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Fixed point theorems in fuzzy metric spaces for mappings with $B_{\gamma,\mu}$ condition

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Abstract:

In this paper we prove some fixed point theorems in fuzzy metric spaces for a class of generalized nonexpansive mappings satisfying $B_{\gamma,\mu}$ condition. We introduce a type of convexity in fuzzy metric spaces with respect to an altering distance function and prove convergence results for some iteration schemes to the fixed point. The results are supported by suitable examples.

Keywords: Fuzzy metric space; $B_{\gamma,\mu}$ condition; Convexity, Fixed point.

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1. Introduction and preliminaries

In 1988, Grabiec [7] introduced fixed point theory in fuzzy metric spacesby extending different existing results to such spaces. During the recent decades, the study of different generalized classes of nonexpansive mappings and the related fixed point theorems in different spaces have found much importance due to many practical applications (refer to [2, 3, 4, 5, 7, 9,10, 11, 12, 13, 17, 18, 20, 25, 26]). Several research workers have interesting contribution (refer to [6, 8, 14, 15, 19, 24]) in this regard. In 2008, Suzuki[2] defined a class of mappings satisfying condition (C) in a Banach space X, which is wider than the class of nonexpansive mappings. In 2011, García-Falsetet al. [5] and in 2018 Patir et al. [18] introduced some new classes ofgeneralized nonexpansive mappings which contain the mappings satisfying(C) condition as a subclass.

In this paper, we have extended these generalized classes of mappings with (C) condition by Suzuki and $B_{\gamma,\mu}$ condition by Patir et al. to fuzzy metric spaces and prove some fixed point theorems. Moreover, defining a new generalized type of convexity with respect to an altering distance function we derive some convergence results of iteration schemes to the fixed point.

First we present the following basic definitions.

Definition 1.1. *[10] Let $T : [0,1] \times [0,1] \longrightarrow [0,1]$. Then the mapping T is said to be a triangular norm (t-norm) if*

i) $T(a,1) = a$ *for all* $a \in [0,1]$,

ii) $T(a,b) = T(b,a)$ *for all* $a,b \in [0,1]$,

iii) $a \geq b, c \geq d \Rightarrow T(a,c) \geq T(b,d)$ *for all* $a,b,c,d \in [0,1]$,

iv) $T(a,T(b,c)) = T(T(a,b),c)$ *for all* $a,b,c \in [0,1]$.

Some basic t-norms are $T_p(a,b) = a.b$, $T_m(a,b) = \min(a,b)$, $T_L(a,b) = \max(a+b-1,0)$.

Definition 1.2. *[6] Let T be a continuous triangular norm on an arbitrary set X and M be a fuzzy set on $X^2 \times (0,\infty)$. Then the3-tuple (X,M,T) is said to be a fuzzy metric space if the following conditions are satisfied :*

a) $M(a,b,t) > 0, \ \forall \ a,b \in X, t > 0$,

b) $M(a,b,t) = 1, \ \forall \ t > 0 \Leftrightarrow a = b$,

c) $M(a,b,t) = M(b,a,t), \ \forall \ a,b \in X, t > 0,$

d) $T(M(a,b,t), M(b,c,s)) \leq M(a,c,t+s), \ \forall \ a,b,c \in X, t,s > 0,$

e) $M(a,b,.) : (0,\infty) \longrightarrow [0,1]$ *is continuous for all* $a,b \in X$.

Example 1.3. *Let* $X = \mathbf{R}$ *with usual metric and for* $t > 0$, $M(x,y,t) = \frac{t}{t+d(x,y)}$. *Then* (X,M,T) *is a fuzzy metric space with respect to the t-norm* $T_m(x,y) = \min(x,y), \quad x,y \in X$.

Lemma 1.4. *[7] For all* x, y *in* X, $M(x,\ y,\ t)$ *is a non-decreasing functionof* t.

Definition 1.5. *[7, 16] Let* (X,M,T) *be a fuzzy metric space.*

(i) *A sequence* $\{x_n\}$ *in* X *is a Cauchy sequence if for all* $\varepsilon \in (0,1)$, $t > 0$, $\exists \ n_0 \ \in \mathbf{N}$ *such that* $M(x_n, x_m, t) > 1 - \varepsilon$, $\forall \ n,m \geq n_0$.

(ii) *A sequence* $\{x_n\}$ *in* X *converges to* x *if for all* $\varepsilon \in (0,1)$, $t > 0$, $\exists \ n_0 \in \mathbf{N}$ *such that* $M(x_n, x, t) > 1 - \varepsilon$, $\forall \ n \geq n_0$.

(iii) *A fuzzy metric space* X *is complete if and only if every Cauchy sequence converges in* X.

(iv) *Then* (X,M,T) *is said to be sequentially compact if every sequence in* X *has a convergent subsequence in* X.

Definition 1.6. *[20] A mapping* $\phi : [0,\ 1] \longrightarrow [0,\ 1]$ *is said to be an alteringdistance function if the following conditions are satisfied:*
(i) ϕ *is strictly decreasing and left continuous.*
(ii) $\phi(\lambda) = 0$ *if and only if* $\lambda = 1$,
i.e., $\lim_{\phi \longrightarrow 1-} \phi(1) = 0$.

Definition 1.7. *[16] Let* E *be a subset of a fuzzy metric space* (X,M,T). *Then* E *is said to bounded if there exists* $t > 0$ *and* $0 < \varepsilon < 1$ *such that* $M(x,y,t) > 1 - \varepsilon$, *for all* $x,y \in E$.

Definition 1.8. *[21] Let* C *be a nonempty subset of a Banach space* X. *A mapping* $T : C \longrightarrow X$ *is said to satisfy condition (C) on* C *if for all* $x,y \in C$, $\frac{1}{2}||x - Tx|| \leq ||x - y||$ *implies* $||Tx - Ty|| \leq ||x - y||$.

Clearly, for every nonexpansive mapping on C, condition (C) is satisfied. But there are also some examples of non continuous mappings which satisfies condition (C) (refer to [21]).

Definition 1.9. *Let E be a nonempty subset of a complete fuzzy metric space (X, M, T). Then $f : E \longrightarrow X$ is said to be fuzzy nonexpansive with respect to an altering distance function ϕ if*

$$\phi(M(f(x), f(y), t)) \leq \phi(M(x, y, t)),$$

for all $t > 0$ and $x, y \in E$.

