Strong Riesz summability of Fourier series

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Abstract:

The notion of strong summability was introduced by Fekete (Math. És Termesz Ertesitö, 34 (1916), 759-786). Dealing with Nörlund summability of Fourier series Mittal (J. Math. Anal. Appl. 314 (2006), 75-84) has established a result on strong summability. We have established a new result on sufficient condition for strong Riesz summability of Fourier series.

Keywords: Strong summability; Cesàro summability; $[\bar{N}, p_n^{(1)}, 2]$ -summability; Fourier series.

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1. Introduction

Let $\sum u_n$ be an infinite series with a sequence of partial sum (s_n) . The sequence-to-sequence transformation,

$$C_n^\alpha = \frac{1}{E_n^{\alpha-1}} \sum_{r=0}^n s_{n-r} E_r^{\alpha-2},$$

where

$$E_n^{\alpha-1} = \frac{\Gamma(n+1)}{\Gamma(n+1)\Gamma(\alpha)}, \alpha > 0,$$

defines the Cesàro mean of order α of (s_n) . The series $\sum u_n$ is said to be summable (C, α) to the sum s if,

$$\lim_{n \rightarrow \infty} C_n^\alpha = s$$

and is said to be strongly summable (C, α) with index q or $[C, \alpha, q], \alpha > 0, q > 0$, to the sum s (see [4]), if

$$\sum_{r=0}^n |C_r^{\alpha-1} - s|^q = o(n) \quad (n \rightarrow \infty).$$

Let (p_n) be a sequence of non-negative numbers such that

$$P_n = \sum_{r=0}^n p_r, (P_{-1} = p_{-1} = 0), p_0 = 0.$$

The sequence to sequence transformation

$$(1.1) \quad t_n = \frac{1}{P_n} \sum_{r=0}^n s_r p_{n-r}, n = 0, 1, 2, \dots,$$

defines the sequence (t_n) of the (N, p_n) mean of the sequence (s_n) generated by the sequence of coefficients (p_n) . The series $\sum u_n$ is said to be strongly summable (N, p_n) with index q or $[N, p_n, q], q > 0$, to the sum s (see [14]), if

$$(1.2) \quad \frac{1}{|P_n|} \sum_{r=0}^n |p_r| |t_r' - s|^q = o(1),$$

where

$$\begin{aligned}
t'_r &= \frac{1}{P_n} \sum_{r=0}^n s_r \nabla p_{n-r}, (\nabla p_{n-r} \equiv p_{n-r} - p_{n-r-1}) \\
&= \frac{1}{p_n} \sum_{r=0}^n u_r p_{n-r}.
\end{aligned}$$

Here, just by taking

$$p_n^\alpha = \sum_{k=0}^n E_{n-k}^{\alpha-1} p_k$$

in place of p_n ; it can further be extended to $[N, p_n^\alpha, q], \alpha \geq 1, q \geq 1$, summability. It is noticed that putting $p_n = E_n^{\alpha-1}$ in (1.1), (t_n) reduces to the Cesàro mean of order α of (s_n) .

Similarly, the sequence to sequence transformation,

$$(1.3) \quad T_n = \frac{1}{P_n} \sum_{r=0}^n s_r p_r$$

defines $(\bar{N}, p_n)$ mean of the sequence (s_n) generated by the sequence of coefficients (p_n) . The series $\sum u_n$ is said to be strongly summable $(\bar{N}, p_n)$ with index q or $[\bar{N}, p_n, q], q > 0$, to the sum s , if

$$(1.4) \quad \frac{1}{|P_n|} \sum_{r=0}^n |p_r| |T'_r - s|^q = o(1),$$

where

$$\begin{aligned}
T'_r &= \frac{1}{P_n} \sum_{r=0}^n s_r \nabla p_r, (\nabla p_r \equiv p_r - p_{r-1}) \\
&= \frac{1}{p_n} \sum_{r=0}^n u_r p_r.
\end{aligned}$$

Similar to $[N, p_n^\alpha, q] (\alpha \geq 1, q \geq 1)$ -summability, $[\bar{N}, p_n^\alpha, q] (\alpha \geq 1, q \geq 1)$ -summability can also be defined.

Let $f(t)$ be a 2π - periodic function such that $f \in L[-\pi, \pi]$. Let $s[f]$ denote the Fourier series of $f(t)$, given by

$$(1.5) \quad s[f] = (a_0/2) + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=1}^{\infty} u_n(t)$$

with a sequence of its partial sums $(s_n(t))$, where the constant term $(a_0/2)$ is assumed to be zero. Moreover, for local property of factored Fourier

series one may refer to the recent work [7].

