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Prime rings with involution involving left multipliers

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Abstract:

Let $\mathcal{R}$ be a prime ring of characteristic different from 2 with involution '$*$' of the second kind and $n \geq 1$ be a fixed positive integer. In the present paper it is shown that if $\mathcal{R}$ admits nonzero left multipliers S and T, then the following conditions are equivalent: $(i)\, \mathcal{R}$ is commutative, $(ii)\, T^n([x,x^*]) \in Z(\mathcal{R})$ for all $x \in \mathcal{R}$; $(iii)\, T^n(x \circ x^*) \in Z(\mathcal{R})$ for all $x \in \mathcal{R}$; $(iv)\, [S(x), T(x^*)] \in Z(\mathcal{R})$ for all $x \in \mathcal{R}$; $(v)\, [S(x), T(x^*)] - (x \circ x^*) \in Z(\mathcal{R})$ for all $x \in \mathcal{R}$; $(vi)\, S(x) \circ T(x^*) \in Z(\mathcal{R})$ for all $x \in \mathcal{R}$; $(vii)\, S(x) \circ T(x^*) - [x, x^*] \in Z(\mathcal{R})$ for all $x \in \mathcal{R}$. The existence of hypotheses in various theorems have been justified by the examples.

Keywords: Prime ring; Derivation; Multiplier; Involution; Commutativity.

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1. Introduction

Throughout this present paper, $\mathcal{R}$ will represent an associative ring with center $Z(\mathcal{R})$. For any $x, y \in \mathcal{R}$, the symbol $[x, y]$ denotes the commutator $xy - yx$; while the symbol xoy stands for the anticommutator $xy + yx$. A ring $\mathcal{R}$ is 2-torsion free if whenever $2x = 0$, with $x \in \mathcal{R}$ implies $x = 0$. A ring $\mathcal{R}$ is said to be prime if $a\mathcal{R}b = \{0\}$, where $a, b \in \mathcal{R}$, implies $a = 0$ or $b = 0$, and $\mathcal{R}$ is called semiprime ring in case $a\mathcal{R}a = \{0\}$ implies $a = 0$. An additive mapping $* : \mathcal{R} \to \mathcal{R}$ is called an involution on $\mathcal{R}$ if $*$ is an anti-automorphism of order 2; that is, $(x^*)^* = x$ for all $x \in \mathcal{R}$. An element x in a ring $\mathcal{R}$ with involution '$*$' is said to be hermitian if $x^* = x$ and skew-hermitian if $x^* = -x$. The sets of all hermitian and skew-hermitian elements of $\mathcal{R}$ will be denoted by $H(\mathcal{R})$ and $S(\mathcal{R})$ respectively. The involution is said to be of the first kind if $Z(\mathcal{R}) \subseteq H(\mathcal{R})$, otherwise it is said to be of the second kind. In the later case, $S(\mathcal{R}) \cap Z(\mathcal{R}) \neq \{0\}$. An element x is normal if $xx^* = x^*x$. If all elements in $\mathcal{R}$ are normal, then $\mathcal{R}$ is called a normal ring (or equivalently, $*$ commuting). A derivation on $\mathcal{R}$ is an additive mapping $d : \mathcal{R} \to \mathcal{R}$ such that $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in \mathcal{R}$. A derivation d is said to be inner, if there exists $a \in \mathcal{R}$ such that $d(x) = ax - xa$ for all $x \in \mathcal{R}$. Over the past three decades, several authors have investigated the relationship between the commutativity of the ring $\mathcal{R}$ and certain special types of mappings on $\mathcal{R}$. Following [9], an additive mapping $T : \mathcal{R} \to \mathcal{R}$ is said to be a left (respectively right) multiplier (centralizer) of $\mathcal{R}$ if $T(xy) = T(x)y$ (respectively $T(xy) = xT(y)$) holds for all $x, y \in \mathcal{R}$. An additive mapping T is called a multiplier if it is both left as well as right multiplier. We shall denote by C the extended centroid of a prime ring $\mathcal{R}$. There has been a great deal of work concerning multipliers on prime and semiprime rings (for reference one can see [3], [1], [5], [8] etc., where further references can be found). Many authors have obtained commutativity of prime or semiprime rings admitting various functions viz. derivations or automorphisms satisfying certain polynomial constraints (see for reference [2], [4], [6], & [7]). Recently the second author together with Ali [3] proved that if a prime ring $\mathcal{R}$ admits a left centralizer (multiplier) $T : \mathcal{R} \to \mathcal{R}$ such that $T([x, y]) = [x, y]$ with $T(x) \neq x$ for all $x, y \in I$, a nonzero ideal of $\mathcal{R}$, then $\mathcal{R}$ is commutative. This result was further extended by Ali and Dar [1] who considered the above problem in the setting of rings with involution '$*$' and obtained the commutativity of rings. In fact, they proved that if $\mathcal{R}$ is a prime ring with involution $*$ such that $char(\mathcal{R}) \neq 2$ and $\mathcal{R}$ admits a left multiplier $T : \mathcal{R} \to \mathcal{R}$

satisfying any one of the conditions: $(i)\, T([x, x^*]) = 0$; $(ii)\, T(x \circ x^*) = 0$; $(iii)\, T([x, x^*]) \pm [x, x^*] = 0$; $(iv)\, T(x \circ x^*) \pm (x \circ x^*) = 0$ for all $x \in \mathcal{R}$, then $\mathcal{R}$ is commutative. In the present paper, our objective is to generalize this result in a more general situation. In fact, we obtain commutativity of a prime ring $\mathcal{R}$ with involution '$*$' satisfying any one of the conditions: $(i)\, T([x, x^*]) \in Z(\mathcal{R})$ for all $x \in \mathcal{R}$; $(ii)\, T(x \circ x^*) \in Z(\mathcal{R})$ for all $x \in \mathcal{R}$; (iii) $[S(x), T(x^*)] \in Z(\mathcal{R})$ for all $x \in \mathcal{R}$; $(iv)\, [S(x), T(x^*)] - (x \circ x^*) \in Z(\mathcal{R})$ for all $x \in \mathcal{R}$; $(v)\, S(x) \circ T(x^*) \in Z(\mathcal{R})$ for all $x \in \mathcal{R}$; and $(vi)\, S(x) \circ T(x^*) - [x, x^*] \in Z(\mathcal{R})$ for all $x \in \mathcal{R}$.

