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# A new approach for solving linear fractional integro-differential equations and multi variable order fractional differential equations

F. Ghomanjani[1] orcid.org/0000-0002-5319-9389

[1]Kashmar Higher Education Institute, Dept. of Mathematics, Kashmar, Iran.
f.ghomanjani@kashmar.ac.ir

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## Abstract:

*In the sequel, the numerical solution of linear fractional integrodifferential equations (LFIDEs) and multi variable order fractional differential equations (MVOFDEs) are found by Bezier curve method (BCM) and operational matrix. Some numerical examples are stated and utilized to evaluate the good and accurate results. een the Hölder inequality and the Cauchy-Schwarz inequality.*

**Keywords:** Fractional integro-differential equations; Bezier curve; Variable order fractional differential equation; Caputo's variable order fractional derivative.

**MSC (2010):** 65K10; 26A33; 49K15.

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## 1. Introduction

In this paper, the following linear fractional integro-differential equations (LFIDEs) are considered

$$(1.1)\quad D^{\alpha}f(x)=g(x)+\int_0^1 K(x,t)f(t)\,dt,\quad 0\leq x,t\leq 1,$$
$$f^{(i)}(0)=\delta_i,\quad i=0,1,\ldots,n_1-1,\quad \exists\, n_1-1<\alpha\leq n_1,\quad n_1\in\mathbf{N},$$

where $D^{\alpha}f(x)$ is the $\alpha$th Caputo fractional derivative (CFD) of $f(x)$, $g(x)$ and $K(x,t)$ are given continues functions, $x$ and $t$ are real variables varying in $[0,1]$, and $f(x)$ is the unknown function. We note that the initial conditions are $f^{(i)}(0)=\delta_i$ (for $i=0,1,\ldots,n_1-1$).
By fractional integro-differential equations, various problems from various sciences can be modeled. Recently, many numerical techniques to solve LFIDEs have been given. Bhrawy and Alghamdi [1] applied collocation method for solving the nonlinear fractional Langevin equation involving two fractional orders in different intervals and fractional Fredholm integro-differential equations. In [9], Mohammed proposed numerical solution of LFIDEs by least squares method and shifted Chebyshev polynomial.

Additionally many papers manage the Bezier curves. In [3] and [4], the authors utilized the Bezier curves for solving delay differential equation (DDE) and optimal control of switched systems numerically. In [5], the authors proposed the utilization of Bezier curves on some linear optimal control systems with pantograph delays. Also, to solve the quadratic Riccati differential equation and the Riccati differential-difference equation, the Bezier control points strategy is utilized (see [6]). Some other uses of the Bezier functions are found in (see [7],[10]). In this sequel, the Bezier curve method are applied to solve LFIDEs of the form (1.1).

The organization of this study is classified as follows: In Section 2, Basic Preliminaries is stated. Problem Statement is introduced in Section 3. Also approximation technique is stated in Section 4. Some numerical examples are solved in Section 5. Also, a remark is given for solving MVOFDEs, then numerical applications for solving MVOFDEs is stated in Section 6. Finally, Section 7 will give a conclusion briefly.

## 2. Basic Preliminaries

**Definition 2.1.** *Let $f : [a, b] \to R$ be a function, $\alpha > 0$ a real number, and $n_1 = \alpha$, where $\lceil \alpha \rceil$ denotes the smallest integer greater than or equal to $\alpha$ (see [13]). The left (left RLFD) and right (right RLFD) Riemann-Liouville fractional derivatives are given according to*

$$
\begin{aligned}
{}_aD_t^{\alpha} f(t) &= \frac{1}{\Gamma(n_1-\alpha)} \frac{d^{n_1}}{dt^{n_1}} \int_a^t (t-\tau)^{n_1-\alpha-1} f(\tau)\, d\tau, \quad \textit{(left RLFD)}, \\
{}_tD_b^{\alpha} f(t) &= \frac{(-1)^{n_1}}{\Gamma(n_1-\alpha)} \frac{d^{n_1}}{dt^{n_1}} \int_t^b (\tau-t)^{n_1-\alpha-1} f(\tau)\, d\tau, \quad \textit{(right RLFD)},
\end{aligned} \tag{2.1}
$$

