)x_0 & = & y_0 \\ b_0x_0+(a_1-\lambda)x_1 & = & y_1 \\ b_1x_1+(a_2-\lambda)x_2 & = & y_2 \\ b_2x_2+(a_0-\lambda)x_3 & = & y_3 \\ b_0x_3+(a_1-\lambda)x_4 & = & y_4 \\ & \vdots & \\ b_0x_{3n}+(a_1-\lambda)x_{3n+1} & = & y_{3n+1} \\ b_1x_{3n+1}+(a_2-\lambda)x_{3n+2} & = & y_{3n+2} \\ b_2x_{3n+2}+(a_0-\lambda)x_{3n+3} & = & y_{3n+3} \\ & \vdots & \end{array} \quad , n \geq 0$$

Solving these equations, we have

$$\begin{aligned} x_0 &= \tfrac{1}{a_0-\lambda}y_0 \\ x_1 &= \tfrac{1}{a_1-\lambda}y_1-\tfrac{b_0}{(a_0-\lambda)(a_1-\lambda)}y_0 \\ x_2 &= \tfrac{1}{a_2-\lambda}y_2-\tfrac{b_1}{(a_1-\lambda)(a_2-\lambda)}y_1+\tfrac{b_0b_1}{(a_0-\lambda)(a_1-\lambda)(a_2-\lambda)}y_0 \\ x_3 &= \tfrac{1}{a_0-\lambda}y_3-\tfrac{b_2}{(a_0-\lambda)(a_2-\lambda)}y_2+\tfrac{b_1b_2}{(a_0-\lambda)(a_1-\lambda)(a_2-\lambda)}y_1-\tfrac{b_0b_1b_2}{(a_0-\lambda)^2(a_1-\lambda)(a_2-\lambda)}y_0 \\ &\vdots \end{aligned}$$

Then, we get

$$\begin{aligned} x_{3n} &= \tfrac{1}{a_0-\lambda}y_{3n}-\tfrac{b_2}{(a_0-\lambda)(a_2-\lambda)}y_{3n-1}+\tfrac{b_1b_2}{(a_0-\lambda)(a_1-\lambda)(a_2-\lambda)}y_{3n-2} \\ &\quad -\tfrac{b_0b_1b_2}{(a_0-\lambda)^2(a_1-\lambda)(a_2-\lambda)}y_{3n-3}+\cdots+(-1)^{3n}\tfrac{b_0^nb_1^nb_2^n}{(a_0-\lambda)^{n+1}(a_1-\lambda)^n(a_2-\lambda)^n}y_0 \\ x_{3n+1} &= \tfrac{1}{a_1-\lambda}y_{3n+1}-\tfrac{b_0}{(a_0-\lambda)(a_1-\lambda)}y_{3n}+\tfrac{b_0b_2}{(a_0-\lambda)(a_1-\lambda)(a_2-\lambda)}y_{3n-1} \\ &\quad -\tfrac{b_0b_1b_2}{(a_0-\lambda)(a_1-\lambda)^2(a_2-\lambda)}y_{3n-3}+\cdots+(-1)^{3n+1}\tfrac{b_0^nb_1^{n-1}b_2^{n-1}}{(a_0-\lambda)^n(a_1-\lambda)^{n-1}(a_2-\lambda)^{n-1}}y_0 \\ x_{3n+2} &= \tfrac{1}{a_2-\lambda}y_{3n+2}-\tfrac{b_0}{(a_1-\lambda)(a_2-\lambda)}y_{3n+1}+\tfrac{b_0b_1}{(a_0-\lambda)(a_1-\lambda)(a_2-\lambda)}y_{3n} \\ &\quad -\tfrac{b_0b_1b_2}{(a_0-\lambda)(a_1-\lambda)(a_2-\lambda)^2}y_{3n-1}+\cdots+(-1)^{3n+2}\tfrac{b_0^nb_1^nb_2^{n-1}}{(a_0-\lambda)^n(a_1-\lambda)^n(a_2-\lambda)^n}y_0. \end{aligned}$$