We use the following notations:

$$\begin{aligned}\phi(t) &= \phi_x(t) = (1/2)[f(x+t) + f(x-t) - 2s]; \quad v_r = rp_r/p_r^{(1)} = rp_r/P_r; \\ \Phi(t) &= \int_0^t |\phi(u)| du; \quad \bar{N}_n(t) = (\pi P_n)^{-1} \sum_{r=0}^n p_r \sin(n+1/2)t / \sin(t/2); \\ \alpha(t) &= \sum_{r=0}^n p_r \cos(rt); \quad \beta(t) = \sum_{r=0}^n p_r \sin(rt); \\ I_1 &= \sum_{m=0}^r \frac{p_m}{P_r} \left(\frac{1}{\pi} \int_0^{1/r} \phi(t) \frac{\sin(m+1/2)t}{\sin t/2} \right) dt; \\ I_2 &= \sum_{m=0}^r \frac{p_m}{P_r} \left(\frac{1}{\pi} \int_{1/r}^\delta \phi(t) \frac{\sin(m+1/2)t}{\sin t/2} \right) dt; \\ I_3 &= \sum_{m=0}^r \frac{p_m}{P_r} \left(\frac{1}{\pi} \int_\delta^\pi \phi(t) \frac{\sin(m+1/2)t}{\sin t/2} \right) dt; \\ l_0 &= x+1; \quad l_1 = \log(x+1); \quad l_2(x) = \log \log(x+1), \dots \text{and so on,} \\ &\text{for } x > 0;\end{aligned}$$

and $\tau = [1/t]$, the largest integer less than or equal to $1/t$.

2. Preliminaries

The idea of strong summability was introduced by Fekete [2], based on a result of Hardy and Littlewood [3] concerning the summability of Fourier series. A few researchers have used strong summability method in the course of their research works. Moreover, Matrix summability or matrix transformation is very important in the study of summability theory in the sense that it generalizes the different summability methods like Cesàro summability, Nörlund summability, Riesz summability etc. Also, the statistical convergence and statistical summability are more general than the ordinary convergence and ordinary summability. For recent works in this direction, see [1], [5], [6], [8], [9], [10], [11], [15], [16], [17], [18], [19] and [20]. Dealing with strong summability of Nörlund mean Mittal [13] has proved a theorem as follows.

Theorem 1. (see [13]) If

$$(2.1) \quad \Phi(t) = o \left(t / \sqrt{\prod_{m=1}^{k+1} l_m(1/t)} \right), \text{ as } t \rightarrow 0,$$

then the series $\sum_{n=0}^{\infty} u_n(t)$ is $[N, p_n^{(1)}, 2]$ -summable to the sum s (same sum), for $t = x$, provided

$$(2.2) \quad p_n = \left[\prod_{m=0}^k l_m(n) \right]^{-1}, \quad k = 0, 1, 2, \dots,$$

holds true.

3. Main Theorem

In an attempt to have further studies in this direction we have established a criterion for the $[\bar{N}, p_n^{(1)}, 2]$ summability and established a new theorem on sufficient condition for strong Riesz summability of Fourier series, as follows.

Theorem 1. *If*

$$(3.1) \quad \Phi(t) = o\left(t / \sqrt{\prod_{m=1}^{k+1} l_m(1/t)}\right) \quad (t \rightarrow 0),$$

then the series $\sum_{n=0}^{\infty} u_n(t)$ is strongly $[\bar{N}, p_n^{(1)}, 2]$ -summable to the sum s (same sum), for $t = x$, provided

$$(3.2) \quad p_n = \left[\prod_{m=0}^k l_m(n) \right]^{-1}, \quad k = 0, 1, 2, \dots,$$

holds true.

4. Required Lemmas

We need to prove the following lemmas for the proof of our theorem.