## 3. Problem Statement

Our strategy is utilizing Bezier curves to approximate the solutions $f(x)$ where $f(x)$ is given below. Define the Bezier polynomials of degree $n$ over the interval $[x_0, x_f]$ as follows:

$$
f(x) = \sum_{r=0}^{n} a_r B_{r,n}\left(\frac{x-x_0}{h}\right), \; x_f = 1, \; x_0 = 0, \tag{3.1}
$$

where $h = x_f - x_0$, and

$$
B_{r,n}\left(\frac{x-x_0}{h}\right) := \binom{n}{r} \frac{1}{h^n} (x_f - x)^{n-r} (x - x_0)^r
$$

is the Bernstein polynomial of degree $n$ over the interval $[x_0, x_f]$, and $a_r$, $r = 0, 1, \ldots, n$, and they are unknown control points. By substituting $f(x)$ in (1.1), one may define $R_1(x, a_0, a_1, \ldots, a_n)$ for $x \in [x_0, x_f]$ as follows:

$$
R_1(x, a_0, a_1, \ldots, a_n) \;\; := \;\; D^{\alpha} f(x) - g(x) - \int_0^1 K(x,t) f(t)\, dt. \tag{3.2}
$$

The convergence of this method is proven where $n \to \infty$ (see [2]). Now, we define the residual function over the interval $[x_0, x_f]$ as follows

$$
R = \int_{x_0}^{x_f} \left(R_1(x, a_0, a_1, \ldots, a_n)\right)^2 dx. \tag{3.3}
$$

Our aim is to solve the following optimization problem over the interval $[x_0, x_f]$ to find the entries of the vectors $a_r$, for $r = 0, 1, \ldots, n$:

$$
\min R.
$$

## 4. Approximation technique

We consider equation (1.1). Now by Eq. (3.1), one may have

$$D^{\alpha}\left(\sum_{r=0}^{n} a_r B_{r,n}(x)\right) = g(x) + \int_0^1 K(x,t)\left(\sum_{r=0}^{n} a_r B_{r,n}(t)\right) dt, \tag{4.1}$$

One may define

$$R_1(x, a_0, a_1, \ldots, a_n) := \sum_{r=0}^{n} a_r D^{\alpha} B_{r,n}(x) - g(x) - \int_0^1 K(x,t)\left(\sum_{r=0}^{n} a_r B_{r,n}(t)\right) dt,$$

(4.2)

now, we have

$$\begin{aligned} R &= \int_0^1 \left(R_1(x, a_0, a_1, \ldots, a_n)\right)^2 dx \\ &= \int_0^1 \left(\sum_{r=0}^{n} a_r D^{\alpha} B_{r,n}(x) - g(x) - \int_0^1 K(x,t)\Big(\sum_{r=0}^{n} a_r B_{r,n}(t) dt\Big)\right)^2 dx, \end{aligned}$$

(4.3)

where (see [8])

$$\begin{aligned} D^{\alpha} B_{r,n}(x) &= \frac{1}{\Gamma(n_1-\alpha)} \int_0^x \frac{B_{r,n}^{(n_1)}(t)}{(x-t)^{\alpha+1-n_1}} dt, \ n_1 \le \alpha < n_1, \\ B_{r,n}(t) &= \binom{n}{r} t^r (1-t)^{n-r} = \sum_{j=0}^{n-r} (-1)^j \binom{n}{r}\binom{n-r}{j} t^{j+r}, \\ D^{\alpha} B_{r,n}(t) &= \sum_{j=\lceil\alpha\rceil}^{n-r} \Omega_{(r,j,n)} \frac{\Gamma(r+j+1)}{\Gamma(r+j-\alpha+1)} t^{r+j-\alpha}, \\ \Omega_{(r,j,n)} &= (-1)^j \binom{n}{r}\binom{n-r}{j}, \end{aligned}$$

because of

$$\begin{aligned} B_{r,n}(x) &= \binom{n}{r} x^r (1-x)^{n-r} \\ &= \binom{n}{r} x^r \sum_{j=0}^{n-r} (-1)^j \binom{n-r}{j} x^j \\ &= \sum_{j=0}^{n-r} (-1)^j \binom{n}{r} \binom{n-r}{j} x^{j+r} \end{aligned} \tag{4.4}$$

By Eq. (4.3) and multiplier lagrange method, to find $a_r$, $r = 0, 1, \ldots, n$, we have $R$ is equivalent as follows:

$$\frac{\partial R}{\partial a_r} = 0, \;\; r = 0, 1, \ldots, n, \tag{4.5}$$

by Eqs. (4.3) and (4.5), we can obtain

$$\begin{aligned} &\int_0^1 \left( \sum_{r=0}^{n} a_r D^\alpha B_{r,n}(x) - g(x) - \int_0^1 K(x,t) \sum_{r=0}^{n} a_r B_{r,n}(t) dt \right) \\ &\times \left( D^\alpha B_{r,n}(x) - \int_0^1 K(x,t) B_{r,n}(t) dt \right) \Bigg) dx = 0, \end{aligned} \tag{4.6}$$

by Eq. (4.6), we can obtain a system of $n+1$ linear equations with $n+1$ unknown coefficients $a_r$.