Example 1.10. *Let $X = \{(0,0), (3,0), (6,0), (6,7)\}$ and d be the Euclidean distance. Let $M : X^2 \times [0, \infty[\longrightarrow [0,1]$ be defined by $M(x, y, t) = \frac{t}{t+d(x,y)}$ $\forall\ x, y \in X,\ t > 0$ and $T(r, s) = \min\{r, s\},\ r, s \in [0,1]$, then (X, M, T) is a complete fuzzy metric space [8].*
Next we consider $f : X \longrightarrow X$ by
$f((\{(0,0), (3,0), (6,0), (6,7)\}) = \{(0,0), (0,0), (3,0), (6,0)\}$ correspondingly and $\phi(a) = 1 - a$.
Then f is fuzzy nonexpansive with respect to ϕ.

Definition 1.11. *Let E be a nonempty subset of a complete fuzzy metric space (X, M, T). Then a mapping $f : E \longrightarrow X$ is said to satisfy fuzzy (C) condition with respect to ϕ if*

$$\frac{1}{2}\phi(M(x, f(x), t)) \leq \phi(M(x, y, t)),$$

implies $\phi(M(f(x), f(y), t)) \leq \phi(M(x, y, t))$, for all $x, y \in E, t > 0$, where ϕ is an altering distance function.

Definition 1.12. *Let (X, M, T) be a complete fuzzy metric space and E be a nonempty subset of X. A mapping $f : E \longrightarrow X$ is said to be fuzzy quasi-nonexpansive with respect to ϕ if*

$$\phi(M(f(x), p, t)) \leq \phi(M(x, p, t)),$$

for all $x \in E$ and $p \in F(f)$ $\Big($where $F(f)$ denotes the set of all fixed points of $f\Big)$, $t > 0$.

In 1970, Takahashi [22] introduced the following concept of convex structure in a metric space.

Definition 1.13. *Let (X, d) be a metric space. A mapping $\mathcal{W} : X^2 \times [0,1] \longrightarrow X$ satisfying*

$$d\,(z, \mathcal{W}(x, y, t)) \leq td(z, x) + (1 - t)d(z, y),$$

for all $x, y, z \in X$ and $t \in [0, 1]$ is called a convex structure on X.

Definition 1.14. *[24] Let $(X,\ M,\ T)$ be a fuzzy metric space. A continuousmapping $W : X \times X \times [0, 1] \longrightarrow X$ is said to be a convex structure on Xif for each $x, y, z \in X, t > 0$ and $\alpha \in [0, 1]$, we have*

$$M(z, W(x, y, \alpha), t) \geq \alpha M(z, x, t) + (1 - \alpha)M(z, y, t).$$

The space X together with a convex structure W is called a fuzzy convex metric space or convex fuzzy metric space.

Definition 1.15. *[21] A complete fuzzy metric space (X, M, T) is said tosatisfy fuzzy Opial property with respect to ϕ if for every sequence $\{a_n\}$ inX with $a_n \longrightarrow u$, we have for each $t > 0$,*

$$\lim_{n \longrightarrow \infty} \inf \phi(M(a_n, u, t)) < \lim_{n \longrightarrow \infty} \inf \phi(M(a_n, v, t)),$$

whenever $v = u$.

2. Results and discussion

First we define a type of convexity with respect to an altering distance function in fuzzy metric space.

Definition 2.1. *Let (X, M, T) be a fuzzy metric space. A continuous mapping $W : X \times X \times [0, 1] \longrightarrow X$ is said to be a ϕ convex structure on X if for each $x, y, z \in X$ and $\alpha \in [0, 1]$, $t > 0$, we have,*

$$\phi(M(z, W(x, y, \alpha), t)) \leq \alpha\phi(M(z, x, t)) + (1 - \alpha)\phi(M(z, y, t)),$$

where ϕ is the altering distance function.
Then (X, M, T) is called ϕ convex fuzzy metric space.

Example 2.2. *Consider $X = \mathbf{R}$ and $d(x, y) = |x - y|$ such that $d(x, y) < k$. Let $g : (0, \infty) \longrightarrow]k, \infty[$ be an increasing continuous function and for $t \in (0, \infty)$, $M(x, y, t) = 1 - \frac{d(x,y)}{g(t)}$. Then (X, M, T_L) is a fuzzy metric space*

[8] with respect to Lukasievicz t-norm, $T_L(a,b) = \max(a+b-1, 0)$ $a, b \in X$. *Also, let the altering distance function* $\phi(t) = 1 - t,\ t \in [0,1]$.

Now, let $W : X \times X \times [0,1] \longrightarrow X$ *be defined by* $W(x,y,\alpha) = \alpha x + (1-\alpha)\, y, \quad \alpha \in [0,1]$ *then*

$$\begin{aligned}\phi(M(z, W(x,y,\alpha),t)) &= \tfrac{d(z,W(x,y,\alpha))}{g(t)} \\ &= \tfrac{|z-(\alpha x+(1-\alpha)y)|}{g(t)} \\ &\leq \tfrac{\alpha|z-x|}{g(t)} + \tfrac{(1-\alpha)|z-y|}{g(t)} \\ &= \alpha\phi(M(z,x,t)) + (1-\alpha)\phi(M(z,y,t)).\end{aligned}$$

Thus (X, M, T_L) *is a* ϕ *convex fuzzy metric space.*

The concept of uniform convexity for metric spaces [15] can also beextended to ϕ convex fuzzy metric spaces.

Definition 2.3. *For a* ϕ *convex fuzzy metric space* (X, M, T), X *is said to be uniformly* ϕ *convex if for arbitrary numbers* $\varepsilon, h \in \mathbf{R}^+$, *there exists* $\beta(\varepsilon) > 0$ *such that*

$$\phi(M(z, W(x,y,\frac{1}{2}),t)) \leq h(1-\beta),$$

whenever $\phi(M(z,x,t)) \leq h, \phi(M(z,y,t)) \leq h$ *and* $\phi(M(x,y,t)) \geq h\varepsilon$, *where* $x, y, z \in X, t > 0$.

Remark : The class of ϕ convex fuzzy metric spaces includes the (usual) convex fuzzy metric spaces (for the altering distance function, $\phi(a) = 1 - a, \forall\ a \in [0,1]$), which follows from the lemma below.

