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On nearly Lindelöf spaces via generalized topology

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Abstract

In this paper a new class of sets termed as ω_μ-regular open sets has been introduced and some of its properties are studied. We have introduced μ-nearly Lindelöfness in μ-spaces. We have shown that under certain conditions a μ-Lindelöf space [7] is equivalent to a μ-nearly Lindelöf space. Some properties of such spaces and some characterizations of such spaces in terms of ω_μ-regular open sets are given.

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1. Introduction

For the last one decade or so, a new area of study has emerged and has been rapidly growing. The area is concerned with the investigations of generalized topological spaces and several classes of generalized types of open sets. The notion of ω-open set in a topological space was introduced by Hdeib [4] which forms a topology finer than the original topology. Our aim here is to study the notion of μ-nearly Lindelöf spaces by using the concept of generalized topology introduced by Á. Császár [2]. We first recall some definitions given in [2]. Let X be a non-empty set and $expX$ denote the power set of X. We call a class $\mu \subseteqq expX$ a generalized topology (briefly, GT) [1, 2], if $\emptyset \in \mu$ and μ is closed under arbitrary union. A set X, with a GT μ on it is said to be a generalized topological space (briefly, GTS) and is denoted by (X, μ). For a GTS (X, μ), the elements of μ are called μ-open sets and the complements of μ-open sets are called μ-closed sets. A GTS (X, μ) is called a μ-space [5] or a strong GTS [2] if $X \in \mu$. A GTS (X, μ) is called a QTS if μ is closed under finite intersection. It is easy to observe that i_μ and c_μ are idempotent and monotonic. It is also well known from [1, 2] that if μ is a GT on X and $A \subseteqq X$, $x \in X$, then $x \in c_\mu(A)$ iff $x \in M \in \mu \Rightarrow M \cap A \neq \emptyset$ and $c_\mu(X \setminus A) = X \setminus i_\mu(A)$. A subset A of a GTS (X, μ) is called μr-open if $A = i_\mu(c_\mu(A))$. The complement of a μr-open set is called a μr-closed set(see [3] for details).

The purpose of this paper is to study the concept of μ-nearly Lindelöfness in a μ-space. We have also investigated several characterizations of such spaces. In the sequel we have studied some special types of functions which preserve μ-nearly Lindelöfness where $X \in \mu$.

2. ω_μ-regular open sets and its properties

Definition 2.1. Let (X, μ) be a GTS. A subset A of X is called an ω_μ-regular open (resp. ω_μ-open [6]) set if for each $x \in A$, there exists a μr-open (μ-open) set U containing x such that $U \setminus A$ is countable. The complement of an ω_μ-regular open (resp. ω_μ-open) set is known as an ω_μ-regular closed (resp. ω_μ-closed [6]) set.

It follows from Definition 2.1 that every ω_μ-regular open set is an ω_μ-open set and every μr-open set is an ω_μ-regular open set but the converses are false as shown in Example 2.3.

Remark 2.2. Let μ be a GT on a set X. Then the following relations hold:

$$\mu r\text{-open set} \Rightarrow \omega_\mu\text{-regular open set} \Rightarrow \omega_\mu\text{-open set}$$

Example 2.3. (a) Let $\Re$ be the set of reals and $\mu = \{\emptyset, Q, I, \Re\}$, where Q is the set of rationals and I is the set of irrationals. Then $(\Re, \mu)$ is a GTS. It can be checked easily that $I \cup \{0\}$ is an ω_μ-regular open set but not a μr-open set.
(b) Let $X = \Re$ be the set of reals and $\mu = \{A \subseteqq X : 0 \in A\} \cup \{\emptyset\}$. Then μ is a GT on X. It can be checked easily that $[0, 1)$ is an ω_μ-open set but not an ω_μ-regular open set.

The family of all ω_μ-regular open sets of a GTS (X, μ) is denoted by ω_μ-$RO(X)$.

Proposition 2.4. In a GTS (X, μ), ω_μ-$RO(X)$ is a GT on X.

Proof. It is obvious that $\emptyset$ is an ω_μ-regular open set. Let $\{A_\alpha : \alpha \in \Lambda\}$ be a collection of ω_μ-regular open subsets of X. Then for each $x \in \cup\{A_\alpha : \alpha \in \Lambda\}$, $x \in A_\alpha$ for some $\alpha \in \Lambda$. Thus there exists a μr-open set U containing x such that $U \setminus A_\alpha$ is countable. Now as $U \setminus (\cup\{A_\alpha : \alpha \in \Lambda\}) \subseteqq U \setminus A_\alpha$, so $U \setminus (\cup\{A_\alpha : \alpha \in \Lambda\})$ is also countable. Thus $\cup\{A_\alpha : \alpha \in \Lambda\}$ is an ω_μ-regular open set.