2. Some preliminaries

We begin this section with the following lemmas which are essential for developing the proof of our main results. The proof of the first lemma is straightforward in the setting of prime rings and the proof of the next two Lemmas can be found in [8].

Lemma 2.1. *Let $\mathcal{R}$ be a prime ring. If z is a nonzero central element such that $xz \in Z(R)$, then $x \in Z(\mathcal{R})$.*

Lemma 2.2. [8, Lemma 2] *Let $\mathcal{R}$ be a prime ring and let $T : \mathcal{R} \to \mathcal{R}$ be a left centralizer. If $T(x) \in Z(R)$ holds for all $x \in \mathcal{R}$, then $T = 0$ or $\mathcal{R}$ is commutative.*

Lemma 2.3. [8, Lemma 3] *Let $\mathcal{R}$ be a noncommutative prime ring and let $S : \mathcal{R} \to \mathcal{R}$, $T : \mathcal{R} \to \mathcal{R}$ be left centralizers. Suppose that $[S(x), T(x)] = 0$ holds for all $x \in \mathcal{R}$. If $T \neq 0$, then there exists $\lambda \in C$ such that $S = \lambda T$.*

3. Main Results

Motivated by the notion of the left multiplier, Shakir and Dar [1] initiated the study of a more general concept by considering differentiate identities. More precisely, they proved that a prime ring $\mathcal{R}$ with involution $'*'$ of the second kind must be commutative if it admits a nonzero left centralizer T satisfying any one of the conditions: $T([x, x^*]) = 0$ for all $x \in \mathcal{R}$, $T(x \circ x^*) = 0$ for all $x \in \mathcal{R}$.

In ([1], Theorem 3.2) it is proved that if $(\mathcal{R}, *)$ is a ring with involution of the second kind provided with a nonzero left centralizer T which satisfies

$T(x \circ x^*) = 0$ for all $x \in \mathcal{R}$, then $\mathcal{R}$ is commutative. However, this result is not complete. Indeed, it is proved by the authors that $\mathcal{R}$ is commutative. In this case, it is obvious to see that $2T(x)x^* = 0$ for all $x \in \mathcal{R}$, since the characteristic of $\mathcal{R}$ is different from two, the last expression yields $T(x)x^* = 0$ for all $x \in \mathcal{R}$ and linearizing this, we find that $T(x)y + T(y^*)x^* = 0$ for all $x, y \in \mathcal{R}$. Replacing y by ys where $s \in S(\mathcal{R}) \cap Z(\mathcal{R})\backslash\{0\}$ and using the primeness of $\mathcal{R}$, we arrive at $T(x)y - T(y^*)x^* = 0$ for all $x, y \in \mathcal{R}$. Combining the last expressions, we find that $2T(x)y = 0$ for all $x, y \in \mathcal{R}$ which, because of the characteristic of $\mathcal{R}$ is different from two, forces $T(x)\mathcal{R}y = \{0\}$ for all $x, y \in \mathcal{R}$. In view of the primeness of $\mathcal{R}$, we conclude that $T = 0$; a contradiction. Our aim in the next theorem is to give a generalization of both results i.e., Theorem 3.1 & Theorem 3.2 obtained in [1].

Theorem 3.1. *Let $\mathcal{R}$ be a prime ring with involution $'*'$ of the second kind such that the characteristic of $\mathcal{R}$ is different from two. If $\mathcal{R}$ admits a nonzero left multiplier T, then the following assertions are equivalent:*

(i) $T([x, x^*]) \in Z(\mathcal{R})$ *for all* $x \in \mathcal{R}$;

(ii) $T(x \circ x^*) \in Z(\mathcal{R})$ *for all* $x \in \mathcal{R}$;

(iii) $\mathcal{R}$ *is commutative.*

Proof. It is easy to verify that $(iii) \Rightarrow (i)$ and $(iii) \Rightarrow (ii)$.
$(i) \Rightarrow (iii)$. By the assumption, we have

$$T([x, x^*]) \in Z(\mathcal{R}) \quad \text{for all } x \in \mathcal{R}. \tag{3.1}$$

Linearizing (3.1) and using the relation so obtained, we find that

$$T([x, y^*]) + T([y, x^*]) \in Z(\mathcal{R}) \quad \text{for all } x, y \in \mathcal{R}. \tag{3.2}$$

Replacing y by y^* in (3.2), we get

$$T([x, y]) + T([y^*, x^*]) \in Z(\mathcal{R}) \quad \text{for all } x, y \in \mathcal{R}. \tag{3.3}$$

Taking ys in place of y in (3.3) where $s \in S(\mathcal{R}) \cap Z(\mathcal{R})\backslash\{0\}$, we arrive at

$$\Big(T([x, y]) - T([y^*, x^*])\Big)s \in Z(\mathcal{R}) \quad \text{for all } x, y \in \mathcal{R}. \tag{3.4}$$

From Lemma 2.1, we have

$$T([x,y]) - T([y^*,x^*]) \in Z(\mathcal{R}) \quad \text{for all } x,y \in \mathcal{R}. \tag{3.5}$$

Combining (3.3) and (3.5) and using the characteristic of $\mathcal{R}$ is different from two, we obtain

$$T([x,y]) \in Z(\mathcal{R}) \quad \text{for all } x,y \in \mathcal{R}. \tag{3.6}$$

Replacing y by yx in (3.6), we get $T([x,y])x \in Z(\mathcal{R})$ for all $x,y \in \mathcal{R}$ and using (3.6) again together with Lemma 2.1, we conclude that either $T([x,y]) = 0$ or $x \in Z(\mathcal{R})$ for all $x,y \in \mathcal{R}$, and hence in both cases, we arrive at

$$T([x,y]) = 0 \quad \text{for all } x,y \in \mathcal{R}. \tag{3.7}$$

This reduces to $T(x)y = T(y)x$ for all $x,y \in \mathcal{R}$ and putting $[u,v]$ instead of y in the last expression and using (3.7), we arrive at

$$T(x)[u,v] = 0 \quad \text{for all } x,u,v \in \mathcal{R} \tag{3.8}$$

Replacing u by yu in (3.8) and using it again, we find that $T(x)y[u,v] = 0$. Since $T \neq 0$, by primeness of $\mathcal{R}$, it follows that $\mathcal{R}$ is commutative.
$(ii) \Rightarrow (iii)$. Suppose that

$$T(x \circ x^*) \in Z(\mathcal{R}) \quad \text{for all } x \in \mathcal{R}. \tag{3.9}$$