Lemma 2.4. *If* $W : X \times X \times [0,1] \longrightarrow X$ *is a convex structure on a fuzzy metric space* (X, M, T), *then it is also* ϕ *convex on the same fuzzy metric space, for* $\phi(a) = 1 - a, a \in [0,1]$, *i.e. the class of* ϕ *convex fuzzy metric spaces is wider than the class of convex fuzzy metric spaces.*

Proof. Since W is a convex structure on a fuzzy metric space (X, M, T), we have for each $x, y, z \in X$ and $\alpha \in [0,1], t > 0$,

$$M(z, W(x,y,\alpha),t) \geq \alpha M(z,x,t) + (1-\alpha)M(z,y,t),$$

$$\begin{aligned}\text{i.e., } 1 - M(z, W(x,y,\alpha),t) &\leq 1 - (\alpha M(z,x,t) + (1-\alpha)M(z,y,t)) \\ &= \alpha(1 - M(z,x,t)) + (1-\alpha)(1 - M(z,y,t)).\end{aligned}$$

Thus, $\phi(M(z, W(x,y,\alpha),t)) \leq \alpha\phi(M(z,x,t)) + (1-\alpha)\phi(M(z,y,t))$,

where $\phi(a) = 1 - a, \forall\ a \in [0,1]$.
Hence, W is also a ϕ convex structure on (X, M, T). $\square$

Considering fuzzy metric space, the definitions of asymptotic radius and asymptotic centre (refer to [5]) can be stated as follows.

Definition 2.5. *Let E be a nonempty subset and $\{x_n\}$ be a bounded sequence of a complete fuzzy metric space (X, M, T). Then for each $x \in X$,*

(i) *asymptotic radius of $\{x_n\}$ with respect to an altering distance ϕ at x is defined by $r(x, \{x_n\}) = \sup_t\{\limsup_{n \longrightarrow \infty} \phi(M(x_n, x, t))\}$.*

(ii) *asymptotic radius of $\{x_n\}$ with respect to ϕ relative to E is defined by $r(E, \{x_n\}) = \inf\{r(x, \{x_n\}) : x \in E\}$.*

(iii) *asymptotic center of $\{x_n\}$ with respect to ϕ relative to E is defined by $\mathcal{C}(E, \{x_n\}) = \{x \in E : r(x, \{x_n\}) = r(E, \{x_n\})\}$.*
We note that $\mathcal{C}(E, \{x_n\})$ is non empty. Again, if X is uniformly ϕ convex, then $\mathcal{C}(E, \{x_n\})$ has exactly one point.

Following is an extension of condition $B_{\gamma,\mu}$ [18] to fuzzy metric space.

Definition 2.6. *Let (X, M, T) be a complete fuzzy metric space, ϕ be an altering distance function and E be a nonempty subset of X. Let $\gamma \in [0,1]$ and $\mu \in [0, \frac{1}{2}]$ such that $2\mu \leq \gamma$. A mapping $f : E \longrightarrow X$ is said to satisfy fuzzy $B_{\gamma,\mu}$ condition with respect to ϕ on E if, for all $x, y \in E$ and $t > 0$,*

$$\gamma\ \phi(M(x, f(x), t)) \leq \phi(M(x, y, t)) + \mu\ \phi(M(y, f(y), t))$$

implies

$$\phi(M(f(x), f(y), t)) \leq (1-\gamma)\phi(M(x, y, t)) + \mu\,(\phi(M(x, f(y), t)) + \phi(M(y, f(x), t))), \tag{2.1}$$

where ϕ satisfies the condition

$$\phi(T(M(x, y, t), M(z, w, s))) \leq \phi(M(x, y, t + s)) + \phi(M(z, w, t + s)), \tag{2.2}$$

for $x, y, z, w \in E$ and $s, t > 0$.

In all our subsequent results, we consider only the altering distance functions satisfying the inequality (2) (on E).

Example 2.7. *Let* $X = [0, \frac{1}{2}], d(x, y) = |x - y|$ *and*

$$M(x, y, t) = \begin{cases} 1, & \text{if } x = y, \\ d(x, y), & \text{if } x = y, \end{cases}$$

$\forall\ t > 0$. *Then* (X, M, T) *is a complete fuzzy metric space [8] with thet-norm* $T_L(a, b) = \max(a + b - 1, 0)\ \ x, y \in X$. *Let the mapping* f *on* X *bedefined by*

$$f(x) = \begin{cases} 0, & \text{if } x = \frac{1}{2}, \\ \frac{1}{4}, & \text{if } x = \frac{1}{2}. \end{cases}$$

Then f *satisfies* $B_{\gamma,\mu}$ *condition on* X *for* $\gamma = 1, \mu = \frac{1}{2}$.

Lemma 2.8. *Let a mapping* f *on a nonempty subset* E *of a complete fuzzy metric space* (X, M, T) *satisfy the fuzzy* $B_{\gamma,\mu}$ *condition with respect to* ϕ. *If* f *has a fixed point, say* p, *on* E, *then* f *is fuzzy quasi-nonexpansive with respect to* ϕ *i.e.,* $\phi(M(f(x), p, t)) \leq \phi(M(x, p, t))$ *for all* $x \in E$, $t > 0$. *The converse is not necessarily true in general.*

Proof. Since p is a fixed point, we have

$$\gamma\ \phi(M(p, f(p), t)) = 0 \leq \phi(M(x, y, t)) + \mu\ \phi(M(y, f(y), t)).$$

So, from the fuzzy $B_{\gamma,\mu}$ condition with respect to ϕ,

$$\phi(M(f(p), f(x), t)) \leq (1-\gamma)\phi(M(p, x, t)) + \mu\,(\phi(M(p, f(x), t)) + \phi(M(x, f(p), t)))$$

$$\Rightarrow \phi(M(p, f(x), t)) \leq (1-\gamma)\phi(M(p, x, t)) + \mu\,\phi(M(p, f(x), t)) + \mu\,\phi(M(x, p, t))$$

$$\Rightarrow \phi(M(p, f(x), t) \leq \left(\frac{1 - \gamma + \mu}{1 - \mu}\right) \phi(M(x, p, t)) \leq \phi(M(x, p, t)) \quad (2\mu \leq \gamma).$$

Thus f is fuzzy quasi-nonexpansive with respect to ϕ. □

For the converse part is not true in general, consider the following example.