Thus we obtain that

$$x_{3n+t}=\frac{1}{a_t-\lambda}\sum_{k=0}^{3n+t}(-1)^{3n+t-k}y_k\prod_{m=0}^{3n+t-k-1}\frac{b_{3n+t-m}}{a_{3n+t-m}-\lambda}, \quad t=0,1,2.$$

Herein $a_x = a_y$, $b_x = b_y$ for $x \equiv y (mod 3)$.
Therefore we get

$$\begin{aligned}
\sum_{v=0}^{\infty} |x_v| &= |x_0| + |x_1| + |x_2| + |x_3| + \cdots \\
&= \sum_{n=0}^{\infty} |x_{3n+t}| \\
&= \sum_{n=0}^{\infty} \left| \frac{1}{a_t - \lambda} \sum_{k=0}^{3n+t} (-1)^{3n+t-k} y_k \prod_{m=0}^{3n+t-k-1} \frac{b_{3n+t-m}}{a_{3n+t-m} - \lambda} \right| \\
&\leq \frac{1}{|a_t - \lambda|} \sum_{n=0}^{\infty} \left[\sum_{k=0}^{3n+t} |y_k| \prod_{m=0}^{3n+t-k-1} \left| \frac{b_{3n+t-m}}{a_{3n+t-m} - \lambda} \right| \right].
\end{aligned}$$

Now, let us take $t = 0$ and consider the series $\sum_{n=0}^{\infty} \left[\sum_{k=0}^{3n} |y_k| \prod_{m=0}^{3n-k-1} \left| \frac{b_{3n+t-m}}{a_{3n+t-m}-\lambda} \right| \right]$.
In Lemma 5 if we take $a_k = y_k$ and $b_{nk} = \prod_{m=0}^{3n+t-k-1} \left| \frac{b_{3n+t-m}}{a_{3n+t-m}-\lambda} \right|$ then we have

$$\begin{aligned}
\sum_{n=0}^{\infty} \left[\sum_{k=0}^{3n} |y_k| \prod_{m=0}^{3n-k-1} \left| \frac{b_{3n+t-m}}{a_{3n+t-m} - \lambda} \right| \right] &= \sum_{k=0}^{\infty} \left[\sum_{n=k}^{\infty} |y_k| \prod_{m=0}^{3n-k-1} \left| \frac{b_{3n+t-m}}{a_{3n+t-m} - \lambda} \right| \right] \\
&= \sum_{k=0}^{\infty} \left[|y_k| \sum_{n=k}^{\infty} \prod_{m=0}^{3n-k-1} \left| \frac{b_{3n+t-m}}{a_{3n+t-m} - \lambda} \right| \right].
\end{aligned}$$

Also since $\prod_{m=0}^{3n-k-1} \frac{b_{3n-m}}{a_{3n-m}-\lambda} = B \left[\frac{b_2 b_1 b_0}{(a_2-\lambda)(a_1-\lambda)(a_0-\lambda)} \right]^{n-j-1}$ from Lemma 5, the last equation turns into the series

$$B \sum_{k=0}^{\infty} \left[|y_k| \sum_{n=k}^{\infty} \left[\frac{b_2 b_1 b_0}{(a_2 - \lambda)(a_1 - \lambda)(a_0 - \lambda)} \right]^{n-j-1} \right]. \tag{2.2}$$

Since $y = (y_n) \in \ell_1$, the series $\sum_{k=0}^{\infty} |y_k|$ is convergent. Hence the series (2.2) is convergent if and only if $\left| \frac{b_2 b_1 b_0}{(a_2-\lambda)(a_1-\lambda)(a_0-\lambda)} \right| < 1$. Consequently, if for $\lambda \in \mathbf{C}$, $|a_2-\lambda|\,|a_1-\lambda|\,|a_0-\lambda| > |b_2|\,|b_1|\,|b_0|$, then $(x_n) \in \ell_1$. Therefore, the operator $(U(a_0,a_1,a_2;b_0,b_1,b_2)-\lambda I)^*$ is onto if $|\lambda-a_0|\,|\lambda-a_1|\,|\lambda-a_2| > |b_0|\,|b_1|\,|b_2|$. Then by Lemma 2 $U(a_0,a_1,a_2;b_0,b_1,b_2)-\lambda I$ has a bounded inverse if $|\lambda-a_0|\,|\lambda-a_1|\,|\lambda-a_2| > |b_0|\,|b_1|\,|b_2|$.