Theorem 2.5. A subset A of a GTS (X, μ) is an ω_μ-regular open set if and only if for each $x \in A$, there exists a μr-open set U_x containing x and a countable subset C such that $U_x \setminus C \subseteqq A$.

Proof. Let A be an ω_μ-regular open set in X and $x \in A$. Then there exists a μr-open set U_x containing x such that $U_x \setminus A$ is countable. Let $C = U_x \setminus A = U_x \cap (X \setminus A)$. Then $U_x \setminus C \subseteqq A$.

Conversely, let $x \in A$ and there exist a μr-open set U_x containing x and a countable subset C such that $U_x \setminus C \subseteqq A$. Thus $U_x \setminus A \subseteqq C$ and hence $U_x \setminus A$ is a countable set. Thus A is an ω_μ-regular open set in X.

Theorem 2.6. Let (X, μ) be a GTS and $C \subseteqq X$. If C is ω_μ-regular closed, then $C \subseteqq K \cup B$ for some μr-closed set K and a countable subset B.

Proof. If C be ω_{μ}-regular closed, then $X \setminus C$ is ω_{μ}-regular open and hence for each $x \in X \setminus C$, there exist a μr-open set U containing x and a countable subset B such that $U \setminus B \subseteqq (X \setminus C)$. Thus $C \subseteqq X \setminus (U \setminus B) = X \setminus (U \cap (X \setminus B)) = (X \setminus U) \cup B$. Let $K = X \setminus U$. Then K is μr-closed such that $C \subseteqq K \cup B$.

3. μ-nearly Lindelöf spaces

Definition 3.1. A μ-space (X, μ) is said to be μ-nearly Lindelöf if every cover of X by μr-open sets has a countable subcover.

A subset A of a μ-space (X, μ) is said to be μ-nearly Lindelöf relative to X if every cover of A by μr-open sets of X has a countable subcover of A.

Theorem 3.2. A μ-space (X, μ) is μ-nearly Lindelöf if and only if every ω_{μ}-regular open cover of X has a countable subcover.

Proof. Necessity. Let $\{U_{\alpha} : \alpha \in \Lambda\}$ be any ω_{μ}-regular open cover of X. Then for each $x \in X$, there exists $\alpha(x) \in \Lambda$ such that $x \in U_{\alpha(x)}$. As $U_{\alpha(x)}$ is ω_{μ}-regular open, there exists a μr-open set $V_{\alpha(x)}$ containing x such that $V_{\alpha(x)} \setminus U_{\alpha(x)}$ is also countable. Then the family $\{V_{\alpha(x)} : x \in X\}$ is a cover of X by μr-open subsets of X. Since (X, μ) is μ-nearly Lindelöf, there exist $\{x_i : i < \omega\} \subseteqq X$ such that $X = \cup\{V_{\alpha(x_i)} : i < \omega\}$. Thus $X = \cup[\{V_{\alpha(x_i)} \setminus U_{\alpha(x_i)} : i < \omega\}] \cup [\cup(U_{\alpha(x_i)} : i < \omega)]$. For each $\alpha(x_i)$, $V_{\alpha(x_i)} \setminus U_{\alpha(x_i)}$ is a countable set and there exists a countable subset $\Lambda_{\alpha(x_i)}$ of Λ such that $V_{\alpha(x_i)} \setminus U_{\alpha(x_i)} \subseteqq \{U_{\alpha} : \alpha \in \Lambda_{\alpha(x_i)}\}$. Thus $X \subseteqq [\cup(\cup\{U_{\alpha} : \alpha \in \Lambda_{\alpha(x_i)}; i < \omega\})] \cup [\cup(U_{\alpha(x_i)} : i < \omega)]$.

Since every μr-open set is ω_{μ}-regular open, the sufficiency is obvious.

Theorem 3.3. A μ-space (X, μ) is μ-nearly Lindelöf if and only if for every family $\{F_{\alpha} : \alpha \in \Lambda\}$ of ω_{μ}-regular closed sets with countable intersection property, $\cap\{F_{\alpha} : \alpha \in \Lambda\} \neq \emptyset$.