Now, the following problem is considered:

$$\begin{aligned} \mathrm{L(f(x), D^\alpha f(x))} &= D^\alpha f(x) - g(x) - \int_0^1 K(x,t) f(t) dt = F(x), \;\; 0 \le x, t \le 1, \\ f(0) &= a, \end{aligned} \tag{4.7}$$

where $a$ is given real number.

**Lemma 4.1.** *For a polynomial in Bezier form*

$$u(x)=\sum_{i=0}^{n_2} a_{i,n_2} B_{i,n_2}(x),$$

*where $a_{i,n_2+m_1}$ is the Bezier coefficient of $u(x)$ after being degree-elevated to degree $n_2+m_1$.*

*we have*

$$\frac{\sum_{i=0}^{n_2} a_{i,n_2}^2}{n_2+1} \geq \frac{\sum_{i=0}^{n_2+1} a_{i,n_2+1}^2}{n_2+2} \geq \ldots \geq \frac{\sum_{i=0}^{n_2+m_1} a_{i,n_2+m_1}^2}{n_2+m_1+1}.$$

**Proof.** See [11].

**Theorem 4.2.** *If the problem (4.7) has a unique $C^{n_1}[0,1]$ continuous solution $\bar{f}$, then the approximate solution obtained by the control-point-based method converges to the exact solution $\bar{f}$ as the degree of the approximate solution tends to infinity.*

**Proof.** Given an arbitrary small positive number $\epsilon>0$, by the Weierstrass Theorem, one can find polynomials $Q_{1,N_1}(x)$ of degree $N_1$ such that (see [7])

$$\|Q_{1,N_1}(x)-\bar{f}(x)\|_\infty \leq \frac{\epsilon}{16\|K(x,t)\|_\infty},$$
$$\|D^\alpha Q_{1,N_1}(x)-D^\alpha \bar{f}(x)\|_\infty \leq \frac{\epsilon}{16\|K(x,t)\|_\infty} \leq \frac{\epsilon}{16},\ n_1-1<\alpha\leq n_1,$$

where $\|.\|_\infty$ stands for the

$L_\infty$-norm over $[0,1]$, one may note $K(x,t)$ is continues function on $[0,1]$ therefore it is bounded. Now, we have

$$\|a-Q_{1,N_1}(0)\|_\infty \leq \frac{\epsilon}{16\|K(x,t)\|_\infty}. \tag{4.8}$$

In general, $Q_{1,N_1}(x)$ does not satisfy the boundary conditions. After a small perturbation with linear and constant polynomials $\beta$ for $Q_{1,N_1}(x)$, we can obtain polynomials $P_{1,N_1}(x) = Q_{1,N_1}(x) + \beta$ such that $P_{1,N_1}(x)$ satisfy the boundary conditions $P_{1,N_1}(0) = a$.

Thus $Q_{1,N_1}(0) + \beta = a$, by utilizing (4.8), one have

$$\|a - Q_{1,N_1}(0)\|_\infty = \|\beta\|_\infty \leq \frac{\epsilon}{16\|K(x,t)\|_\infty}.$$

Now, we have

$$\begin{aligned}
\|P_{1,N_1}(x) - \bar{f}(x)\|_\infty &= \|Q_{1,N_1}(x) + \beta - \bar{f}(x)\|_\infty \\
&\leq \|Q_{1,N_1}(x) - \bar{f}(x)\|_\infty + \|\beta\|_\infty \\
&\leq \frac{2\epsilon}{16\|K(x,t)\|_\infty}, \\
\|D^\alpha P_{1,N_1}(x) - D^\alpha \bar{f}(x)\|_\infty &= \|D^\alpha Q_{1,N_1}(x) - D^\alpha \bar{f}(x)\|_\infty < \frac{\epsilon}{16},
\end{aligned}$$

Now, let define

$$\begin{aligned}
LP_N(x) = L\Bigg(P_{1,N_1}(x), D^\alpha P_{1,N_1}(x)\Bigg) &= D^\alpha P_{1,N_1}(x) - g(x) \\
&- \int_0^1 K(x,t) P_{1,N_1}(t)dt = F(t)
\end{aligned}$$

for every $x \in [0,1]$. Thus for $N \geq N_1$, one may find an upper bound for the following residual:

$$\begin{aligned}
\|LP_N(x) - F(x)\|_\infty &= \|L\Bigg(P_{1,N_1}(x), D^\alpha P_{1,N_1}(x)\Bigg) - F(x)\|_\infty \\
&\leq \|D^\alpha P_{1,N_1}(x) - D^\alpha \bar{f}(x)\|_\infty \\
&+ \int_0^1 \|K(x,t)\|_\infty \|P_{1,N_1}(t) - \bar{f}(t)\|_\infty dt \\
&\leq \frac{\epsilon}{16} + \|K(x,t)\|_\infty \frac{\epsilon}{16\|K(x,t)\|_\infty} \leq \epsilon
\end{aligned}$$