So, $\sigma_c(U(a_0,a_1,a_2;b_0,b_1,b_2),c_0) \subseteq \{\lambda \in \mathbf{C} : |\lambda-a_0|\,|\lambda-a_1|\,|\lambda-a_2| \leq |b_0|\,|b_1|\,|b_2|\}$.

Since $\sigma(L,c_0)$ is the disjoint union of $\sigma_p(L,c_0)$, $\sigma_r(L,c_0)$ and $\sigma_c(L,c_0)$, therefore

$$\sigma(U(a_0,a_1,a_2;b_0,b_1,b_2),c_0) \subseteq \{\lambda \in \mathbf{C} : |\lambda-a_0|\,|\lambda-a_1|\,|\lambda-a_2| \leq |b_0|\,|b_1|\,|b_2|\}.$$

By Theorem 1, we get
$\{\lambda \in \mathbf{C} : |\lambda-a_0|\,|\lambda-a_1|\,|\lambda-a_2| < |b_0|\,|b_1|\,|b_2|\} = \sigma_p(U(a_0,a_1,a_2;b_0,b_1,b_2),c_0) \subset \sigma(U(a_0,a_1,a_2;b_0,b_1,b_2),c_0)$.

Since, $\sigma(L,c_0)$ is a compact set, so it is closed and thus,

$$\begin{aligned}
\overline{\{\lambda \in \mathbf{C} : |\lambda-a_0|\,|\lambda-a_1|\,|\lambda-a_2| < |b_0|\,|b_1|\,|b_2|\}} &\subset \overline{\sigma(U(a_0,a_1,a_2;b_0,b_1,b_2),c_0)} \\
&= \sigma(U(a_0,a_1,a_2;b_0,b_1,b_2),c_0)
\end{aligned}$$

and $\{\lambda \in \mathbf{C} : |\lambda-a_0|\,|\lambda-a_1|\,|\lambda-a_2| \leq |b_0|\,|b_1|\,|b_2|\} \subset \sigma(U(a_0,a_1,a_2;b_0,b_1,b_2),c_0)$.

Hence, $\sigma(U(a_0,a_1,a_2;b_0,b_1,b_2),c_0) = \{\lambda \in \mathbf{C} : |\lambda-a_0|\,|\lambda-a_1|\,|\lambda-a_2| \leq |b_0|\,|b_1|\,|b_2|\}$ and so $\sigma_c(U(a_0,a_1,a_2;b_0,b_1,b_2),c_0) = \{\lambda \in \mathbf{C} : |\lambda-a_0|\,|\lambda-a_1|\,|\lambda-a_2| = |b_0|\,|b_1|\,|b_2|\}$.
□

Theorem 5. *If $|\lambda-a_0|\,|\lambda-a_1|\,|\lambda-a_2| < |b_0|\,|b_1|\,|b_2|$, then $\lambda \in I_3\sigma(U(a_0,a_1,a_2;b_0,b_1,b_2),c_0)$.*

Proof. Suppose that $|\lambda-a_0|\,|\lambda-a_1|\,|\lambda-a_2| < |b_0|\,|b_1|\,|b_2|$ and so from Theorem 1, $\lambda \in \sigma_p(U(a_0,a_1,a_2;b_0,b_1,b_2),c_0)$. Hence, λ satisfies Golberg's condition 3.We shell show that $U(a_0,a_1,a_2;b_0,b_1,b_2)-\lambda I$ is onto when $|\lambda-a_0|\,|\lambda-a_1|\,|\lambda-a_2| < |b_0|\,|b_1|\,|b_2|$.