Example 2.9. *Let $X = [0,5]$, $d(x,y) = |x-y|$ and for $t > 0$, $M(x,y,t) = \frac{t}{t+d(x,y)}$. Then with respect to the t-norm $T_p(x,y) = x.y$, (X,M,T) is a complete fuzzy metric space, where $x,y \in X$. Let the mapping f on $[0,5]$ be defined by*

$$f(x) = \begin{cases} 0, & \text{if } x \neq 5, \\ 4, & \text{if } x = 5, \end{cases}$$

and the altering distance function ϕ be defined by $\phi(a) = 1-a, \forall\, a \in [0,1]$. Clearly $x = 0$ is the only fixed point of f, and therefore $\phi(M(f(x),p,t)) \leq \phi(M(x,p,t))$ for $p = 0$ and $\forall\, x \in [0,5]$.
Thus f is fuzzy quasi-nonexpansive.
Also for $x = 5$ and $y = 4$,

$$\phi(M(x,f(x),t)) \leq \phi(M(x,y,t)) + \mu\, \phi(M(y,f(y),t))$$
$$\text{where } \gamma \in [0,1], \mu \in [0,\tfrac{1}{2}], t > 0.$$

But

$$\phi(M(f(x),f(y),t)) \leq (1-\gamma)\phi(M(x,y,t)) + \mu\, (\phi(M(x,f(y),t)) + \phi(M(y,f(x),t)))$$

$$\begin{aligned} \Rightarrow \tfrac{4}{t+4} &\leq \tfrac{(1-\gamma)}{t+1} + \tfrac{5\,\mu}{t+5} \\ &\leq \tfrac{2.5}{t+5} \quad (\text{for } \gamma = 1, \mu = \tfrac{1}{2}), \end{aligned}$$

a contradiction.
Hence, f does not satisfy fuzzy $B_{\gamma,\mu}$ condition on X.

It is also seen that every mapping satisfying fuzzy (C) condition with respect to an altering distance function ϕ satisfies fuzzy $B_{\gamma,\mu}$ condition (for $\gamma = \mu = 0$) with respect to ϕ, but the converse does not hold.

Example 2.10. *For $X = [0,4]$, we consider the complete fuzzy metric space (X,M,T_L) as in Example 2.2. Let the mapping f on X be defined by*

$$f(x) = \begin{cases} , & \text{if } x \neq 4, \\ 2, & \text{if } x = 4, \end{cases}$$

and $\phi(a) = 1-a, \forall\, a \in [0,1]$. Now, for $x = 2.4, y = 4$, we have
$\frac{1}{2}\phi(M(x,f(x),t)) \leq \phi(M(x,y,t))$.
But $\phi(M(f(x),f(y),t)) \leq \phi(M(x,y,t))\ \forall\, t > 0$.
So, f does not satisfy fuzzy (C) condition with respect to ϕ.
Again, for all $x,y \in X$ and $\gamma = 1, \mu = \frac{1}{2}$, we have
$\phi(M(f(x),f(y),t)) \leq (1-\gamma)\phi(M(x,y,t)) + \mu\, (\phi(M(x,f(y),t)) + \phi(M(y,f(x),t)))$
$\forall\, t > 0$.
Hence f satisfies fuzzy $B_{\gamma,\mu}$ condition with respect to ϕ.

Proposition 2.11. *Let (X, M, T) be a complete fuzzy metric space and f be a mapping on a subset E of X. Suppose that f satisfies fuzzy $B_{\gamma,\mu}$ condition with respect to ϕ on E. Then the following holds for $x, y \in E$ and $c \in [0, 1]$.*

(i) $\phi(M(f(x), f^2(x), t)) \leq \phi(M(x, f(x), t))$,

(ii) *Either* $\frac{c}{2}\phi(M(x, f(x), t)) \leq \phi(M(x, y, t)))$
or $\frac{c}{2}\phi(M(f(x), f^2(x), t)) \leq \phi(M(f(x), y, t))$ *holds,*
which implies
either $\phi(M(f(x), f(y), t)) \leq \left(1 - \dfrac{c}{2}\right)\phi(M(x, y, t)) + \mu(\phi(M(x, f(y), t)) + \phi(M(y, f(x), t)))$
or $\phi(M(f^2(x), f(y), t)) \leq \left(1 - \dfrac{c}{2}\right)\phi(M(f(x), y, t)) + \mu(\phi(M(f(x), f(y), t)) + \phi(M(y, f^2(x), t)))$ *holds,*

(iii) $\phi(M(x, f(y), t)) \leq (3 - c)\phi(M(x, f(x), t)) + \left(1 - \dfrac{c}{2}\right)\phi(M(x, y, t)) + \mu(2\phi(M(x, f(x), t)) + \phi(M(x, f(y), t)) + \phi(M(y, f(x), t)) + 2\phi(M(f(x), f^2(x), t)))$.

Theorem 2.12. *Let (X, M, T) be a complete fuzzy metric space and f be a self mapping on a compact and ϕ convex subset E of X. Let f satisfy fuzzy $B_{\gamma,\mu}$ condition with respect to an altering distance function ϕ on E. For $s \in [0, 1)$ and $x_1 \in E$, define a sequence $\{x_n\}$ in E such that*

$$x_{n+1} = W(x_n, f(x_n), s),$$

where W is the ϕ convex structure on E and $\lim_{n\to\infty} \phi(M(x_n, f(x_n), t)) = 0$, $n \in \mathbf{N}$. Then the sequence $\{x_n\}$ converges to a fixed point of f.