Since the residual $R(P_N) := LP_N(x) - F(x)$ is a polynomial, we can represent it by a Bezier form. Thus we have

$$R(P_N) := \sum_{i=0}^{m} d_{i,m} B_{i,m}(x).$$

Then from Lemma 1 in [11], there exists an integer $M(\geq N)$ such that when $m > M$, we have

$$|\frac{1}{m+1}\sum_{i=0}^{m} d_{i,m}^2 - \int_0^1 (R(P_N))^2 \, dx| < \epsilon,$$

which gives

$$\frac{1}{m+1}\sum_{i=0}^{m} d_{i,m}^2 \quad < \quad \epsilon + \int_0^1 (R(P_N))^2 \, dt \leq \epsilon \tag{4.9}$$

Suppose $f(x)$ is approximated solution of (4.7) obtained by the control-point-based method of degree $k(k \geq m \geq M)$. Let

$$\begin{aligned} R(f(x), D^\alpha f(x)) &= L(f(x), D^\alpha f(x)) - F(x) \\ &= \sum_{i=0}^{k} c_{i,k} B_{i,k}(x), \quad k \geq m \geq M, \quad x \in [0,1]. \end{aligned}$$

Define the following norm for difference approximated solution $f(x)$ and exact solution $\bar{f}(x)$:

$$\|f(x) - \bar{f}(x)\| := |f(x) - \bar{f}(x)|. \tag{4.10}$$

It is easy to show that:

$$\begin{aligned} \|f(x) - \bar{f}(x)\| &\leq C(|f(0) - \bar{f}(0)| + \|R(f(x), D^\alpha f(x))\|_2^2) \\ &= C\int_0^1 \sum_{i=0}^{k} (c_{i,k} B_{i,k}(t))^2 \, dx \leq \frac{C}{k+1}\sum_{i=0}^{k} c_{i,k}^2 \end{aligned} \tag{4.11}$$

Last inequality in (4.11) is obtained from Lemma 1 in [11] in which $C$ is a constant positive number. Now from Lemma 4.1 and (4.9), one can easily show that:

$$
\begin{aligned}
\|f(x)) - \bar{f}(x)\| &\leq \frac{C}{k+1}\sum_{i=0}^{k} c_{i,k}^2 \\
&\leq \frac{C}{k+1}\sum_{i=0}^{k} d_{i,k}^2 \leq \dots \leq \frac{C}{m+1}\sum_{i=0}^{m} d_{i,m}^2 \\
&\leq C(\epsilon) = \epsilon_1, \ m \geq M, \qquad (4.12)
\end{aligned}
$$

where last inequality in (4.12) is coming from (4.9).
This completes the proof.

## 5. Numerical examples

Now, some numerical examples of LFIDEs are stated to illustrate the Bezier curve method. All results are obtained by utilizing Maple 14.

**Example 1.** *The following LFIDEs is considered (see [9])*

$$
\begin{aligned}
&D^{1/2}f(x) = \frac{(8/3)x^{3/2} - 2x^{1/2}}{\sqrt{\pi}} + \frac{x}{12} + \int_0^1 xtf(t)\ dt, \ 0 \leq x, t \leq 1, \\
&f(0) = 0,
\end{aligned}
$$

*where the exact solution is $f(x) = x^2 - x$. Applying the proposed technique with $n = 3, 4, 5$, one may have*

$$
\begin{aligned}
&f_{approx} = -x + x^2 - 6.6 \times 10^{-16}x^3, \quad \text{for } n = 3, \\
&f_{approx} = -x + x^2 - 4.44 \times 10^{-14}x^3 + 1.82 \times 10^{-14}x^4, \quad \text{for } n = 4, \\
&f_{approx} = x^2 - x, \quad \text{for } n = 5.
\end{aligned}
$$

Table 1: The absolute error of the this method for Example 1

| $x$ | error for $n = 4$ | error for $n = 3$ |
|---|---|---|
| 0.1 | $5.925164873 \times 10^{-16}$ | $1.734723476 \times 10^{-17}$ |
| 0.2 | $7.112366252 \times 10^{-16}$ | $5.551115123 \times 10^{-17}$ |
| 0.3 | $5.724587471 \times 10^{-16}$ | $8.326672685 \times 10^{-17}$ |
| 0.4 | $3.053113318 \times 10^{-16}$ | $1.110223025 \times 10^{-16}$ |
| 0.5 | $8.326672685 \times 10^{-17}$ | $1.387778781 \times 10^{-16}$ |
| 0.6 | $6.938893904 \times 10^{-17}$ | $1.665334537 \times 10^{-16}$ |
| 0.7 | $9.714451465 \times 10^{-17}$ | $1.665334537 \times 10^{-16}$ |
| 0.8 | $6.938893904 \times 10^{-17}$ | $1.110223025 \times 10^{-16}$ |
| 0.9 | $2.775557562 \times 10^{-17}$ | $6.938893904 \times 10^{-17}$ |
| 1.0 | 0.0 | $2.225073859 \times 10^{-308}$ |