Let $y=(y_n) \in c_0$ be such that $(U(a_0,a_1,a_2;b_0,b_1,b_2)-\lambda I)x = y$ for $x=(x_n)$. Then,

$$\begin{aligned}
(a_0-\lambda)x_0 + b_0x_1 &= y_0 \\
(a_1-\lambda)x_1 + b_1x_2 &= y_1 \\
&\vdots \\
(a_{n-1}-\lambda)x_{n-1} + b_{n-1}x_n &= y_{n-1} \\
&\vdots
\end{aligned}$$

and so

$$(2.3)\quad x_n = \frac{1}{b_{n-1}}\left[y_{n-1} + \sum_{k=0}^{n-2} y_k \prod_{v=1}^{n-k-1} \frac{\lambda - a_{n-v}}{b_{n-v-1}}\right] + x_0 \prod_{v=1}^{n} \frac{\lambda - a_{n-v}}{b_{n-v}}$$

where $a_x = a_y$, $b_x = b_y$ for $x \equiv y(mod3)$.

$$0 \leq |x_n| \leq \frac{1}{|b_{n-1}|}\left[|y_{n-1}| + \sum_{k=0}^{n-2} |y_k| \prod_{v=1}^{n-k-1} \left|\frac{\lambda - a_{n-v}}{b_{n-v-1}}\right|\right] + |x_0| \prod_{v=1}^{n} \left|\frac{\lambda - a_{n-v}}{b_{n-v}}\right|.$$
(2.4)

If we take $q := \left|\frac{(a_2-\lambda)(a_1-\lambda)(a_0-\lambda)}{b_2 b_1 b_0}\right|^{1/3}$, then for $q < 1$ from Stolz Theorem we have

$$\sum_{k=0}^{n-2} |y_k| \prod_{v=1}^{n-k-1} \left|\frac{\lambda - a_{n-v}}{b_{n-v-1}}\right| \sim M \sum_{k=0}^{n-2} |y_k|\, q^{n-k-1} = M \frac{\sum_{k=0}^{n-2} \frac{|y_k|}{q^k}}{q^{1-n}} \sim M \frac{|y_n|}{1-q},\ n \to \infty$$
(2.5)

since $\prod_{v=1}^{n} \left|\frac{\lambda - a_{n-v}}{b_{n-v}}\right| \sim Mq^n$, $n \to \infty$. If we take limit as $n \to \infty$ in (2.5), then we get $\lim_{n\to\infty} \sum_{k=0}^{n-2} |y_k| \prod_{v=1}^{n-k-1} \left|\frac{\lambda - a_{n-v}}{b_{n-v-1}}\right| = 0$ since $(y_n) \in c_0$. Thus, it must be $\left|\frac{(\lambda-a_0)(a_1-\lambda)(a_2-\lambda)}{b_0 b_1 b_2}\right| < 1$ to (x_n) be in c_0 from (2.4).

Therefore,

$U(a_0, a_1, a_2; b_0, b_1, b_2) - \lambda I$ is onto. So, $\lambda \in I$. Hence we get the required result. □

3. Partition of the Spectrum

Also the spectrum $\sigma(L, X)$ is partitioned into three sets which are not necessarily disjoint as follows:

If there exists a sequence (x_n) in X shuch that $\|x_n\| = 1$ and $\|Lx_n\| \to 0$ as $n \to \infty$ then (x_n) is called Weyl sequence for L.

We call the set

$$(3.1)\ \sigma_{ap}(L, X) := \{\lambda \in \mathbf{C} : \text{there exists a Weyl sequence for } \lambda I - L\}$$

the approximate point spectrum of L. Moreover, the set

$$\sigma_\delta(L, X) := \{\lambda \in \sigma(L, X) : \lambda I - L \text{ is not surjective}\} \tag{3.2}$$

is called defect spectrum of L. Finally, the set

$$\sigma_{co}(L, X) = \{\lambda \in \mathbf{C} : \overline{R(\lambda I - L)} \neq X\} \tag{3.3}$$

is called compression spectrum in the literature.