Proof. Since E is compact, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and $p \in E$ such that $\{x_{n_j}\}$ converges to p. Also by Proposition 2.11 (iii) we have,

$$\begin{aligned}
\phi(M(x_{n_j}, f(p), t)) &\leq (3-c)\phi(M(x_{n_j}, f(x_{n_j}), t)) + \left(1 - \frac{c}{2}\right)\phi(M(x_{n_j}, p, t)) \\
&+ \mu(2\phi(M(x_{n_j}, f(x_{n_j}), t)) + \phi(M(x_{n_j}, f(p), t)) + \phi(M(p, f(x_{n_j}), t)) \\
&+ 2\phi(M(f(x_{n_j}), f^2(x_{n_j}), t))) \\
&\leq (3-c)\phi(M(x_{n_j}, f(x_{n_j}), t)) + \left(1 - \frac{c}{2}\right)\phi(M(x_{n_j}, p, t)) \\
&+ \mu(4\phi(M(x_{n_j}, f(x_{n_j}), t)) + \phi(M(x_{n_j}, f(p), t)) + \phi(M(p, x_{n_j}, \tfrac{t}{2})) \\
&+ \phi(M(x_{n_j}, f(x_{n_j}), \tfrac{t}{2}))),
\end{aligned}$$

(Using proposition 2.11 (i) and inequality (2)).

for all $j \in \mathbf{N}$ and $c \in [0, 1]$. Taking $n_j \longrightarrow \infty$, we have,

$$\begin{aligned}
(1-\mu)\lim_{n_j\to\infty} \phi(M(x_{n_j}, f(p), t)) &\leq (3-c)\ \lim_{n_j\to\infty} \phi(M(x_{n_j}, f(x_{n_j}), t)) \\
&+ \left(1 - \frac{c}{2}\right)\lim_{n_j\to\infty} \phi(M(x_{n_j}, p, t)) \\
&+ \mu(4\lim_{n_j\to\infty} \phi(M(x_{n_j}, f(x_{n_j}), t)) \\
&+ \lim_{n_j\to\infty} \phi(M(p, x_{n_j}, \tfrac{t}{2})) \\
&+ \lim_{n_j\to\infty} \phi(M(x_{n_j}, f(x_{n_j}), \tfrac{t}{2}))) \\
\Rightarrow \lim_{n_j\to\infty} \phi(M(x_{n_j}, f(p), t)) &= 0,
\end{aligned}$$

i.e., $\{x_{n_j}\}$ converges to $f(p)$.

Therefore $f(p) = p$ and hence p is a fixed point of f. Again, by the ϕ convexity of E,

$$\begin{aligned}
\phi(M(p, x_{n+1}, t)) &= \phi(M(p, W(x_n, f(x_n), s), t)) \\
&\leq s\ \phi(M(p, x_n, t)) + (1-s)\phi(M(p, f(x_n), t)) \\
&\leq s\ \phi(M(p, x_n, t)) + (1-s)\phi(M(p, x_n, t)) \quad \text{(by Lemma 2.8)} \\
&= \phi(M(p, x_n, t)), \quad \text{for } n \in \mathbf{N},
\end{aligned}$$

which implies that $\{\phi(M(p, x_n, t))\}$ is a monotonic decreasing bounded sequence and therefore,

$$\lim_{n\to\infty} \phi(M(p, x_n, t)) = 0$$
$$\Rightarrow \lim_{n\to\infty} M(p, x_n, t) = 1.$$

Therefore $\{x_n\}$ converges to p. □

Proposition 2.13. *Let f be a self mapping on a subset E of a complete fuzzy metric space X with Opial condition. Let f satisfy the fuzzy $B_{\gamma,\mu}$ condition with respect to ϕ on E. If $\{x_n\}$ is a sequence in X such that*

(i) *$\{x_n\}$ converges to p,*

(ii) $\lim\limits_{n\to\infty} \phi(M(x_n, f(x_n), t)) = 0,$

then $f(p) = p$.

Proof. From Proposition 2.11 (iii) (for $\gamma = \frac{c}{2},\ c \in [0,1]$),

$$\begin{aligned}
\phi(M(x_n, f(p), t)) \quad &\leq \quad (3-c)\phi(M(x_n, f(x_n), t)) + \left(1 - \frac{c}{2}\right)\phi(M(x_n, p, t)) \\
&+ \quad \mu(2\phi(M(x_n, f(x_n), t)) + \phi(M(x_n, f(p), t)) + \phi(M(p, f(x_n), t)) \\
&+ \quad 2\phi(M(f(x_n), f^2(x_n), t))) \\
&\leq \quad (3-c)\phi(M(x_n, f(x_n), t)) + \left(1 - \frac{c}{2}\right)\phi(M(x_n, p, t)) \\
&+ \quad \mu(4\phi(M(x_{n_j}, f(x_{n_j}), t)) + \phi(M(x_{n_j}, f(p), t)) + \phi(M(p, x_{n_j}, \tfrac{t}{2})) \\
&+ \quad \phi(M(x_{n_j}, f(x_{n_j}), \tfrac{t}{2}))),
\end{aligned}$$

for all $n \in \mathbf{N}$ and hence taking limit $n \longrightarrow \infty$ and using condition (ii) we have,

$$\phi(M(x_n, f(p), t)) \leq \frac{1 - \frac{c}{2} + \mu}{1 - \mu}\phi(M(x_n, p, t)) \leq \phi(M(x_n, p, t))$$

$$\Rightarrow \lim_{n\to\infty} \inf \phi(M(x_n, f(p), t)) \leq \lim_{n\to\infty} \inf \phi(M(x_n, p, t)). \tag{2.3}$$

Let $f(p) \neq p$. Since $\{x_n\} \longrightarrow p$, from the Opial condition we have

$$\lim_{n\to\infty} \inf \phi(M(x_n, p, t)) < \lim_{n\to\infty} \inf \phi(M(x_n, f(p), t)),$$

which is a contradiction to (2.3).
Hence $f(p) = p$. $\square$

Example 2.14. *Let* $X = C(\subset l^p, 1 < p < \infty) = \{\{x_n\} \in l^p : |x_1| \leq 1, x_j = 0\ \forall j \neq 1\}, d(x,y) = ||x-y||_p$ *and* $M(x,y,t) = \frac{t}{t+d(x,y)},\ t > 0$. *Then* (X, M, T) *is a complete fuzzy metric space for the t-norm,* $T_p(x,y) = x.y$. *Let* $\{a_n\} \in C$ *be such that*

$$a_1 = \{\frac{1}{3}, 0, 0...\}, a_2 = \{\frac{1}{2}, 0, 0...\}, a_3 = \{\frac{3}{5}, 0, 0...\}.........a_n = \{\frac{n}{n+2}, 0, 0....\}.$$

Clearly $\{a_n\}$ *converges to* $z = \{1, 0, 0....\}$.
Let $X_1 = \{x_1, 0, 0.....\} \in X$ *and* f *be a self-map on* X *defined by* $f(X_1) = f(\{x_1, 0, 0....\}) = \{x_1^3, 0, 0....\}$. *Let* $\phi(t) = 1 - t$. *Then* f *satisfies fuzzy* $B_{\gamma,\mu}$

condition with respect to ϕ for $\gamma = \mu = 0$.