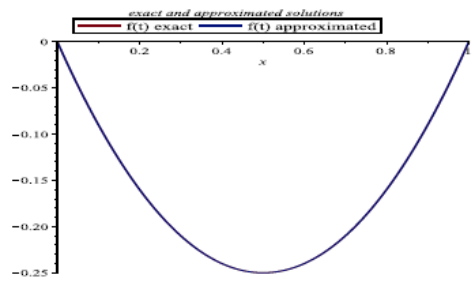


Figure 1: The graphs of approximated and exact solution $f(x)$ $(n = 5)$ for Example 1

**Example 2.** *The following LFIDEs is considered (see [9])*

$$
\begin{aligned}
&D^{5/6} f(x) = g(x) + \int_0^1 x e^t f(t)\, dt, \quad 0 \le x, t \le 1,\\
&g(x) = \frac{-3}{91} \times \frac{x^{1/6}\Gamma(5/6)(-91 + 216x^2)}{\pi} + (5 - 2e)x,\\
&f(0) = 0,
\end{aligned}
$$

*where the exact solution is* $f(x) = -(x^3 - x)$. *Applying the proposed technique with* $n = 4, 5$, *we obtain*

$$
\begin{aligned}
&f_{approx} = x + 7.12 \times 10^{-15} x^2 - x^3 + 3.56 \times 10^{-15} x^4, \quad \text{for } n = 4,\\
&f_{approx} = x - x^3, \quad \text{for } n = 5.
\end{aligned}
$$

Table 2: The absolute error of the this method for Example 2

| $x$ | error for $n = 4$ | error for $n = 3$ |
|---|---|---|
| 0.1 | $1.105886216 \times 10^{-16}$ | $4.510281038 \times 10^{-17}$ |
| 0.2 | $1.301042607 \times 10^{-16}$ | $8.326672685 \times 10^{-17}$ |
| 0.3 | $1.110223025 \times 10^{-16}$ | $1.110223025 \times 10^{-16}$ |
| 0.4 | $6.938893904 \times 10^{-17}$ | $1.110223025 \times 10^{-16}$ |
| 0.5 | 0.0 | $5.551115123 \times 10^{-17}$ |
| 0.6 | $2.775557562 \times 10^{-17}$ | $5.551115123 \times 10^{-17}$ |
| 0.7 | $8.326672685 \times 10^{-17}$ | $5.551115123 \times 10^{-17}$ |
| 0.8 | $8.326672685 \times 10^{-17}$ | $1.665334537 \times 10^{-16}$ |
| 0.9 | $2.775557562 \times 10^{-17}$ | $1.387778781 \times 10^{-16}$ |
| 1.0 | 0.0 | $2.220446049 \times 10^{-16}$ |

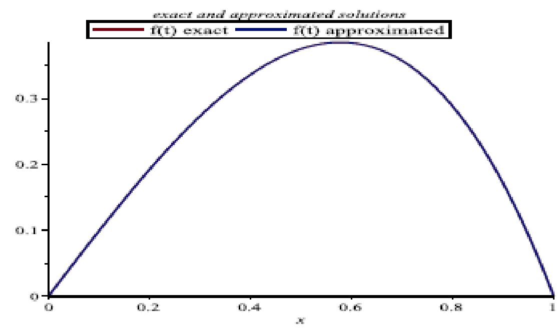


Figure 2: The graphs of approximated and exact solution $f(x)$ $(n = 5)$ for Example 2

**Example 3.** *The following LFIDEs is considered (see [9])*

$$D^{5/3}f(x) = \frac{2\sqrt{3}\Gamma(2/3)x^{1/3}}{\sqrt{\pi}} - \frac{1}{5}x^2 - \frac{1}{4}x + \int_0^1 (xt + x^2t^2)f(t)\ dt,\ 0 \le x, t \le 1,$$
$$f(0) = f'(0) = 0,$$

*where the exact solution is* $f(x) = x^2$*. Applying the proposed technique with* $n = 4, 5$*, we have*

$$\begin{aligned}
f_{approx} &= -6.661338148 \times 10^{-16} + x^2 - 7.861338148 \times 10^{-16} x^3, \quad \textit{for } n = 3,\\
f_{approx} &= 3.108624469 \times 10^{-15} + x^2 + 2.310587341 \times 10^{-14} x^3\\
&- 1.066862447 \times 10^{-14} x^4, \quad \textit{for } n = 4,\\
f_{approx} &= -4.440892098 \times 10^{-15} + 1.000000000 \times x^2 - 2.664535259 \times 10^{-14} x^3\\
&+ 1.776356839 \times 10^{-14} x^4 - 4.440892098 \times 10^{-15} x^5, \quad \textit{for } n = 5,
\end{aligned}$$

The graphs of approximated and exact solution $f(x)$ are plotted in Fig. 3 (with $n = 5$).