The following Proposition is quitly useful for calculating the separation of the spectrum of linear operator in Banach spaces.

Proposition 1 ([2], Proposition 1.3). *The spectra and subspectra of an operator $L \in B(X)$ and its adjoint $L^* \in B(X^*)$ are related by the following relations:*
(a) $\sigma(L^*, X^*) = \sigma(L, X)$, (b) $\sigma_c(L^*, X^*) \subseteq \sigma_{ap}(L, X)$,
(c) $\sigma_{ap}(L^*, X^*) = \sigma_\delta(L, X)$, (d) $\sigma_\delta(L^*, X^*) = \sigma_{ap}(L, X)$,
(e) $\sigma_p(L^*, X^*) = \sigma_{co}(L, X)$, (f) $\sigma_{co}(L^*, X^*) \supseteq \sigma_p(L, X)$,
(g) $\sigma(L, X) = \sigma_{ap}(L, X) \cup \sigma_p(L^*, X^*) = \sigma_p(L, X) \cup \sigma_{ap}(L^*, X^*)$.

By the definitions given above, we can write following table

		1	2	3
		L_λ^{-1} exists and is bounded	L_λ^{-1} exists and is unbounded	L_λ^{-1} does not exists
I	$R(\lambda I - L) = X$	$\lambda \in \rho(L, X)$	–	$\lambda \in \sigma_p(L, X)$ $\lambda \in \sigma_{ap}(L, X)$
II	$\overline{R(\lambda I - L)} = X$	$\lambda \in \rho(L, X)$	$\lambda \in \sigma_c(L, X)$ $\lambda \in \sigma_{ap}(L, X)$ $\lambda \in \sigma_\delta(L, X)$	$\lambda \in \sigma_p(L, X)$ $\lambda \in \sigma_{ap}(L, X)$ $\lambda \in \sigma_\delta(L, X)$
III	$\overline{R(\lambda I - L)} \neq X$	$\lambda \in \sigma_r(L, X)$ $\lambda \in \sigma_\delta(L, X)$ $\lambda \in \sigma_{co}(L, X)$	$\lambda \in \sigma_r(L, X)$ $\lambda \in \sigma_{ap}(L, X)$ $\lambda \in \sigma_\delta(L, X)$ $\lambda \in \sigma_{co}(L, X)$	$\lambda \in \sigma_p(L, X)$ $\lambda \in \sigma_{ap}(L, X)$ $\lambda \in \sigma_\delta(L, X)$ $\lambda \in \sigma_{co}(L, X)$

Table 1: Subdivisions of the spectra of a linear operator.

Quite recently, many authors have examined spectral divisions of generalized difference matrices. For example, Akhmedov and El-Shabrawy, [1] have studied the spectrum and fine spectrum of the generalized lower triangle double-band matrix Δ_v on the sequence spaces c_0, c and ℓ_p, where $1 < p < \infty$.

The above-mentioned articles, concerned with the decomposition of the spectrum which defined by Goldberg. However, in [7] Durna and Yildirim have investigated subdivision of the spectra for factorable matrices on c_0 and in [3] Basar, Durna and Yildirim have investigated subdivisions of the spectra for generalized difference operator on the sequence spaces c_0 and c and in [8] Durna, have studied subdivision of the spectra for the generalized upper triangular double-band matrices Δ^{uv} over the sequence spaces c_0 and c. In [5], Das has calculated the spectrum and fine spectrum of the upper triangular matrix $U(r_1, r_2; s_1, s_2)$ over the sequence space c_0. In [20], the fine spectrum of the upper triangular matrix $U(r, 0, 0, s)$ over the squence spaces c_0 and c was studied by Tripathy and Das.

Corollary 2. $III_1\sigma(U(a_0, a_1, a_2; b_0, b_1, b_2), c_0)$
$= III_2\sigma(U(a_0, a_1, a_2; b_0, b_1, b_2), c_0) = \emptyset$.