Linearizing (3.9) and using it again, we obtain

$$T(x \circ y^*) + T(y \circ x^*) \in Z(\mathcal{R}) \quad \text{for all } x,y \in \mathcal{R}. \tag{3.10}$$

Replacing y by y^* in (3.10), we get

$$T(x \circ y) + T(y^* \circ x^*) \in Z(\mathcal{R}) \quad \text{for all } x,y \in \mathcal{R}. \tag{3.11}$$

Taking ys instead of y in (3.11) where $s \in S(\mathcal{R}) \cap Z(\mathcal{R}) \backslash \{0\}$, we arrive at

$$(T(x \circ y) - T(y^* \circ x^*))s \in Z(\mathcal{R}) \quad \text{for all } x,y \in \mathcal{R}.$$

By Lemma 2.1, the above relation can be further written as

$$T(x \circ y) - T(y^* \circ x^*) \in Z(\mathcal{R}) \quad \text{for all } x,y \in \mathcal{R}. \tag{3.12}$$

Calculating the sum of equations (3.11), (3.12) and using the fact that the characteristic of $\mathcal{R}$ is different from two yields

$$T(x \circ y) \in Z(\mathcal{R}) \quad \text{for all} \quad x, y \in \mathcal{R}. \tag{3.13}$$

Replacing y by yx in (3.13), we obtain that $T(x \circ y)x \in Z(\mathcal{R})$ for all $x, y \in \mathcal{R}$ and using (3.13) again together with Lemma 2.1, the later expression can be rewritten as

$$T(x \circ y) = 0 \quad \text{or} \quad x \in Z(\mathcal{R}) \quad \text{for all} \quad x, y \in \mathcal{R}. \tag{3.14}$$

If there exists $x_0 \in \mathcal{R}$ such that $T(x_0 \circ y) = 0$ for all $y \in \mathcal{R}$, then $T(x_0)y + T(y)x_0 = 0$ for all $y \in \mathcal{R}$ and putting $x_0 \circ y$ in place of y in the last equation and using $T(x_0 \circ y) = 0$, we are forced to conclude that

$$T(x_0)(x_0 \circ y) = 0 \quad \text{for all} \quad y \in \mathcal{R}. \tag{3.15}$$

Replacing y by yt in (3.15) and using it, we find that $T(x_0)\mathcal{R}[x_0, t] = \{0\}$. By primeness of $\mathcal{R}$, we conclude that either $T(x_0) = 0$ or $x_0 \in Z(\mathcal{R})$. In the latter case, (3.14) becomes

$$T(x) = 0 \quad \text{or} \quad x \in Z(\mathcal{R}) \quad \text{for all} \quad x \in \mathcal{R}. \tag{3.16}$$

The sets $H = \{x \in \mathcal{R} \mid T(x) = 0\}$ and $K = \{x \in \mathcal{R} \mid x \in Z(\mathcal{R})\}$ are additive subgroups of $\mathcal{R}$. But a group cannot be the union of proper subgroups. Hence we get $H = \mathcal{R}$ or $K = \mathcal{R}$ which force that $\mathcal{R}$ is commutative because $T \neq 0$. This completes the proof of theorem.

If we put $T = I_{\mathcal{R}}$, we obtain the following result:

Corollary 3.1. *Let $\mathcal{R}$ be a prime ring with involution $'*'$ of the second kind such that the characteristic of $\mathcal{R}$ is different from two, then the following assertions are equivalent:*

(i) $[x, x^*] \in Z(\mathcal{R})$ *for all* $x \in \mathcal{R}$;

(ii) $x \circ x^* \in Z(\mathcal{R})$ *for all* $x \in \mathcal{R}$;

(iii) $\mathcal{R}$ *is commutative.*

It is straightforward to see that T is a left centralizer of a ring $\mathcal{R}$ if and only of T^n is a left centralizer of $\mathcal{R}$, where $n \geq 1$ is a fixed positive integer. Hence in view of the above result we have the following:

Corollary 3.2. *Let $\mathcal{R}$ be a prime ring with involution $'*'$ of the second kind such that the characteristic of $\mathcal{R}$ is different from two. If $\mathcal{R}$ admits a left multiplier T, then for a fixed positive integer $n \geq 1$ the following assertions are equivalent:*

(i) $T^n([x, x^*]) \in Z(\mathcal{R})$ *for all* $x \in \mathcal{R}$;

(ii) $T^n(x \circ x^*) \in Z(\mathcal{R})$ *for all* $x \in \mathcal{R}$;

(iii) $\mathcal{R}$ *is commutative.*

If a ring $\mathcal{R}$ admits a left multiplier T then it can be seen easily that T is a left multiplier on $\mathcal{R}$ if and only if $T + I_{\mathcal{R}}$ (resp. $T - I_{\mathcal{R}}$), where $I_{\mathcal{R}}$ is the identity map on $\mathcal{R}$, is a left multiplier on $\mathcal{R}$. By using induction on n, more generally, one can see that for any fixed positive integer $n \geq 1$, T is a left multiplier on $\mathcal{R}$ if and only if $T^n \pm I_{\mathcal{R}}$ is a left multiplier on $\mathcal{R}$. In view of the above theorem, our aim in the next theorem is to give a suitable conditions that assures the commutativity of $\mathcal{R}$.