Now,

$$\begin{aligned} & \lim_{n\to\infty} \phi(M(a_n, f(a_n), t)) \\ = \;& \lim_{n\to\infty} \tfrac{|a_n - f(a_n)|}{t+|a_n - f(a_n)|} = \lim_{n\to\infty} \tfrac{|\frac{n}{n+2} - (\frac{n}{n+2})^3|}{t+|\frac{n}{n+2} - (\frac{n}{n+2})^3|} \\ = \;& 0. \end{aligned}$$

Thus by Proposition 2.13, $f(z) = z$, which is clearly the fixed point here.

Theorem 2.15. *Let (X, M, T) be a complete fuzzy metric space and f be a self mapping on a compact ϕ convex subset E of X with the Opial property with respect to ϕ. Suppose that f satisfies fuzzy $B_{\gamma,\mu}$ condition with respect to ϕ. Define a sequence $\{x_n\}$ in E such that*

$$x_{n+1} = W(x_n, f(x_n), s)\ ,\ x_1 \in E,\ \ s \in [0, 1)$$

and $\lim_{n\to\infty} \phi(M(x_n, f(x_n), t)) = 0, \quad n \in \mathbf{N}$. Then $\{x_n\}$ converges to a fixed point of f.

Proof. Since E is compact, we have a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and $p \in E$ such that $\{x_{n_j}\}$ converges to p. Now, by Proposition 2.13, p is a fixed point of f.

We assume that $\{x_n\}$ does not converge to p. Then there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $q \in C$ such that $\{x_{n_k}\}$ converges to q and $q \neq p$. Again, $Tq = q$ (by Proposition 2.13). From the Opial property,

$$\begin{aligned} \lim_{n\to\infty} \inf \phi(M(x_n, p, t)) &= \lim_{j\to\infty} \inf \phi(M(x_{n_j}, p, t)) \\ &< \lim_{j\to\infty} \inf \phi(M(x_{n_j}, q, t)) \\ &= \lim_{k\to\infty} \inf \phi(M(x_{n_k}, q, t)) \\ &< \lim_{k\to\infty} \inf \phi(M(x_{n_k}, p, t)) \\ &= \lim_{n\to\infty} \inf \phi(M(x_n, p, t)), \end{aligned}$$

which is a contradiction.

Hence $\{x_n\}$ converges to the fixed point p. $\square$

Next, we discuss the convergence of following type of extended Mann iteration scheme with respect to the ϕ convex structure W on X:

$$(2.4)\quad \begin{array}{ll} x_1 & \in X, \\ y_n & = W(x_n, f(x_n), \xi_n), \\ x_{n+1} & = W(y_n, f(y_n), \eta_n), \\ n \in \mathbf{N} & , \xi_n, \eta_n \in [0,1] \text{ for each } n \text{ and } \sum_n \xi_n = \infty, \sum_n \eta_n = \infty. \end{array}$$

Lemma 2.16. *Let E be a nonempty closed ϕ convex subset of a complete fuzzy metric space (X, M, T) and f be a self-mapping on E satisfying the fuzzy $B_{\gamma,\mu}$ condition with respect to ϕ on E. Let $x_1 \in E$ and $\{x_n\}$ be a sequence in E defined by the above iteration scheme (2.4).Then $\lim_{n\to\infty} \phi(M(x_n, p, t))$ exists for all $p \in F(f)$.*

Proof. For $p \in F(f)$, by Lemma 2.8,
$\phi(M(f(x_n), p, t)) \leq \phi(M(x_n, p, t))$ for all $n \in \mathbf{N}$.

Now,

$$\begin{aligned}
\phi(M(p, x_{n+1}, t)) &= \phi(M(p, W(y_n, f(y_n), \eta_n), t)) \\
&\leq \eta_n\ \phi(M(p, y_n, t)) + (1-\eta_n)\phi(M(p, f(y_n), t)) \\
&\leq \eta_n\ \phi(M(p, y_n, t)) + (1-\eta_n)\phi(M(p, y_n, t)) \\
&= \phi(M(p, y_n, t)) \\
&= \phi(M(p, W(x_n, f(x_n), \xi_n), t)) \\
&\leq \xi_n\ \phi(M(p, x_n, t)) + (1-\xi_n)\phi(M(p, f(x_n), t)) \\
&\leq \xi_n\ \phi(M(p, x_n, t)) + (1-\xi_n)\phi(M(p, x_n, t)) \\
&= \phi(M(p, x_n, t))
\end{aligned}$$

i.e. $\phi(M(p, x_{n+1}, t)) \leq \phi(M(p, x_n, t))$,

which implies $\{\phi(M(p, x_n, t))\}$ is a monotonic decreasing bounded sequence. Hence $\lim_{n\to\infty} \phi(M(x_n, p, t))$ exists for all $p \in F(f)$. □

The next Lemma is the fuzzy counterpart of the lemma 2.11 of [1] (referto [23, 27]).

Lemma 2.17. *Let X be a uniformly ϕ convex and complete fuzzy metric space. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X such that $\limsup_{n\to\infty} \phi(M(x_n, x, t)) \leq r$, $\limsup_{n\to\infty} \phi(M(y_n, x, t)) \leq r$ and $\lim_{n\to\infty} \phi(M(W(x_n, y_n, \lambda_n), x, t)) = r$ for all $n \in \mathbf{N}$, $x \in X$ and some $r \geq 0$, where $\{\lambda_n\}$ is a sequence of real numbers with $0 < a \leq \lambda_n \leq b < 1$ $\forall n \in \mathbf{N}, t > 0$. Then $\lim_{n\to\infty} \phi(M(x_n, y_n, t)) = 0$.*

Theorem 2.18. *Let E be a nonempty closed ϕ convex subset of a uniformly ϕ convex and complete fuzzy metric space (X, M, T) and f be a self-mapping on E which satisfy the fuzzy $B_{\gamma,\mu}$ condition with respect to ϕ on E. Let $x_1 \in E$ and $\{x_n\}$ be a sequence in E defined by the above iteration scheme (2.4) where $\xi_n, \eta_n \in (0,1)$. Then $F(f) = \phi$ if and only if $\{x_n\}$ is bounded and $\lim_{n\to\infty} \phi(M(x_n, f(x_n), t)) = 0,\ t > 0$.*

Proof. Suppose, $F(f) \neq \phi$ and $p \in F(f)$.
From Lemma 2.16 we have $\lim_{n\to\infty} \phi(M(x_n, p, t))$ exists and $\{x_n\}$ is bounded.