Proof. Since $\sigma_r(L, c_0) = III_1\sigma(L, c_0) \cup III_2\sigma(L, c_0)$ from Table 1, the required result is obtained from Theorem 3. $\square$

Corollary 3. $II_3\sigma(U(a_0, a_1, a_2; b_0, b_1, b_2), c_0)$
$= III_3\sigma(U(a_0, a_1, a_2; b_0, b_1, b_2), c_0) = \emptyset$.

Proof. Since $\sigma_p(L, c_0) = I_3\sigma(L, c_0) \cup II_3\sigma(L, c_0) \cup III_3\sigma(L, c_0)$ from Table 1, the required result is obtained from Theorem 1 and Theorem 5. $\square$

Theorem 6.
(a) $\sigma_{ap}(U(a_0, a_1, a_2; b_0, b_1, b_2), c_0) = \{\lambda \in \mathbf{C} : |\lambda - a_0|\,|\lambda - a_1|\,|\lambda - a_2| \le |b_0|\,|b_1|\,|b_2|\}$,
(b) $\sigma_\delta(U(a_0, a_1, a_2; b_0, b_1, b_2), c_0) = \{\lambda \in \mathbf{C} : |\lambda - a_0|\,|\lambda - a_1|\,|\lambda - a_2| = |b_0|\,|b_1|\,|b_2|\}$,
(c) $\sigma_{co}(U(a_0, a_1, a_2; b_0, b_1, b_2), c_0) = \emptyset$.

Proof. (a) From Table 1, we get $\sigma_{ap}(L, c_0) = \sigma((L, c_0)\backslash III_1\sigma(L, c_0)$. And so
$\sigma_{ap}(U(a_0, a_1, a_2; b_0, b_1, b_2), c_0) = \{\lambda \in \mathbf{C} : |\lambda - a_0|\,|\lambda - a_1|\,|\lambda - a_2| \le |b_0|\,|b_1|\,|b_2|\}$ from Corollary 2. (b) From Table 1, we have $\sigma_\delta(L, c_0) = \sigma(L, c_0)\backslash I_3\sigma(L, c_0)$. So using Theorem 4 and 5, we get the required result. (c) By Proposition 1 (e), we have $\sigma_p(L^*, c_0^*) = \sigma_{co}(L, c_0)$.

Using Theorem 2, we get the required result. $\square$

Corollary 4. *(a)* $\sigma_{ap}(U(a_0, a_1, a_2; b_0, b_1, b_2)^*, c_0^* \cong \ell_1)$
$= \{\lambda \in \mathbf{C} : |\lambda - a_0|\,|\lambda - a_1|\,|\lambda - a_2| = |b_0|\,|b_1|\,|b_2|\}$
(b) $\sigma_{\delta}(U(a_0, a_1, a_2; b_0, b_1, b_2)^*, c_0^* \cong \ell_1)$
$= \{\lambda \in \mathbf{C} : |\lambda - a_0|\,|\lambda - a_1|\,|\lambda - a_2| \leq |b_0|\,|b_1|\,|b_2|\}$.

Proof. Using Proposition 1 (c) and (d), we have
$\sigma_{ap}(U(a_0, a_1, a_2; b_0, b_1, b_2)^*, c_0^* \cong \ell_1) = \sigma_{\delta}(U(a_0, a_1, a_2; b_0, b_1, b_2), c_0)$
and
$\sigma_{\delta}(U(a_0, a_1, a_2; b_0, b_1, b_2)^*, c_0^* \cong \ell_1) = \sigma_{ap}(U(a_0, a_1, a_2; b_0, b_1, b_2), c_0)$.
Using Theorem 6 (a) and (b), we get the required results. □

4. Results

We have obtained subsequences x_{3k}, x_{3k+1}, x_{3k+2} investigating subsets of the spectrum in calculations which are in before sections. And we have examined when these sequences belong to spaces c_0 or ℓ_1. For this, we have proved Theorem 1 and Lemma 4.