Corollary 3.3. *Let $\mathcal{R}$ be a prime ring with involution $'*'$ of the second kind such that the characteristic of $\mathcal{R}$ is different from two. If $\mathcal{R}$ admits a left multiplier T such that $T(x) \neq \pm x$, for all $x \in \mathcal{R}$, then for a fixed positive integer $n \geq 1$ the following assertions are equivalent:*

(i) $T^n([x, x^*]) - [x, x^*] \in Z(\mathcal{R})$ *for all* $x \in \mathcal{R}$;

(ii) $T^n(x \circ x^*) - (x \circ x^*) \in Z(\mathcal{R})$ *for all* $x \in \mathcal{R}$;

(iii) $T^n([x, x^*]) + [x, x^*] \in Z(\mathcal{R})$ *for all* $x \in \mathcal{R}$;

(iv) $T^n(x \circ x^*) + (x \circ x^*) \in Z(\mathcal{R})$ *for all* $x \in \mathcal{R}$;

(v) $\mathcal{R}$ *is commutative.*

Theorem 3.2. *Let $\mathcal{R}$ be a prime ring with involution $'*'$ of the second kind such that the characteristic of $\mathcal{R}$ is different from two. If $\mathcal{R}$ admits a left multiplier T, then for a fixed positive integer $n \geq 1$ the following assertions are equivalent:*

(i) $T^n([x, x^*]) - (x \circ x^*) \in Z(\mathcal{R})$ *for all* $x \in \mathcal{R}$;

(ii) $T^n(x \circ x^*) - [x, x^*] \in Z(\mathcal{R})$ *for all* $x \in \mathcal{R}$;

(iii) $\mathcal{R}$ *is commutative.*

Proof. It is obvious that (iii) implies both of (i) and (ii). Hence it remains to prove that $(i) \Rightarrow (iii)$ and $(ii) \Rightarrow (iii)$. If $T = 0$, according to our Corollary 3.1, we conclude that $\mathcal{R}$ is commutative. Hence, we suppose that $T \neq 0$.
$(i) \Rightarrow (iii)$. Suppose that $\mathcal{R}$ satisfies

$$T^n([x,x^*]) - (x \circ x^*) \in Z(\mathcal{R}) \quad \text{for all} \;\; x \in \mathcal{R}. \tag{3.17}$$

Substituting $x + y^*$ in place of y in (3.17), we obtain

$$T^n([x,y]) - (x \circ y) + T^n([y^*,x^*]) - (y^* \circ x^*) \in Z(\mathcal{R}) \quad \text{for all} \;\; x, y \in \mathcal{R}. \tag{3.18}$$

Taking ys instead of y in (3.18) where $s \in S(\mathcal{R}) \cap Z(\mathcal{R}) \backslash \{0\}$ and using Lemma 2.1, we arrive at

$$T^n([x,y]) - (x \circ y) - T^n([y^*,x^*]) + (y^* \circ x^*) \in Z(\mathcal{R}) \quad \text{for all} \;\; x, y \in \mathcal{R}. \tag{3.19}$$

Combining (3.18), (3.19) and using the fact that characteristic of $\mathcal{R}$ is different from two, we deduce that

$$T^n([x,y]) - (x \circ y) \in Z(\mathcal{R}) \quad \text{for all} \;\; x, y \in \mathcal{R}. \tag{3.20}$$

In particular for $x = y$, (3.20) implies $x^2 \in Z(\mathcal{R})$ for all $x \in \mathcal{R}$. Replacing x by $x^2 + y$ in the last expression and using it with $char(\mathcal{R}) \neq 2$, we obtain $x^2 y \in Z(\mathcal{R})$ for all $x, y \in \mathcal{R}$. By Lemma 2.1, it is easy to see that $\mathcal{R}$ is commutative.
$(ii) \Rightarrow (iii)$. By the hypothesis, we have

$$T^n(x \circ x^*) - [x,x^*] \in Z(\mathcal{R}) \quad \text{for all} \;\; x, y \in \mathcal{R}. \tag{3.21}$$

Taking $x + y^*$ in place of y in (3.21), we obtain

$$T^n(x \circ y) - [x,y] + T^n(y^* \circ x^*) - [y^*,x^*] \in Z(\mathcal{R}) \quad \text{for all} \;\; x, y \in \mathcal{R}. \tag{3.22}$$

Letting ys in place of y in (3.22) where $s \in S(\mathcal{R}) \cap Z(\mathcal{R}) \backslash \{0\}$ and using Lemma 2.1, we arrive at

$$T^n(x \circ y) - [x,y] - T^n(y^* \circ x^*) + [y^*,x^*] \quad \text{for all} \;\; x, y \in \mathcal{R}. \tag{3.23}$$

Combining (3.22) and (3.23), we deduce

$$T^n(x \circ y) - [x, y] \in Z(\mathcal{R}) \quad \text{for all } x, y \in \mathcal{R}. \tag{3.24}$$

In particular for $x = y$, by using the characteristic of $\mathcal{R}$ is different from two, (3.24) implies $T^n(x)x \in Z(\mathcal{R})$ for all $x \in \mathcal{R}$. Replacing y by x^2 in (3.24) and using Lemma 2.1, we arrive at either $T^n(x^2) = 0$ or $x \in Z(\mathcal{R})$ for all $x \in \mathcal{R}$. If there is $x_0 \in \mathcal{R}$ such that $T^n(x_0^2) = 0$, choosing $x = x_0^2$ and $y = x_0^2 y$ in (3.24), we arrive at $x_0^2[x_0^2, y] \in Z(\mathcal{R})$ for all $y \in \mathcal{R}$. By Lemma 2.1, we obtain $x_0^2 \in Z(\mathcal{R})$ and from the above, we conclude that $x^2 \in Z(\mathcal{R})$ for all $x \in \mathcal{R}$ which forces that $\mathcal{R}$ is commutative. This completes the proof of theorem.

The following example demonstrates that the restriction of the *second kind involution* in the hypotheses of the above theorem is indispensable.