Let

$$\lim_{n\to\infty} \phi(M(x_n, p, t)) = q. \tag{2.5}$$

Now,

$$\begin{aligned}
\phi(M(y_n, p, t)) &= \phi(M(W(x_n, f(x_n), \xi_n), p, t)) \\
&\leq \xi_n\ \phi(M(x_n, p, t)) + (1 - \xi_n)\phi(M(f(x_n), p, t)) \\
&\leq \xi_n\ \phi(M(x_n, p, t)) + (1 - \xi_n)\phi(M(x_n, p, t)) \\
&= \phi(M(x_n, p, t)) \text{ (by Lemma 2.8)}
\end{aligned}$$

$$\Rightarrow \lim_{n\to\infty} \phi(M(y_n, p, t)) \leq \lim_{n\to\infty} \phi(M(x_n, p, t)) = q. \tag{2.6}$$

Again,

$$\begin{aligned}
\phi(M(x_{n+1}, p, t)) &= \phi(M(W(y_n, f(y_n), \eta_n), p, t)) \\
&\leq \eta_n\ \phi(M(y_n, p, t)) + (1 - \eta_n)\phi(M(f(y_n), p, t)) \\
&\leq \eta_n\ \phi(M(y_n, p, t)) + (1 - \eta_n)\phi(M(y_n, p, t)) \\
&= \phi(M(y_n, p, t))
\end{aligned}$$

$$\Rightarrow \lim_{n\to\infty} \phi(M(x_{n+1}, p, t)) \leq \lim_{n\to\infty} \phi(M(y_n, p, t)) \Rightarrow q \leq \lim_{n\to\infty} \phi(M(y_n, p, t)). \tag{2.7}$$

Therefore, from (2.6) and (2.7)

$$\lim_{n\to\infty} \phi(M(y_n, p, t)) = q. \tag{2.8}$$

Now, from Lemma 2.8,

$$\begin{aligned}
&\phi(M(f(y_n), p, t)) \leq \phi(M(y_n, p, t)) \\
\Rightarrow\ &\limsup_{n\to\infty} \phi(M(f(y_n), p, t)) \leq \limsup_{n\to\infty} \phi(M(y_n, p, t)) = q.
\end{aligned} \tag{2.9}$$

Also,

$$\begin{aligned}
&\phi(M(W(y_n, f(y_n), \eta_n), p, t)) = \phi(M(x_{n+1}, p, t)) \\
\Rightarrow\ &\lim_{n\to\infty} \phi(M(W(y_n, f(y_n), \eta_n), p, t)) = \lim_{n\to\infty} \phi(M(x_{n+1}, p, t)) = q.
\end{aligned} \tag{2.10}$$

Now, from equation (2.8), (2.9), (2.10) and Lemma 2.17,

$$\lim_{n\to\infty} \phi(M(y_n, f(y_n), t)) = 0.$$

Again, from Lemma 2.8,

$$\begin{aligned}&\phi(M(f(x_n),p,t)) \leq \phi(M(x_n,p,t))\\ \Rightarrow&\limsup_{n\to\infty}\phi(M(f(x_n),p,t)) \leq \limsup_{n\to\infty}\phi(M(x_n,p,t)) = q.\end{aligned} \tag{2.11}$$

Also,

$$\begin{aligned}&\phi(M(W(x_n,f(x_n),\xi_n),p,t)) = \phi(M(y_n,p,t))\\ &\Rightarrow \lim_{n\to\infty}\phi(M(W(x_n,f(x_n),\xi_n),p,t)) = \lim_{n\to\infty}\phi(M(y_n,p,t)) = q.\end{aligned} \tag{2.12}$$

Hence, from equation (2.5), (2.11), (2.12) and Lemma 2.17,
$\lim_{n\to\infty}\phi(M(x_n,f(x_n),t)) = 0.$
Conversely, suppose that $\{x_n\}$ is bounded and $\lim_{n\to\infty}\phi(M(x_n,f(x_n),t)) = 0$.
Let $p \in \mathcal{C}(E,\{x_n\})$.

Then by Proposition 2.11 (iii) (for $\gamma = \frac{c}{2}, c \in [0,1]$)

$$\begin{aligned}\phi(M(x_n,f(p),t)) &\leq (3-c)\phi(M(x_n,f(x_n),t)) + \left(1-\frac{c}{2}\right)\phi(M(x_n,p,t))\\ &+\mu(2\phi(M(x_n,f(x_n),t)) + \phi(M(x_n,f(p),t))\\ &+\phi(M(p,f(x_n),t)) + 2\phi(M(f(x_n),f^2(x_n),t)))\\ \Rightarrow (1-\mu)\limsup_{n\to\infty}\phi(M(x_n,f(p),t)) &\leq \left(1-\frac{c}{2}+\mu\right)\limsup_{n\to\infty}\phi(M(x_n,p,\tfrac{t}{2}))\end{aligned}$$

$\Rightarrow \sup_t\{\limsup_{n\to\infty}\phi(M(x_n,f(p),t))\} \leq \sup_t\{\limsup_{n\to\infty}\phi(M(x_n,p,t))\}$
$\Rightarrow r(f(p),\{x_n\}) \leq r(p,\{x_n\})$.
So, $f(p) \in \mathcal{C}(E,\{x_n\})$.
Again since X is uniformly ϕ convex, $f(p) = p$, i.e., $p \in F(f)$.
Hence, $F(f) = \phi$.‘ □

Example 2.19. *Let* $X = [0,3]$, $d(x,y) = |x-y|$ *and* $M(x,y,t) = 1 - \frac{d(x,y)}{g(t)}$ *where* $g : (0,\infty) \longrightarrow]3,\infty[$. *Let* $W(x,y,\alpha) = \alpha x + (1-\alpha)\,y,\quad \alpha \in [0,1]$ *bethe* ϕ *convex structure on* X *for* $\phi(l) = 1-l,\ l \in [0,1]$. *Then* (X, M, T) *isa* ϕ *convex and complete fuzzy metric space [8] with the t-norm* $T_L(a,b) = \max(a+b-1,0)\ \ x,y \in X$.
Define f *on* X *by*

$$f(x) = \begin{cases} 0, & \textit{if } x = 3,\\ 1, & \textit{if } x = 3.\end{cases}$$

Let $x_0 = 1$, *then we can construct a sequence* $\{x_n\}$ *by iteration (2.4) with* $\xi_n = \xi \in (0,1)$ *and* $\eta_n = \eta \in (0,1)$, *where* $x_n = (1-\xi)^n(1-\eta)^n$, $\forall\ n \in \mathbf{N}$.

$$\begin{aligned} \textit{Now, } \lim_{n\to\infty} \phi(M(x_n, f(x_n), t)) &= \lim_{n\to\infty} \tfrac{|x_n - f(x_n)|}{g(t)} \\ &= \lim_{n\to\infty} \tfrac{|(1-\xi)^n(1-\eta)^n|}{g(t)} \\ &= 0. \end{aligned}$$

Thus $F(f) \neq \phi$, *by the above theorem. Also, clearly* $0 \in F(f)$.