Let the upper triangular matrix $U(a_0, a_1, \ldots, a_{n-1}; b_0, b_1, \ldots, b_{n-1})$ be an infinitite matrix with the non-zero diagonals are the entries of an oscillatory sequence.of the form

$$U(a_0, a_1, \ldots, a_{n-1}; b_0, b_1, \ldots, b_{n-1}) = \begin{bmatrix} a_0 & b_0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & a_1 & b_1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & \ddots & \ddots & 0 & 0 & \cdots \\ 0 & 0 & 0 & a_{n-1} & b_{n-1} & 0 & \cdots \\ 0 & 0 & 0 & 0 & a_0 & b_0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & a_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$
(4.1)

where b_0, $b_1, \ldots, b_{n-1}$ are nonrzero.

For calculating subsets of spectrum of matrix $U(a_0, a_1, \ldots, a_{n-1}; b_0, b_1, \ldots, b_{n-1})$, we obtain subsequences x_{nk}, x_{nk+1},…,x_{nk+n-1} using same techniques in before section. And for investigating when these sequences belong to spaces c_0 or ℓ_1, if we generalize Lemma 2 and Lemma 5, we get the following spectral decompositions:

Theorem 7. Let $S = \left\{ \lambda \in \mathbf{C} : \prod_{k=0}^{n-1} \left| \frac{\lambda - a_k}{b_k} \right| \leq 1 \right\}$, $\overset{\circ}{S}$ be the interior of the set S and ∂S be the boundary of the set S. Then the following are provided

1. $\sigma_p(U(a_0, a_1, \dots, a_{n-1}; b_0, b_1, \dots, b_{n-1}), c_0) = \overset{\circ}{S}$,
2. $\sigma_p(U(a_0, a_1, \dots, a_{n-1}; b_0, b_1, \dots, b_{n-1})^*, c_0^* \cong \ell_1) = \emptyset$,
3. $\sigma_r(U(a_0, a_1, \dots, a_{n-1}; b_0, b_1, \dots, b_{n-1}), c_0) = \emptyset$,
4. $\sigma_c(U(a_0, a_1, \dots, a_{n-1}; b_0, b_1, \dots, b_{n-1}), c_0) = \partial S$,
5. $\sigma(U(a_0, a_1, \dots, a_{n-1}; b_0, b_1, \dots, b_{n-1}), c_0) = S$,
6. $I_3\sigma(U(a_0, a_1, \dots, a_{n-1}; b_0, b_1, \dots, b_{n-1}), c_0) = \overset{\circ}{S}$,
7. $III_1\sigma\,(U(a_0, a_1, \dots, a_{n-1}; b_0, b_1, \dots, b_{n-1}), c_0) = \emptyset$,
8. $III_2\sigma(U(a_0, a_1, \dots, a_{n-1}; b_0, b_1, \dots, b_{n-1}), c_0) = \emptyset$,
9. $III_3\sigma(U(a_0, a_1, \dots, a_{n-1}; b_0, b_1, \dots, b_{n-1}), c_0) = \emptyset$,
10. $II_3\sigma(U(a_0, a_1, \dots, a_{n-1}; b_0, b_1, \dots, b_{n-1}), c_0) = \emptyset$,
11. $\sigma_{ap}(U(a_0, a_1, \dots, a_{n-1}; b_0, b_1, \dots, b_{n-1}), c_0) = S$,
12. $\sigma_\delta(U(a_0, a_1, \dots, a_{n-1}; b_0, b_1, \dots, b_{n-1}), c_0) = \partial S$,
13. $\sigma_{co}(U(a_0, a_1, \dots, a_{n-1}; b_0, b_1, \dots, b_{n-1}), c_0) = \emptyset$,
14. $\sigma_{ap}(U(a_0, a_1, \dots, a_{n-1}; b_0, b_1, \dots, b_{n-1})^*, c_0^* \cong \ell_1) = \partial S$,
15. $\sigma_\delta(U(a_0, a_1, \dots, a_{n-1}; b_0, b_1, \dots, b_{n-1})^*, c_0^* \cong \ell_1) = S$.

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