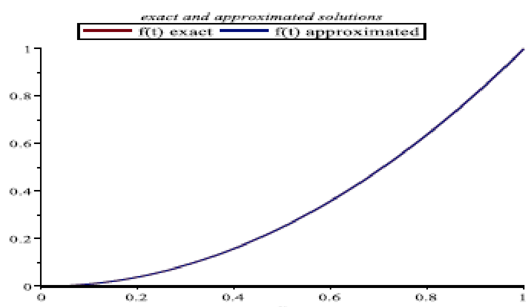


Figure 3: The graphs of approximated and exact solution $f(x)$ with $n = 5$ for Example 3

Table 3: The absolute error of the this method for Example 3

| $x$ | error for $n = 4$ | error for $n = 3$ |
|---|---|---|
| 0.1 | $1.775516583 \times 10^{-16}$ | $5.486062993 \times 10^{-17}$ |
| 0.2 | $1.693523793 \times 10^{-16}$ | $8.500145032 \times 10^{-17}$ |
| 0.3 | $7.285838599 \times 10^{-17}$ | $9.714451465 \times 10^{-17}$ |
| 0.4 | $3.816391647 \times 10^{-17}$ | $9.714451465 \times 10^{-17}$ |
| 0.5 | $1.110223025 \times 10^{-16}$ | $1.249000903 \times 10^{-16}$ |
| 0.6 | $1.110223025 \times 10^{-16}$ | $8.326672685 \times 10^{-17}$ |
| 0.7 | $2.775557562 \times 10^{-17}$ | $1.110223025 \times 10^{-16}$ |
| 0.8 | 0.0 | $1.110223025 \times 10^{-16}$ |
| 0.9 | $1.110223025 \times 10^{-16}$ | $2.220446049 \times 10^{-16}$ |
| 1.0 | 0.0 | $1.110223025 \times 10^{-16}$ |

**Remark 5.1.** *Any arbitrary fractional order can be derivative of fractional calculus, therefore it may be a function of time* **[12]** *such as $\alpha(t)$ where this problem exists in various physical problems [13]. In the recent years, variable-order fractional derivatives were stated by different physical models. The advantages of utilizing these derivatives were studied in [13] when it is a new study in the fractional calculus [13].*
*In [14], authors stated a technique based on second kind Chebyshev wavelets (SKCWs) for solving a class of nonlinear Fredholm integro-differential equations of fractional order. Recently, In [15], authors have stated an efficient SKCWs technique to solve the nonlinear diffusion equation for the steady-state condition. In [16], a numerical method based on SKCWs was stated to solve time fractional fifth-order Sawada-Kotera equation.*
*In the current work, the following MVOFDEs is considered*

$$\begin{aligned}
&\min\ I = \int_0^1 F(t, x(t), u(t))\, dt, \\
&\text{such that} \\
&D^{\alpha(t)} x(t) = H(t, x(t), u(t)), \quad p-1 < \alpha(t) \le p,\ p \in \mathbf{N},\ t \in [0,1], \\
&x(0) = x_0,\ x'(0) = x'_0, \ldots, x^{p-1}(0) = x_0^{p-1},
\end{aligned} \tag{5.1}$$

*where $x_0,\ x'_0, \ldots, x_0^{p-1}$ are given constants, $F$ and $H$ are continuous functions, also*

$$D^{\alpha(t)} x(t) = \begin{cases} \frac{1}{\Gamma(p-\alpha(t))} \int_0^1 (t-s)^{p-\alpha(t)-1} \frac{d^p x(s)}{ds^p} ds & \text{if } p-1 < \alpha(t) \le p \\ \frac{d^p x(t)}{dt^p} & \text{if } \alpha(t) = p \end{cases}$$

*Our strategy is to utilize BCM to approximate the solutions $x(t)$ and $u(t)$. Define the Bezier polynomials of degree $n$ over the interval $[t_0, t_f]$ as follows:*

$$\begin{aligned}
x(t) &= \sum_{r=0}^{n} a_r B_{r,n}\Big(\frac{t-t_0}{h}\Big), \\
u(t) &= \sum_{r=0}^{n} b_r B_{r,n}\Big(\frac{t-t_0}{h}\Big),\ t_f = 1,\ t_0 = 0,
\end{aligned} \tag{5.2}$$