Proof. Suppose that (X, μ) be a μ-nearly Lindelöf space and suppose that $\{F_{\alpha} : \alpha \in \Lambda\}$ be a family of ω_{μ}-regular closed subsets of X with countable intersection property such that $\cap\{F_\alpha : \alpha \in \Lambda\} = \emptyset$. Let us consider the ω_μ-regular open sets $U_\alpha = X \setminus F_\alpha$. Then the family $\{U_\alpha : \alpha \in \Lambda\}$ is an ω_μ-regular open cover of X. Since X is μ-nearly Lindelöf, there exists a countable subcover $\{U_{\alpha_i} : \alpha_i \in \mathbf{N}\}$ such that $X = \cup\{U_{\alpha_i} : \alpha_i \in \mathbf{N}\} = \cup\{X \setminus F_{\alpha_i} : \alpha_i \in \mathbf{N}\} = X \setminus \cap\{F_{\alpha_i} : \alpha_i \in \mathbf{N}\}$. Hence $\cap\{F_{\alpha_i} : \alpha_i \in \mathbf{N}\} = \emptyset$. Thus if the family $\{F_\alpha : \alpha \in \Lambda\}$ of ω_μ-regular closed sets be with the countable intersection property, then $\cap\{F_\alpha : \alpha \in \Lambda\} \neq \emptyset$.

Conversely, let $\{U_\alpha : \alpha \in \Lambda\}$ be a cover of X by ω_μ-regular open sets and that for every family $\{F_\alpha : \alpha \in \Lambda\}$ of ω_μ-regular closed sets with countable intersection property, $\cap\{F_\alpha : \alpha \in \Lambda\} \neq \emptyset$. Then $X = \cup\{U_\alpha : \alpha \in \Lambda\}$. Therefore $\emptyset = \cap\{X \setminus U_\alpha : \alpha \in \Lambda\}$ and the family $\{X \setminus U_\alpha : \alpha \in \Lambda\}$ is a family of ω_μ-regular closed sets with an empty intersection. Thus there exists a countable collection $\{X \setminus U_{\alpha_i} : i \in \mathbf{N}\}$ such that $\cap\{X \setminus U_{\alpha_i} : i \in \mathbf{N}\} = \emptyset$. Hence $X = \cup\{U_{\alpha_i} : i \in \mathbf{N}\}$. Thus X is μ-nearly Lindelöf.

Theorem 3.4. Every ω_μ-regular closed subset of a μ-nearly Lindelöf space (X, μ) is μ-nearly Lindelöf relative to X.

Proof. Let A be an ω_μ-regular closed subset of X. Let $\{U_\alpha : \alpha \in \Lambda\}$ be a cover of A by μr-open subsets of X. Then for each $x \in X \setminus A$, there exists a μr-open set V_x such that $V_x \cap A$ is countable. Since X is μ-nearly Lindelöf and the collection $\{U_\alpha : \alpha \in \Lambda\} \cup \{V_x : x \in X \setminus A\}$ is a cover of X by μr-open sets, there exists a countable subcover $\{U_{\alpha_i} : i \in \mathbf{N}\} \cup \{V_{x_i} : i \in \mathbf{N}\}$. Since $\cup\{V_{x_i} \cap A : i \in \mathbf{N}\}$ is countable, so for each $x_j \in \cup\{V_{x_i} \cap A : i \in \mathbf{N}\}$, there is $U_{\alpha x_j} \in \{U_\alpha : \alpha \in \Lambda\}$ such that $x_j \in U_{\alpha x_j}$ and $j \in \mathbf{N}$. Hence $\{U_{\alpha_i} : i \in \mathbf{N}\} \cup \{U_{\alpha x_j} : j \in \mathbf{N}\}$ is a countable subcover of $\{U_\alpha : \alpha \in \Lambda\}$ and it covers A. Therefore A is μ-nearly Lindelöf relative to X.

Definition 3.5. A GTS (X, μ) is said to be ω_μ-regular if for each $x \in X$ and each μ-open set G_x in X containing x, there exists an ω_μ-regular open set H_x such that $x \in H_x \subseteqq c_\mu(H_x) \subseteqq G_x$.

Definition 3.6. A GTS (X, μ) is said to be weakly ω_μ-regular if for each $x \in X$ and each μ-open set U_x containing x there exists an ω_μ-regular open set H_x such that $x \in H_x \subseteqq U_x$.

We note from Definitions 3.5 and 3.6 that every ω_μ-regular space is weakly ω_μ-regular.

Definition 3.7. A μ-space (X, μ) is said to be μ-Lindelöf [7] if every cover of X by μ-open sets has a countable subcover.

A subset A of a μ-space (X, μ) is said to be μ-Lindelöf relative to X [7] if every cover of A by μ-open sets of X has a countable subcover of A.

Theorem 3.8. A weakly ω_μ-regular space is μ-nearly Lindelöf if and only if it is μ-Lindelöf.