Example 3.1. *Let* $\mathcal{R} = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mid \alpha, \beta, \gamma, \delta \in \mathbf{Z} \right\}$. *It is obvious that R is prime ring. Next, define mappings* $T : \mathcal{R} \to \mathcal{R}$ *by* $T \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ 0 & 0 \end{pmatrix}$, *and* $* : \mathcal{R} \to \mathcal{R}$ *such that* $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^* = \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix}$. *Obviously,* $Z(\mathcal{R}) = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \mid a \in \mathbf{Z} \right\}$. *Then* $x^* = x$ *for all* $x \in Z(\mathcal{R})$, *and hence* $Z(\mathcal{R}) \subseteq H(\mathcal{R})$, *which shows that the involution* $'*'$ *is of the first kind. Moreover,* T *is a nonzero left multiplier on* $\mathcal{R}$ *which satisfies the conditions:*

(i) $T^n([x, x^*]) \in Z(\mathcal{R})$ *for all* $x \in \mathcal{R}$;

(ii) $T^n(x \circ x^*) \in Z(\mathcal{R})$ *for all* $x \in \mathcal{R}$;

(iii) $T^n(x \circ x^*) \pm [x, x^*] \in Z(\mathcal{R})$ *for all* $x \in \mathcal{R}$;

(iv) $T^n([x, x^*]) \pm (x \circ x^*) \in Z(\mathcal{R})$ *for all* $x \in \mathcal{R}$.

However, $\mathcal{R}$ *is not commutative.*

Theorem 3.3. *Let* $\mathcal{R}$ *be a prime ring with involution* $'*'$ *of the second kind such that the characteristic of* $\mathcal{R}$ *is different from two. If* $\mathcal{R}$ *admits nonzero left multipliers* S *and* T, *then the following assertions are equivalent:*

(i) $[S(x), T(x^*)] \in Z(\mathcal{R})$ *for all* $x \in \mathcal{R}$;

(ii) $[S(x), T(x^*)] - x \circ x^* \in Z(\mathcal{R})$ *for all* $x \in \mathcal{R}$;

(iii) $\mathcal{R}$ *is commutative.*

Proof. Note that (iii) implies both of (i) and (ii) is clear. It remains to show that $(i) \Rightarrow (iii)$ and $(ii) \Rightarrow (iii)$.
$(i) \Rightarrow (iii)$. In view of the hypothesis, we have

$$[S(x), T(x^*)] \in Z(\mathcal{R}) \quad \text{for all } \ x \in \mathcal{R}. \tag{3.25}$$

Replacing x by $x + y^*$ in (3.25) and using it with a simple calculation, we obtain

$$[S(x), T(y)] + [S(y^*), T(x^*)] \in Z(\mathcal{R}) \quad \text{for all } \ x, y \in \mathcal{R}. \tag{3.26}$$

Putting ys instead of y where $s \in S(\mathcal{R}) \cap Z(\mathcal{R}) \backslash \{0\}$ in (3.26) and using Lemma 2.1, we arrive at

$$[S(x), T(y)] - [S(y^*), T(x^*)] \in Z(\mathcal{R}) \quad \text{for all } \ x, y \in \mathcal{R}. \tag{3.27}$$

Calculating the sum of the two relations (3.26), (3.27) and applying the fact that the characteristic of $\mathcal{R}$ is different from two, we get $[S(x), T(y)] \in Z(\mathcal{R})$ for all $x, y \in \mathcal{R}$. Replacing y by $yS(x)$, we get $[S(x), T(y)]S(x) \in Z(\mathcal{R})$ for all $x, y \in \mathcal{R}$, and by Lemma 2.1, we deduce that $[S(x), T(y)] = 0$ or $S(x) \in Z(\mathcal{R})$ for all $x, y \in \mathcal{R}$, both cases lead to $[S(x), T(y)] = 0$ for all $x, y \in \mathcal{R}$. Substituting yrz in place of y in the last expression, we can easily conclude that $T(y)\mathcal{R}[S(x), z] = \{0\}$ for all $x, y, z \in \mathcal{R}$. Since $T \neq \{0\}$, by primeness of $\mathcal{R}$, we find that $S(x) \in Z(\mathcal{R})$ for all $x \in \mathcal{R}$ and Lemma 2.2 forces that $\mathcal{R}$ is commutative.
$(ii) \Rightarrow (iii)$ We have, from assumption

$$[S(x), T(x^*)] - x \circ x^* \in Z(\mathcal{R}) \quad \text{for all } \ x \in \mathcal{R}. \tag{3.28}$$

Linearizing (3.28) and using the same techniques as we have already used above, we find that

$$[S(x), T(y)] - x \circ y \in Z(\mathcal{R}) \quad \text{for all } \ x, y \in \mathcal{R}. \tag{3.29}$$

Replacing y by $yS(x)$ in (3.29), we arrive at

$$[S(x), T(y)]S(x) - x \circ yS(x) \in Z(\mathcal{R}) \quad \text{for all } \ x, y \in \mathcal{R} \tag{3.30}$$

and therefore

$$([S(x),T(y)] - x \circ y)S(x) + y([S(x),x]) \in Z(\mathcal{R}) \quad \text{for all } x,y \in \mathcal{R}. \tag{3.31}$$

Applying (3.29) in the relation (3.31), we obtain

$$y[S(x),x]S(x) = S(x)y[S(x),x] \quad \text{for all } x,y \in \mathcal{R}. \tag{3.32}$$

Replacing y by yt in (3.32) and using it again, we arrive at

$$[S(x),y]t[S(x),x] = 0 \quad \text{for all } x,y,t \in \mathcal{R}$$

which can be written as

$$[S(x),x]\mathcal{R}[S(x),x] = \{0\} \quad \text{for all } x \in \mathcal{R}.$$

Since $\mathcal{R}$ is prime, we find that $[S(x),x] = 0$ for all $x \in \mathcal{R}$. In this case, (3.31) becomes

$$([S(x),T(y)] - x \circ y)S(x) \in Z(\mathcal{R}) \quad \text{for all } x,y \in \mathcal{R}. \tag{3.33}$$

Invoking Lemma 2.1, in the above relation, we find that

$$[S(x),T(y)] - x \circ y = 0 \ \text{ or } \ S(x) \in Z(\mathcal{R}) \quad \text{for all } x,y \in \mathcal{R}. \tag{3.34}$$