Now, we discuss the convergence of following type of extended Picard-Mann hybrid iteration scheme with respect to the ϕ convex structure W for a fuzzy metric space X:

$$\begin{aligned} & x_1 \in X, \\ & y_n = W(x_n, f(x_n), u_n), \\ (2.13)\quad & z_n = W(x_n, f(y_n), v_n), \\ & x_{n+1} = f(z_n), \\ & n \in \mathbf{N}, u_n, v_n \in [0,1] \text{ for each } n \text{ and } \textstyle\sum_n u_n = \infty, \sum_n v_n = \infty. \end{aligned}$$

Lemma 2.20. *Let* E *be a nonempty closed* ϕ *convex subset of a complete fuzzy metric space* (X, M, T) *and* f *be a self-mapping on* E *satisfying the fuzzy* $B_{\gamma,\mu}$ *condition with respect to* ϕ *on* E. *Let* $x_1 \in E$ *and* $\{x_n\}$ *be a sequence in* E *defined by the iteration scheme (2.13). Then* $\lim\limits_{n\to\infty} \phi(M(x_n, p, t))$ *exists for all* $p \in F(f)$.

Proof. Similar to Lemma 2.16 □

Theorem 2.21. *Let* f *be a self-mapping on a nonempty closed* ϕ *convex subset* E *of a uniformly* ϕ *convex and complete fuzzy metric space* (X, M, T). *Let* f *satisfy the fuzzy* $B_{\gamma,\mu}$ *condition with respect to* ϕ *on* E. *For* $x_1 \in E$, $\{x_n\}$ *be the sequence in* E *defined by the above iteration scheme (2.13) where* $u_n, v_n \in (0,1)$ *and* $\lim\limits_{n\to\infty} v_n = j(\neq 1)$. *Then* $F(f) \neq \phi$ *if and only if* $\{x_n\}$ *is bounded and* $\lim\limits_{n\to\infty} \phi(M(x_n, f(x_n), t)) = 0$.

Proof. Suppose that $F(f) \neq \phi$ and $p \in F(f)$.

From Lemma 2.20 we have $\lim\limits_{n\to\infty} \phi(M(x_n, p, t))$ exists and $\{x_n\}$ is bounded.

Let

$$\lim_{n\to\infty} \phi(M(x_n, p, t)) = q. \tag{2.14}$$

From Lemma 2.8,

$$\begin{aligned} & \phi(M(f(x_n),p,t)) \leq \phi(M(x_n,p,t)) \\ \Rightarrow \quad & \limsup_{n\to\infty} \phi(M(f(x_n),p,t)) \leq \limsup_{n\to\infty} \phi(M(x_n,p,t)) = q. \end{aligned} \tag{2.15}$$

Now, using ϕ convexity of E,

$$\begin{aligned} \phi(M(y_n,p,t)) &= \phi(M(W(x_n,f(x_n),u_n),p,t)) \\ &\leq u_n\ \phi(M(x_n,p,t)) + (1-u_n)\phi(M(f(x_n),p,t)) \\ &\leq u_n\ \phi(M(x_n,p,t)) + (1-u_n)\phi(M(x_n,p,t)) \\ &= \phi(M(x_n,p,t)) \\ \Rightarrow \limsup_{n\to\infty} \phi(M(y_n,p,t)) &\leq \limsup_{n\to\infty} \phi(M(x_n,p,t)) = q. \end{aligned} \tag{2.16}$$

Again,

$$\begin{aligned} \phi(M(x_{n+1},p,t)) &= \phi(M(f(z_n),p,t)) \\ &\leq \phi(M(z_n,p,t)) \\ &= \phi(M(W(x_n,f(y_n),v_n),p,t)) \\ &\leq v_n\ \phi(M(x_n,p,t)) + (1-v_n)\phi(M(f(y_n),p,t)) \\ &\leq v_n\ \phi(M(x_n,p,t)) + (1-v_n)\phi(M(y_n,p,t)) \end{aligned}$$

$$\begin{aligned} &\Rightarrow \liminf_{n\to\infty} \phi(M(x_{n+1},p,t)) \leq \liminf_{n\to\infty} v_n\ \phi(M(x_n,p,t)) + \liminf_{n\to\infty} (1-v_n)\phi(M(y_n,p,t)) \\ &\Rightarrow (1-j)q \leq\ (1-j)\ \liminf_{n\to\infty} \phi(M(y_n,p,t)) \\ &\Rightarrow q \leq \liminf_{n\to\infty} \phi(M(y_n,p,t)). \end{aligned} \tag{2.17}$$

Therefore, from (2.16) and (2.17),

$$\begin{aligned} q \leq\ & \liminf_{n\to\infty} \phi(M(y_n,p,t)) \leq \limsup_{n\to\infty} \phi(M(y_n,p,t)) \leq q \\ \Rightarrow\ & \lim_{n\to\infty} \phi(M(y_n,p,t)) = q \\ \Rightarrow\ & \lim_{n\to\infty} \phi(M(W(x_n,f(x_n),u_n),p,t)) = q. \end{aligned} \tag{2.18}$$

So, from (2.14),(2.15),(2.18) and Lemma 2.17 we have,
$\lim_{n\to\infty} \phi(M(x_n,f(x_n),t)) = 0.$
The Converse part follows similar to Theorem 2.18. □

Remark:
We have proved Theorem 2.18 and Theorem 2.21 assuming the uniform ϕ convexity of the fuzzy metric space X. It is interesting to ask whether similar results hold without using this condition.

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