Lemma 1. *If*

$$(4.1) \quad \Phi(t) = o(t), \quad t \rightarrow 0$$

then,

$$I_1 = o(1) \quad \text{and} \quad I_3 = o(1).$$

Proof. By using Riemann - Lebesgue lemma and the regularity condition of summation, $I_3 = o(1)$. Next, since $\bar{N}_r(t) = O(r)$ is uniform in $[0, 1/r]$, so by (4.1) we have

$$\begin{aligned} I_1 &= O \left[\int_0^{1/r} |\phi(t)| \bar{N}_r(t) dt \right] \\ &= O(r) \Phi(1/r) \\ &= O(r) o(1/r) = o(1) \quad (t \rightarrow 0). \end{aligned} \quad \square$$

Lemma 2. Let $\{p_n\}$ is a non-negative and non-increasing sequence. If condition (4.1) of Lemma 1 holds true, then

$$\begin{aligned} I_2 &= \frac{1}{\pi P_r} \int_{1/r}^{\delta} \phi(t) \alpha(t) dt \\ &\quad + \frac{1}{\pi P_r} \int_{1/r}^{\delta} \phi(t) \beta(t) \cot(t/2) dt + o(1). \end{aligned}$$

Proof. Since $p_r \geq p_{r+1}$, $v_n = O(1)$, so we have

$$\begin{aligned} I_2 &= \frac{1}{\pi P_r} \int_{1/r}^{\delta} \phi(t) \left[\sum_{m=0}^{\tau} + \sum_{m=\tau+1}^r \right] p_m \frac{\sin(m+1/2)t}{\sin(t/2)} dt \\ &= \frac{1}{\pi P_r} \int_{1/r}^{\delta} \frac{\phi(t)}{\sin(t/2)} [\alpha(t) \sin(t/2) + \beta(t) \cos(t/2)] dt \\ &\quad + \frac{1}{\pi P_r} \int_{1/r}^{\delta} \frac{\phi(t)}{\sin(t/2)} \left[\sum_{m=\tau+1}^r p_m \sin(m+1/2)t \right] dt \\ &= I_{2,1} + I_{2,2} \quad (\text{say}). \end{aligned}$$

Thus for proving the validity of (2) it suffices to show $I_{2,2} = o(1)$. We have,

$$\begin{aligned} I_{2,2} &= \frac{1}{\pi P_r} \int_{1/r}^{\delta} \frac{\phi(t)}{\sin(t/2)} \left[\sum_{m=\tau+1}^r p_m \sin(m+1/2)t \right] dt \\ &= O \left(\frac{1}{\pi P_r} \int_{1/r}^{\delta} \frac{\phi(t)}{t^2} p_{\tau} dt \right). \\ &\quad \left(\sum_{m=\tau+1}^{\infty} p_m \sin(m+1/2)t = O\{p_{\tau}/t\}, \text{ for } p_m \geq p_{m+1} \right). \end{aligned}$$

Now using (3.2) and (4.1) and using integration by parts, we have

$$\begin{aligned}
I_{2,2} &= O \left[\frac{1}{\pi P_r} \left(\left[\frac{\Phi(t)p(1/t)}{t^2} \right]_{1/r}^\delta - \int_{1/r}^\delta \Phi(t) \left(\frac{d}{dt} \frac{1}{t \prod_{m=1}^k l_m(1/t)} \right) dt \right) \right] \\
&= O \left[\frac{1}{\pi P_r} \left(\left\{ o \left(\frac{p(1/t)}{t} \right) \right\}_{1/r}^\delta + \int_{1/r}^\delta \frac{\Phi(t)}{t^2 \prod_{m=1}^k l_m(1/t)} dt \right) \right] \\
&\quad - O \left[\frac{1}{\pi P_r} \left(\int_{1/r}^\delta \frac{\Phi(t)}{t} \frac{d}{dt} \left(\frac{1}{\prod_{m=1}^k l_m(1/t)} \right) dt \right) \right].
\end{aligned}$$

Furthermore, since, k is fixed, so we can write $P_r = l_{k+1}(r) = l(r)$.

Clearly, we have

$$\begin{aligned}
I_{2,2} &= o(v_r) + o(1) + o \left(\frac{1}{\pi l(r)} \right) \\
&\quad \times \left(\int_{1/r}^\delta \frac{1}{t \prod_{m=1}^k l_m(1/t)} dt + \int_{1/r}^\delta \frac{d}{dt} \left\{ \frac{1}{\prod_{m=1}^k l_m(1/t)} \right\} dt \right) \\
&= o(1) + o \left(\frac{1}{\sqrt{l(r)}} \right) \int_{1/r}^\delta \left(\frac{1}{\left\{ t \prod_{m=1}^k l_m(1/t) \right\} \sqrt{l(1/t)}} \right) dt \\
&\quad + o \left\{ \frac{1}{l(r)} \left[\frac{1}{\prod_{m=1}^k l_m(1/t)} \right]_{1/r}^\delta \right\} \\
&= o(1) + o(\sqrt{l(r)}/\sqrt{l(r)}) + o(1/l(r)) \\
&= o(1) \quad (\text{using (4.1) and } p_n \geq p_{n+1})
\end{aligned}$$