Suppose there exists $x_0 \in \mathcal{R}$ such that

$$[S(x_0),T(y)] = x_0 \circ y \text{ for all } y \in \mathcal{R}. \tag{3.35}$$

Replacing y by yt in (3.35), we can easily find that $T(y)[S(x_0),t] = y[x_0,t]$ for all $y,t \in \mathcal{R}$. Taking again y^2r in place of y, we arrive at $[T(y),y]\mathcal{R}[S(x_0),t] = \{0\}$ for all $y,t \in \mathcal{R}$. Since $\mathcal{R}$ is prime we obtain either $[T(y),y] = 0$ for all $y \in \mathcal{R}$ or $S(x_0) \in Z(\mathcal{R})$. In this case, (3.34) becomes $[T(y),y] = 0$ for all $y \in \mathcal{R}$ or $\mathcal{R}$ is commutative by Lemma 2.2.
Now assume the first case and replacing y by $S(x_0)$ in (3.35), we get $x_0 \circ S(x_0) = 0$. Since $[S(x_0),x_0] = 0$, by the last conclusion we can conclude $x_0S(x_0) = S(x_0)x_0 = S(x_0^2) = 0$.

In this case, for $x = x_0^2$, (3.29) implies $x_0^2 \circ y \in Z(\mathcal{R})$ for all $y \in \mathcal{R}$. Replacing y by x_0y in the last expression and using it again with Lemma 2.1, we arrive at $x_0^2 \circ y = 0$ or $x_0 \in Z(\mathcal{R})$ for all $y \in \mathcal{R}$. Putting ty instead of y, we get either $\mathcal{R}[x_0^2,y] = \{0\}$ or $x_0 \in Z(\mathcal{R})$ for all $y \in \mathcal{R}$ which can be further written as $x_0^2 \in Z(\mathcal{R})$ this with (3.35) forces that $x^2 \in Z(\mathcal{R})$ for all $x \in \mathcal{R}$ and in this case $\mathcal{R}$ is commutative.

Theorem 3.4. *Let $\mathcal{R}$ be a prime ring with involution $'*'$ of the second kind such that the characteristic of $\mathcal{R}$ is different from two. If $\mathcal{R}$ admits nonzero left multipliers S and T, then the following assertions are equivalent:*

(i) $S(x) \circ T(x^*) \in Z(\mathcal{R})$ *for all* $x \in \mathcal{R}$;

(ii) $S(x) \circ T(x^*) - [x, x^*] \in Z(\mathcal{R})$ *for all* $x \in \mathcal{R}$;

(iii) $\mathcal{R}$ *is commutative.*

Proof. It is immediate that (iii) implies both of (i) and (ii). It remains to show that $(i) \Rightarrow (iii)$ and $(ii) \Rightarrow (iii)$.
$(i) \Rightarrow (iii)$. We are given that

$$S(x) \circ T(x^*) \in Z(\mathcal{R}) \quad \text{for all} \quad x \in \mathcal{R}.$$

Replacing x by $x + y^*$ in the above expression and using it, we get

$$S(x) \circ T(y) + S(y^*) \circ T(x^*) \in Z(\mathcal{R}) \quad \text{for all} \quad x, y \in \mathcal{R}. \tag{3.36}$$

Putting ys instead of y where $s \in S(\mathcal{R}) \cap Z(\mathcal{R}) \backslash \{0\}$ in (3.36) and using Lemma 2.1, we arrive at

$$S(x) \circ T(y) - S(y^*) \circ T(x^*) \in Z(\mathcal{R}) \quad \text{for all} \quad x, y \in \mathcal{R}. \tag{3.37}$$

Combining (3.36), (3.37) and applying the fact that the characteristic of $\mathcal{R}$ is different from two, we get

$$S(x) \circ T(y) \in Z(\mathcal{R}) \quad \text{for all} \quad x, y \in \mathcal{R}. \tag{3.38}$$

Replacing y by $yS(x)$ in (3.38) one can see that $(S(x) \circ T(y))S(x) \in Z(\mathcal{R})$ for all $x, y \in \mathcal{R}$. By Lemma 2.1, we deduce

$$S(x) \circ T(y) = 0 \;\; \text{or} \;\; S(x) \in Z(\mathcal{R}) \quad \text{for all} \quad x, y \in \mathcal{R}. \tag{3.39}$$

If there is $x_0 \in \mathcal{R}$ such that $S(x_0) \in Z(\mathcal{R})$, then (3.38) implies that $2S(x_0)T(y) \in Z(\mathcal{R})$ for all $y \in \mathcal{R}$ and by Lemma 2.1, we arrive at $S(x_0) = 0$ or $T(y) \in Z(\mathcal{R})$ for all $y \in \mathcal{R}$. Since $T \neq 0$, either $S(x_0) = 0$ or $\mathcal{R}$ is commutative by Lemma 2.2. In this case, (3.39) becomes

$$S(x) \circ T(y) = 0 \quad \text{for all} \quad x, y \in \mathcal{R} \;\; \text{or} \;\; \mathcal{R} \text{ is commutative.} \tag{3.40}$$

If $S(x) \circ T(y) = 0$ for all $x, y \in \mathcal{R}$, then $S(x)T(y) = -T(y)S(x)$ for all $x, y \in \mathcal{R}$. Substituting yz instead of y, we arrive at $T(y)[S(x), z] = 0$ for all

$x, y, z \in \mathcal{R}$, from which we can easily conclude that $T(y)\mathcal{R}[S(x), z] = \{0\}$ for all $x, y, z \in \mathcal{R}$. Since $T \neq 0$, in view of the primeness of $\mathcal{R}$, we find that $S(x) \in Z(\mathcal{R})$ for all $x \in \mathcal{R}$ which forces that $\mathcal{R}$ is commutative.
$(ii) \Rightarrow (iii)$. Suppose that

$$S(x) \circ T(x^*) - [x, x^*] \in Z(\mathcal{R}) \in Z(\mathcal{R}) \quad \text{for all } x \in \mathcal{R}. \tag{3.41}$$

Replacing x by $x + y^*$ in (3.41) and using it again, we find that

$$S(x) \circ T(y) - [x, y] \in Z(\mathcal{R}) \quad \text{for all } x, y \in \mathcal{R}. \tag{3.42}$$