Proof. Let X be a weakly ω_μ-regular and μ-nearly Lindelöf μ-space and $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ be a μ-open cover of X. Then for each $x \in X$, there exists $\alpha(x) \in \Lambda$ such that $x \in U_{\alpha(x)}$. Since X is weakly ω_μ-regular, there exists an ω_μ-regular open set $H_{\alpha(x)}$ such that $x \in H_{\alpha(x)} \subseteqq U_{\alpha(x)}$. Then $\{H_{\alpha(x)} : x \in X\}$ is an ω_μ-regular open cover of X. Thus by Theorem 3.2, there exists a countable subcover $\{H_{\alpha_{x_i}} : i \in \mathbf{N}\}$. Hence X is a μ-Lindelöf space. The converse part is obvious.

Example 3.9. Let $\Re$ be the set of real numbers and $\mu = \{A \subseteqq \Re : 0 \in A\} \cup \{\emptyset\}$. Then $(\Re, \mu)$ is a GTS. It is easy to see that $(\Re, \mu)$ is μ-nearly Lindelöf but not μ-Lindelöf as $\{\{0, x\} : x \in \Re\}$ is a cover of $\Re$ by μ-open sets having no countable subcover.

Corollary 3.10. If (X, μ) is ω_μ-regular and μ-nearly Lindelöf, then it is μ-Lindelöf.

Definition 3.11. A GTS (X, μ) is said to be almost ω_μ-regular if for each $x \in X$ and each ω_μ-regular closed set F disjoint from $\{x\}$, there exist disjoint μ-open sets U and V in X such that $F \subseteqq U$ and $x \in V$.

Example 3.12. (a) Let $X = \{a, b, c\}$ and $\mu = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$. Then (X, μ) is a GTS. It can be checked easily that (X, μ) is weakly ω_μ-regular but neither ω_μ-regular nor almost ω_μ-regular.
(b) Let $X = \{a, b, c\}$ and $\mu = \{\emptyset, \{c\}, \{a, b\}, X\}$. Then (X, μ) is a GTS. It can be checked easily that (X, μ) is ω_μ-regular but not almost ω_μ-regular.

Theorem 3.13. A GTS (X, μ) is almost ω_μ-regular if and only if for each $x \in X$ and each ω_μ-regular open set U_x containing x, there exists a μr-open set V_x in X such that $x \in V_x \subseteqq c_\mu(V_x) \subseteqq U_x$.

Proof. Necessity : Let U be an ω_μ-regular open set with $x \in U$. Then $x \notin X \setminus U$, where $X \setminus U$ is ω_μ-regular closed. Thus there exist disjoint

$G, H \in \mu$ such that $x \in G$, $X \setminus U \subseteqq H$. So, $x \in i_\mu(c_\mu(G)) = V_x$ (say), $X \setminus U \subseteqq i_\mu(c_\mu(H)) = U_x$ (say). Thus $x \in V_x \subseteqq c_\mu(V_x) \subseteqq U$ (as $U_x \cap V_x = \emptyset$).

The sufficient part is obvious from the hypothesis.

Theorem 3.14. Let (X, μ) be an almost ω_μ-regular, μ-nearly Lindelöf QTS. Then for every disjoint ω_μ-regular closed sets C_1 and C_2, there exist two disjoint μ-open sets U and V such that $C_1 \subseteqq U$ and $C_2 \subseteqq V$.

Proof. Since (X, μ) is an almost ω_μ-regular space, for each $x \in C_1$ there exists a μr-open set U_x containing x such that $c_\mu(U_x) \cap C_2 = \emptyset$ (by Theorem 3.13). Then the family $\{U_x : x \in C_1\} \cup \{X \setminus C_1\}$ is an ω_μ-regular open cover of X. Since X is μ-nearly Lindelöf, there exist $\{x_i : i < \omega\} \subseteqq X$ such that $X = [\cup\{U_{x_i} : i < \omega\}] \cup (X \setminus C_1)$. Thus $C_1 \subseteqq \cup\{U_{x_i} : i < \omega\}$ and $c_\mu(U_{x_i}) \cap C_2 = \emptyset$. Similarly, there exist a family of μr-open subset V_{y_i} with $\{y_i : i < \omega\} \subseteqq X$ such that $C_2 \subseteqq \cup\{V_{y_i} : i < \omega\}$ and $c_\mu(V_{y_i}) \cap C_1 = \emptyset$. Let $G_k = U_{x_k} \setminus \cup\{c_\mu(V_{y_i}) : i = 1, 2, ..., k\}$ and $H_k = V_{y_k} \setminus \cup\{c_\mu(U_{x_i}) : i = 1, 2, ..., k\}$ and $U = \cup\{G_i : i < \omega\}$ and $V = \cup\{H_i : i < \omega\}$. Then U and V are two disjoint μ-open sets containing C_1 and C_2 respectively.