*where* $h = t_f - t_0$, *and*

$$B_{r,n}\left(\frac{t-t_0}{h}\right) = \binom{n}{r}\frac{1}{h^n}(t_f - t)^{n-r}(t-t_0)^r, \tag{5.3}$$

*is the Bernstein polynomial of degree* $n$ *over the interval* $[t_0, t_f]$, $a_r$ *and* $b_r$ *are the control points. Suppose that* $M_B$ *is the coefficient matrix of* $B_{r,n}(t)$, $r = 0, 1, \ldots, n$, *where* $M_B(i, j)$ *is the coefficient of the* $B_{i,n}(t)$ *with respect to the monomial* $t^{j-1}$, *then by Eq. (5.3) we have*

$$M_B(i,j) = (-1)^{i+j}\binom{n}{i}\binom{n-i}{j-i}, i = 0, 1, \ldots, n, j = i, \ldots, n,$$

*also*

$$\begin{aligned} B_{r,n}(t) &= \binom{n}{r} t^i (1 - t^{n-i}) \\ &= \binom{n}{r} t^i \left(\sum_{k=0}^{n-r}(-1)^k \binom{n-r}{k} t^k\right) \\ &= \sum_{k=0}^{n-r}(-1)^k \binom{n}{r}\binom{n-r}{k} t^{r+k}, \;\; r = 0, 1, \ldots, n, \end{aligned}$$

*hence*

$$A_{r+1} = \Big\{\; 0, 0, \ldots, 0, (-1)^0\tbinom{n}{r}, (-1)^1\tbinom{n}{r}\tbinom{n-r}{1}, \ldots, (-1)^{n-r}\tbinom{n}{r}\tbinom{n-r}{n-r}$$

*where* $B_{r,n}(t) = A_{r+1}T_n(t)$, $r = 0, 1, \ldots, n$, *and*

$$T_n(t) = \begin{bmatrix} 1 \\ t \\ \vdots \\ t^n \end{bmatrix}_{(n+1)\times 1}$$

*now*

$$A = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_{n+1} \end{bmatrix}_{(n+1)\times(n+1)}$$

$$\phi(t) = AT_n(t)$$

$$\phi(t) = \begin{bmatrix} B_{0,n}(t) \\ B_{1,n}(t) \\ \vdots \\ B_{n,n}(t) \end{bmatrix},$$

*therefore* $\frac{d}{dt}\phi(t) = D\phi(t)$ *for* $0 \leq t \leq 1$, *and*

$$\frac{d}{dt}\phi(t) = A \begin{bmatrix} 0 \\ 1 \\ 2t \\ \vdots \\ nt^{n-1} \end{bmatrix}$$

$$= A \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & n \end{bmatrix} \begin{bmatrix} 1 \\ t \\ t^2 \\ \vdots \\ t^{n-1} \end{bmatrix}$$

$$= A\Lambda' T'$$

where

$$\Lambda' = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & \dots & n \end{bmatrix},$$

$$T' = \begin{bmatrix} 1 \\ t \\ t^2 \\ \vdots \\ t^{n-1} \end{bmatrix},$$

$$B' = \begin{bmatrix} A_1^{-1} \\ A_2^{-1} \\ A_2^{-1} \\ \vdots \\ A_n^{-1} \end{bmatrix},$$

therefore

$$\frac{d}{dt}\phi(t) \quad = \quad A\Lambda' B'\phi(t),$$

hence $D = A\Lambda' B'$ which is operational matrix of differentiation for Bezier curve.

By substituting $x(t)$ and $u(t)$ in system (5.1), one may define $R_1(t, a_0, a_1, \ldots, a_n, b_0, b_1, \ldots, b_n)$ for $t \in [t_0, t_f]$ as follows:

$$\begin{aligned}
&\min \ R_1(t, a_0, a_1, \ldots, a_n, b_0, \ldots, b_n) = \int_0^1 F(t, \sum_{r=0}^{n} a_r B_{r,n}(\frac{t-t_0}{h}), \sum_{r=0}^{n} b_r B_{r,n}(\frac{t-t_0}{h}))\ dt, \\
&\text{such that } D^{\alpha(t)} \sum_{r=0}^{n} a_r B_{r,n}(\frac{t-t_0}{h}) = H(t, \sum_{r=0}^{n} a_r B_{r,n}(\frac{t-t_0}{h}), \sum_{r=0}^{n} b_r B_{r,n}(\frac{t-t_0}{h})), \\
&x(0) = x_0,\ x'(0) = x'_0, \ldots, x^{p-1}(0) = x_0^{p-1},
\end{aligned}$$

(5.4)

using NLPSolve in Maple Software and stated operational matrix, one may solve defined system new2.

## 6. Numerical applications for MVOFDEs

In this section, we consider some numerical examples to illustrate the efficiency and reliability of the proposed method for MVOFDEs. Also, we report the absolute errors of the proposed method for the under consideration examples.