Which completes the proof of Lemma 2. $\square$

5. Proof of Main Theorem

Following Zygmund [[21], p. 50], we have

$$\begin{aligned}
T_r(x) - s &= \sum_{m=0}^r \frac{p_m}{P_r} \{s_m(x) - s\} \\
&= \sum_{m=0}^r \frac{p_m}{P_r} \left(\frac{1}{\pi} \int_0^\pi \phi(t) \frac{\sin(m+1/2)t}{\sin t/2} dt \right) \\
&= \sum_{m=0}^r \frac{p_m}{P_r} \left\{ \int_0^{1/r} + \int_{1/r}^\delta + \int_\delta^\pi \right\} \phi(t) \frac{\sin(m+1/2)t}{\sin t/2} dt \\
&= I_1 + I_2 + I_3 \text{ (say).}
\end{aligned}$$

In order to establish the theorem, it is enough to prove,

$$(5.1) \quad \sum_{r=0}^n |T_r(x) - s|^2 = o(n) \quad (n \rightarrow \infty).$$

By Lemma 1 and Lemma 2, we have

$$\begin{aligned}
|T_r(x) - s| &= \frac{1}{\pi P_r} \int_{1/r}^{\delta} \phi(t) \alpha(t) dt + \frac{1}{\pi P_r} \int_{1/r}^{\delta} \phi(t) \beta(t) \cot(t/2) dt + o(1) \\
&\leq \frac{1}{\pi P_r} \int_{1/n}^{\delta} \phi(t) \alpha(t) dt \\
&\quad + \frac{1}{\pi P_r} \int_{1/n}^{\delta} \phi(t) \beta(t) \cot(t/2) dt + o(1) (r \leq n).
\end{aligned}$$

Since, $\alpha(t) = O(P_r) = \beta(t)$, using monotonicity of $[l(1/t)]^{-1}$, we have

$$\begin{aligned}
\left| \frac{1}{\pi P_r} \int_{1/n}^{1/r} \phi(t) \alpha(t) dt \right| &= O \left(\frac{1}{\pi P_r} \int_{1/n}^{1/r} |\phi(t)| P(1/t) dt \right) \\
&= O \left(\frac{1}{\pi l(r)} \right) \int_{1/n}^{1/r} |\phi(t)| l(1/t) dt \\
&= O \left(\frac{1}{\pi l(r)} \right) \left[(\Phi(t) l(1/t))_{1/n}^{1/r} - \int_{1/n}^{1/r} \frac{\Phi(t) dt}{t \prod_{m=1}^k l_m(1/t)} \right] \\
&= O \left(\frac{1}{l(r)} \right) \left[o \left(\frac{tl(1/t)}{\sqrt{\prod_{m=1}^{k+1} l_m(1/t)}} \right)_{1/n}^{1/r} \right] \\
&\quad + O \left(\frac{1}{l(r)} \right) \left[\int_{1/n}^{1/r} o \left(\frac{t}{\sqrt{\prod_{m=1}^{k+1} l_m(1/t)}} \right) \frac{dt}{t \prod_{m=1}^k l_m(1/t)} \right] \\
&= o \left(\frac{1}{r} \right) + o \left(\frac{1}{l(r) \left\{ \prod_{m=1}^k l_m(r) \right\}^{3/2}} \right) \int_{1/n}^{1/r} dt = o(1).
\end{aligned}$$

Similarly,

$$\left| \frac{1}{\pi P_r} \int_{1/r}^{1/n} \phi(t) \beta(t) \cot(t/2) dt \right| = o(1).$$

Now, using Minkowskis' inequality, we get

$$\begin{aligned}
&\left\{ \sum_{r=0}^n |T_r(x) - s|^2 \right\}^{\frac{1}{2}} \\
&\leq \left[\sum_{r=0}^n \left\{ \frac{1}{\pi P_r} \left(\int_{1/r}^{\delta} \phi(t) \alpha(t) dt + \frac{1}{\pi P_r} \int_{1/r}^{\delta} \phi(u) \beta(u) \cot(u/2) dt \right) \right\} \right]^{1/2} \\
&\quad \times \left[\sum_{r=0}^n \left\{ \frac{1}{\pi P_r} \left(\int_{1/r}^{\delta} \phi(u) \alpha(u) du + \frac{1}{\pi P_r} \int_{1/r}^{\delta} \phi(u) \beta(u) \cot(u/2) du \right) \right\} \right]^{1/2} \\
&\quad + o(n)^{1/2} \\
&= [j_1 + j_2 + j_3 + j_4]^{1/2} + o(n)^{1/2}.
\end{aligned}$$