Replacing y by $yS(x)$ in (3.42), we arrive at

$$(S(x) \circ T(y) - [x, y])S(x) - y([S(x), x]) \in Z(\mathcal{R}) \quad \text{for all } x, y \in \mathcal{R}. \tag{3.43}$$

This implies that

$$y([S(x), x])S(x) = S(x)y([S(x), x]) \quad \text{for all } x, y \in \mathcal{R}. \tag{3.44}$$

Since this equation is the same as (3.32), by reasoning as above, it is obvious to see that $[S(x), x] = 0$ for all $x \in \mathcal{R}$ in this case and by using Lemma 2.1, the equation (3.43) becomes

$$S(x) \circ T(y) = [x, y] \text{ or } S(x) \in Z(\mathcal{R}) \quad \text{for all } x, y \in \mathcal{R}. \tag{3.45}$$

Suppose there is $x_0 \in \mathcal{R}$ such that $S(x_0) \in Z(\mathcal{R})$, then replacing x by x_0 and y by yx_0 respectively in (3.42), we obtain $(2S(x_0)T(y) - [x_0, y])x_0 \in Z(\mathcal{R})$ for all $y \in \mathcal{R}$ from which and with Lemma 2.1, we obtain $2S(x_0)T(y) - [x_0, y] = 0$ or $x_0 \in Z(\mathcal{R})$ for all $y \in \mathcal{R}$. Suppose we have the second case, then (3.42) becomes $2S(x_0)T(y) \in Z(\mathcal{R})$ for all $y \in \mathcal{R}$. Using Lemma 2.1 with the fact that the characteristic of $\mathcal{R}$ is different from two, we find that $S(x_0) = 0$ or $T(y) \in Z(\mathcal{R})$ for all $y \in \mathcal{R}$. By Lemma 2.2, we arrive at $S(x_0) \circ T(y) = [x_0, y]$ for all $y \in \mathcal{R}$ or $\mathcal{R}$ is commutative. In this case, it follows from (3.45) that

$$S(x) \circ T(y) = [x, y] \quad \text{for all } x, y \in \mathcal{R} \text{ or } \mathcal{R} \text{ is commutative.} \tag{3.46}$$

Now suppose that

$$S(x) \circ T(y) = [x, y] \quad \text{for all } x, y \in \mathcal{R}. \tag{3.47}$$

Replacing y by yt in (3.47), we obtain

$$(S(x) \circ T(y))t + T(y)[t, S(x)] = [x, yt] \quad \text{for all } x, y, t \in \mathcal{R}.$$

In light of (3.47), the above relation yields that

$$[x, y]t + T(y)[t, S(x)] = [x, yt] \quad \text{for all } x, y, t \in \mathcal{R}.$$

By a simple calculation, we can conclude that

$$T(y)[t, S(x)] = [x, yt] - [x, y]t = y[x, t] \quad \text{for all } x, y, t \in \mathcal{R}. \tag{3.48}$$

Putting yu instead of y in the latter equation, we get

$$T(y)u[t, S(x)] = yu[x, t] \quad \text{for all } u, x, y, t \in \mathcal{R}.$$

Using (3.47), we have

$$T(y)u[t, S(x)] = yT(u)[t, S(x)] \quad \text{for all } u, x, y, t \in \mathcal{R}.$$

Replacing t by rt, The last equation can be further written as

$$(T(y)u - yT(u))\mathcal{R}[t, S(x)] = \{0\} \quad \text{for all } u, x, y, t \in \mathcal{R}.$$

As $\mathcal{R}$ is prime, by Lemma 2.2, we obtain $T(y)u = yT(u)$ for all $u, y \in \mathcal{R}$ or $\mathcal{R}$ is commutative. Suppose we have the first case, then T is commuting and putting $T(x)$ instead of t in (3.48), we obtain $T(y)\mathcal{R}[T(x), S(x)] = \{0\}$ for all $x, y \in \mathcal{R}$. The primeness of $\mathcal{R}$ gives $[T(x), S(x)] = 0$ for all $x \in \mathcal{R}$. Taking $y = x$ in (3.47), we have $S(x) \circ T(x) = 0$ for all $x \in \mathcal{R}$. Calculating the sum of these two relations yields $S(x)T(x) = T(x)S(x) = 0$ for all $x \in \mathcal{R}$. Then, a linearization of $T(x)S(x) = 0$ for all $x \in \mathcal{R}$ forces $T(x)S(y) + T(y)S(x) = 0$ for all $x, y \in \mathcal{R}$ and by left multiplying it by $S(x)$, we arrive at $S(x)T(y)S(x) = 0$ for all $x, y \in \mathcal{R}$. Putting yt instead of y and using the primeness of $\mathcal{R}$, we find that $S(x)T(y) = 0$ for all $x, y \in \mathcal{R}$. Proceeding as above, we can deduce that $S = T = 0$, yielding a contradiction. Thereby the proof is completed.

Theorem 3.5. *Let $\mathcal{R}$ be a noncommutative prime ring with extended centroid C and involution $'*'$ of the second kind such that the characteristic of $\mathcal{R}$ is different from two and let S, T be nonzero left multipliers on $\mathcal{R}$. Then there exists $\lambda \in C$ such that $S = \lambda T$ if $\mathcal{R}$ has one of the following properties:*

(i) $[S(x) \circ T(x^*)] - [x, x^*] \in Z(\mathcal{R})$ *for all* $x \in \mathcal{R}$.

(ii) $S(x) \circ T(x^*) - x \circ x^* \in Z(\mathcal{R})$ *for all* $x \in \mathcal{R}$.