Definition 3.15. A function $f : (X, \mu) \to (Y, \lambda)$ is said to be δ_{ω_μ}-continuous if for each λ-regular open set V of Y containing $f(x)$, there exists a ω_μ-regular open set U of X containing x such that $f(U) \subseteqq V$.

Theorem 3.16. Let $f : (X, \mu) \to (Y, \lambda)$ be a surjective δ_{ω_μ}-continuous function. If X is μ-nearly Lindelöf, then Y is λ-nearly Lindelöf.

Proof. Let $\mathcal{V} = \{V_\alpha : \alpha \in \Lambda\}$ be a λr-open cover of Y. Then for each $x \in X$, there exists $V_{\alpha(x)} \in \mathcal{V}$ such that $f(x) \in V_{\alpha(x)}$. Thus there exists an ω_μ-regular open set $U_{\alpha(x)}$ of X containing x with $f(U_{\alpha(x)}) \subseteqq V_{\alpha(x)}$. So $\{U_{\alpha(x)} : x \in X\}$ is a cover of X. Hence there exists $\{x_i : i < \omega\} \subseteqq X$ such that $X = \cup\{U_{\alpha(x_i)} : i < \omega\}$. Thus $Y = f(X) = f(\cup\{U_{\alpha(x_i)} : i < \omega\}) = \cup\{f(U_{\alpha(x_i)}) : i < \omega\} \subseteqq \cup\{V_{\alpha(x_i)} : i < \omega\}$.

Definition 3.17. A function $f : (X, \mu) \to (Y, \lambda)$ is said to be $\omega_\mu R$-continuous if $f^{-1}(V)$ is ω_μ-regular open in X for each λ-open set V of Y.

Theorem 3.18. Let $f : (X, \mu) \to (Y, \lambda)$ be a surjective $\omega_\mu R$-continuous function. If X is μ-nearly Lindelöf, then Y is λ- Lindelöf.

Proof. Let $\{V_\alpha : \alpha \in I\}$ be a cover of Y by λ-open subsets of Y. Then $\{f^{-1}(V_\alpha) : \alpha \in I\}$ is a cover of X by ω_μ-regular open sets in X. Thus there exists a countable subset I_0 of I such that $X = \cup\{f^{-1}(V_\alpha) : \alpha \in I_0\}$. Thus $Y = f(X) = \cup\{V_\alpha : \alpha \in I_0\}$ showing Y to be λ-Lindelöf.

Definition 3.19. A function $f : (X, \mu) \to (Y, \lambda)$ is said to be $\omega_\mu R$-closed if for each μr-closed set F of X, $f(F)$ is ω_λ-regular closed in Y.

Definition 3.20. A GTS (X, μ) is said to be a P^*_r-space if every countable union of μr-open sets is a μr-open set.

Theorem 3.21. Let (X, μ) be a P^*_r-space and $f : (X, \mu) \to (Y, \lambda)$ be an $\omega_\mu R$-closed surjection such that for each $y \in Y$, $f^{-1}(y)$ is μ-nearly Lindelöf relative to X. If (Y, λ) is λ-nearly Lindelöf then (X, μ) is μ-nearly Lindelöf.

Proof. Let $\{U_\alpha : \alpha \in \Lambda\}$ be a μr-open cover of X. For each $y \in Y$, $f^{-1}(y)$ is μ-nearly Lindelöf relative to X, so there exists a countable subset $\Lambda(y)$ of Λ such that $f^{-1}(y) \subseteqq \cup\{U_\alpha : \alpha \in \Lambda(y)\}$. Let $U(y) = \cup\{U_\alpha : \alpha \in \Lambda(y)\}$ and $V(y) = Y \setminus f(X \setminus U(y))$. Then $V(y)$ is ω_λ-regular open in Y containing y such that $f^{-1}(V(y)) \subseteqq U(y)$ (as f is $\omega_\mu R$-closed). Since $\{V(y) : y \in Y\}$ is an ω_λ-regular open cover of Y, there exist $\{y_i : i < \omega\} \subseteqq Y$ such that $Y = \cup\{V_{y_k} : k < \omega\}$. Therefore $X = f^{-1}(Y) = \cup\{f^{-1}(V_{y_k}) : k < \omega\} \subseteqq \cup\{U_{y_k} : k < \omega\} = \cup\{U_\alpha : \alpha \in \Lambda(y_k), k < \omega\}$. Thus (X, μ) is μ-nearly Lindelöf.

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