Let

$$j_1 = \frac{1}{\pi^2} \int_{1/n}^{\delta} \int_{1/n}^{\delta} \phi(t) \alpha(t) \phi(u) \alpha(u) \left\{ \sum_{r=0}^n \frac{1}{P_r^2} \right\} dt du.$$

As we have,

$$\begin{aligned} \int_{1/n}^{\delta} |\phi(t)| l(1/t) dt &= [\Phi(t) l(1/t)]_{1/n}^{\delta} + \int_{1/n}^{\delta} \Phi(t) \frac{1}{t \prod_{m=1}^k l_m(1/t)} dt \\ &= o \left(\frac{t l(1/t)}{\sqrt{\prod_{m=1}^{k+1} l_m(1/t)}} \right)_{1/n}^{\delta} + \int_{1/n}^{\delta} \left(\frac{1}{\sqrt{\prod_{m=1}^{k+1} l_m(1/t)}} \right) \left(\frac{1}{\prod_{m=1}^k l_m(1/t)} \right) dt \\ &= o \left(\frac{t l(1/t)}{\sqrt{\prod_{m=1}^{k+1} l_m(1/t)}} \right)_{1/n}^{\delta} + \left(\frac{1}{\sqrt{\prod_{m=1}^{k+1} l_m(1/t)} \prod_{m=1}^k l_m(1/t)} \right) \int_{1/n}^{\delta} dt \\ &= o(1). \end{aligned}$$

So, for $\alpha(t) = O(P_{\tau})$, and by using (3.1), we get

$$j_1 \leq \sum_{r=0}^n \frac{1}{P_r^2} \int_{1/n}^{\delta} |\phi(t)| l(1/t) dt \int_{1/n}^{\delta} |\phi(u)| l(1/u) du = o(1).$$

Next, as

$$\begin{aligned} \int_{1/n}^{\delta} |\phi(u)| \alpha(u) |\cot(u/2)| du &= \int_{1/n}^{\delta} \frac{|\phi(u)| \alpha(u)}{u} du \\ &= \left[\Phi(u) \frac{l(u)}{u} \right]_{1/n}^{\delta} - \int_{1/n}^{\delta} \left[\Phi(u) \frac{d}{du} \left(\frac{l(1/u)}{u} \right) \right] du \\ &= \left[\Phi(u) \frac{l(u)}{u} \right]_{1/n}^{\delta} + \int_{1/n}^{\delta} \Phi(u) \frac{1}{u} \left(\frac{1}{u \prod_{m=1}^k l_m(1/u)} \right) du \\ &= o \left(\frac{l(1/u)}{\sqrt{\prod_{m=1}^{k+1} l_m(1/u)}} \right)_{1/n}^{\delta} \\ &\quad + o \left(\frac{1}{(\sqrt{\prod_{m=1}^{k+1} l_m(1/u)}) (u \prod_{m=1}^k l_m(1/u))} \right) \int_{1/n}^{\delta} du \\ &= o(1); \end{aligned}$$

so it implies

$$\begin{aligned}
j_2 &= \frac{1}{\pi^2} \int_{1/n}^{\delta} \int_{1/n}^{\delta} \phi(t) \alpha(t) \phi(u) \beta(u) \cot(u/2) \left\{ \sum_{r=0}^n \frac{1}{P_r^2} \right\} dt du \\
&\leq \frac{1}{\pi^2} \sum_{r=0}^n \frac{1}{P_r^2} \left(\int_{1/n}^{\delta} |\phi(t)| \alpha(t) dt \int_{1/n}^{\delta} |\phi(u)| \alpha(u) |\cot(t/2)| du \right) \\
&= o(1) \int_{1/n}^{\delta} |\phi(t)| \alpha(t) dt \int_{1/n}^{\delta} |\phi(u)| \alpha(u) |\cot(t/2)| du \\
&= o(1).
\end{aligned}$$

Similarly, we can obtain

$$j_3 = o(1) \quad \text{and} \quad j_4 = o(1).$$

Hence, the proof of the theorem is completed.

Corollary 1. *If the condition (3.1) of main theorem and the condition (1.8) of [12] are satisfied, then the series $\sum_{n=0}^{\infty} u_n(t)$ is $[C, 1, 2]$ -summable to the sum s for $t = x$.*

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