**Example 4.** *First, the following MVOFDEs is considered (see [17])*

$$
\begin{aligned}
&D^{\alpha(t)}u(t)+\sin(t)D^{\beta(t)}u(t)+\cos(t)=\frac{6t^{3-\alpha(t)}}{\Gamma(4-\alpha(t))}+\frac{6\sin(t)t^{3-\beta(t)}}{\Gamma(4-\beta(t))}+t^3\cos(t),\\
&u(0)=u'(0)=0,\\
&1<\alpha(t)\leq 2,\\
&0<\beta(t)\leq 1,\\
&u_{exact}(t)=t^3,
\end{aligned}
$$

*We achieve $u_{approx}(t)=t^3$ with this technique by $n=3$, so the error is zero.*

**Example 5.** *Second, the following MVOFDEs is considered*

$$
\begin{aligned}
&\min\int_0^1 (x-t^2)^2+(u-te^{-t}+\frac{1}{2}e^{t^2-t})^2\ dt\\
&\text{such that } D^{\alpha}x=e^x+2e^t u,\\
&\text{for } \alpha=1 \quad \text{then} \quad x_{exact}=t^2,\ u_{exact}=te^{-t}-\frac{1}{2}e^{t^2-t},\\
&0<\alpha\leq 1.
\end{aligned}
$$

*We obtain*

$$
\begin{aligned}
u_{approx}(t) &= -0.5(1-t)^3-0.0822318325083609t(1-t)^2\\
&+\ 0.0252718969369193t^2(1-t)-0.132120558828558t^3,\ \text{for } n=3,\\
x_{approx}(t) &= -2.355566179\times10^{-9}t+1.000000007t^2-4.711136179\times10^{-9}t^3
\end{aligned}
$$

*Table 4 demonstrates the absolute error of this technique.*

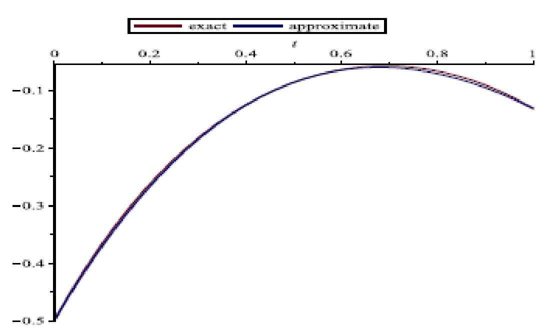


Figure 4: The approximate and exact solution of $x(t)$ for Example 5

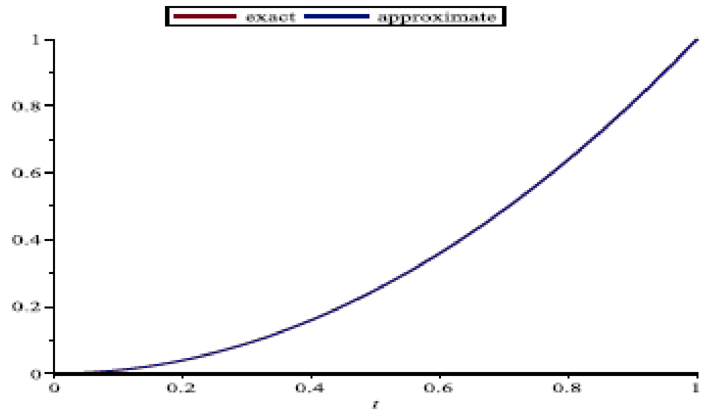


Figure 5: The approximate and exact solution of $u(t)$ for Example 5

Table 4: The absolute error of the this method for Example 5

| $t$ | error of $x$ | error of $u$ |
|---|---|---|
| 0.1 | $1.696006452 \times 10^{-10}$ | 0.004583601100 |
| 0.2 | $2.261341739 \times 10^{-10}$ | 0.004448194400 |
| 0.3 | $1.978673857 \times 10^{-10}$ | 0.002516548200 |
| 0.4 | $1.130670557 \times 10^{-10}$ | 0.0006850853 |
| 0.5 | $5.551115123 \times 10^{-17}$ | 0.0 |
| 0.6 | $1.130671390 \times 10^{-10}$ | 0.0007661945700 |
| 0.7 | $1.978675557 \times 10^{-10}$ | 0.002600577980 |
| 0.8 | $2.261343335 \times 10^{-10}$ | 0.004433618750 |
| 0.9 | $1.696006668 \times 10^{-10}$ | 0.004456051330 |
| 1.0 | 0.0 | 0.0 |

## Conclusions

This paper deals with the approximate solution of LFIDEs and a class of MVOFDEs via BCM and stated operational matrix. Illustrative examples are included to demonstrate the validity and applicability of the technique.

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