Proof. (i) By assumption, we have

$$[S(x), T(x^*)] - [x, x^*] \in Z(\mathcal{R}) \quad \text{for all } \ x \in \mathcal{R}. \tag{3.49}$$

Linearizing (3.49) and using similar argument as we have used above, we get

$$[S(x), T(y)] - [x, y] \in Z(\mathcal{R}) \quad \text{for all } \ x, y \in \mathcal{R}. \tag{3.50}$$

Replacing y by $yS(x)$ in (3.50), we arrive at

$$[S(x), T(y)]S(x) - [x, yS(x)] \in Z(\mathcal{R}) \quad \text{for all } \ x, y \in \mathcal{R}. \tag{3.51}$$

Thus our identity reduces to

$$([S(x), T(y)] - [x, y])S(x) - y([x, S(x)]) \in Z(\mathcal{R}) \quad \text{for all } \ x, y \in \mathcal{R}. \tag{3.52}$$

It now follows from (3.50) and (3.52) that

$$y([x, S(x)]S(x) = y([x, S(x)] \quad \text{for all } \ x, y \in \mathcal{R}. \tag{3.53}$$

Moreover, since this equation is the same as (3.32), then reasoning as above we can show that $[x, S(x)] = 0$ for all $x \in \mathcal{R}$. Therefore, by using Lemma 2.1, equation (3.52) becomes

$$[S(x), T(y)] = [x, y] \ \text{ or } \ S(x) \in Z(\mathcal{R}) \quad \text{for all } \ x, y \in \mathcal{R}. \tag{3.54}$$

If there is $x_0 \in \mathcal{R}$ such that $S(x_0) \in Z(\mathcal{R})$ by (3.50), we obtain $[x_0, y] \in Z(\mathcal{R})$ for all $y \in \mathcal{R}$ from which it is very easy to prove that $x_0 \in Z(\mathcal{R})$. Therefore, by (3.54), we must have

$$[S(x), T(y)] = [x, y] \quad \text{for all } \ x, y \in \mathcal{R}. \tag{3.55}$$

Replacing y by yt in (3.55), thus we can write

$$[S(x), T(y)]t + T(y)[S(x), t] = [x, yt] \quad \text{for all } \ x, y, t \in \mathcal{R}.$$

Using (3.55), we get

$$[x, y]t + T(y)[S(x), t] = [x, yt] \quad \text{for all } \ x, y, t \in \mathcal{R}.$$

By a simple calculation, it is very easy to conclude that

$$(3.56) \quad T(y)[S(x),t] = [x,yt] - [x,y]t = y[x,t] \quad \text{for all } x,y,t \in \mathcal{R}.$$

Since (3.56) is the same as (3.48), then proceeding on similar lines after (3.48), we can prove $[S(x),T(x)] = 0$ for all $x \in \mathcal{R}$, in this case Lemma 2.2 forces the required result.

(ii) Assuming that

$$(3.57) \qquad S(x) \circ T(x^*) - x \circ x^* \in Z(\mathcal{R}) \quad \text{for all } x \in \mathcal{R}.$$

Replacing x by $x + y^*$ in (3.57) and using it, we obtain

$$(3.58) \qquad S(x) \circ T(y) - x \circ y \in Z(\mathcal{R}) \quad \text{for all } x,y \in \mathcal{R}.$$

Replacing y by $yS(x)$ in (3.58) and developing this expression, we obtain

$$(3.59) \qquad (S(x) \circ T(y) - (x \circ y)S(x) + y[S(x),x] \in Z(\mathcal{R}) \quad \text{for all } x,y \in \mathcal{R}.$$

Which implies that $S(x)y[S(x),x] = y[S(x),x]S(x)$ for all $x,y \in \mathcal{R}$, using the same previous techniques, we arrive at $[S(x),x] = 0$ for all $x \in \mathcal{R}$, in this case, (3.59) becomes

$$(3.60) \qquad (S(x) \circ T(y) - x \circ y)S(x) \in Z(\mathcal{R}) \quad \text{for all } x,y \in \mathcal{R}.$$

By Lemma 2.1, (3.60) forces that

$$(3.61) \qquad S(x) \circ T(y) = x \circ y \ \text{ or } \ S(x) \in Z(\mathcal{R}) \quad \text{for all } x,y \in \mathcal{R}.$$

We may assume that there exists $x_0 \in \mathcal{R}$ such that $S(x_0) \in Z(\mathcal{R})$. Using (3.58) and choosing $x = x_0$ and $y = ux_0$, then we get $(2S(x_0)T(u) - x_0 \circ u)x_0 \in Z(\mathcal{R})$ for all $u \in \mathcal{R}$ and by Lemma 2.1, it follows that either $2S(x_0)T(u) - x_0 \circ u = 0$ or $x_0 \in Z(\mathcal{R})$ for all $u \in \mathcal{R}$. Suppose we have the second case, then (3.58) becomes $2S(x_0)T(y) - 2x_0y \in Z(\mathcal{R})$ for all $y \in \mathcal{R}$ and replacing y by yt and using the fact that the characteristic of $\mathcal{R}$ is different from two, we obtain $(S(x_0)T(y) - x_0y)t \in Z(\mathcal{R})$ for all $y,t \in \mathcal{R}$ by Lemma 2.1, we find that $S(x_0)T(y) = x_0y$ for all $y \in \mathcal{R}$ or $\mathcal{R}$ is commutative. In this case, (3.61) becomes

$$(3.62) \quad S(x) \circ T(y) = x \circ y \quad \text{for all } x,y \in \mathcal{R} \ \text{ or } \ \mathcal{R} \text{ is commutative.}$$

Suppose that

$$S(x) \circ T(y) = x \circ y \quad \text{for all } x, y \in \mathcal{R}. \tag{3.63}$$

Replacing y by yt in (3.63), we obtain

$$(S(x) \circ T(y))t + T(y)[t, S(x)] = x \circ yt \quad \text{for all } x, y, t \in \mathcal{R}.$$

Using (3.63), the last expression becomes

$$(x \circ y)t + T(y)[t, S(x)] = x \circ yt \quad \text{for all } x, y, t \in \mathcal{R}.$$

Accordingly, we get

$$T(y)[t, S(x)] = y[t, x] \quad \text{for all } x, y, t \in \mathcal{R}. \tag{3.64}$$

Since (3.64) is the same as (3.56), proceeding in the similar manner as above, we conclude the required result.

The following corollaries are immediate consequences of the above result.

Corollary 3.4. *Let $\mathcal{R}$ be a prime ring with involution $'*'$ of the second kind such that the characteristic of $\mathcal{R}$ is different from two. If $\mathcal{R}$ admits nonzero left multipliers S and T, then the following assertions are equivalent:*

(i) $S(x) \circ T(x^*) + [x, x^*] \in Z(\mathcal{R})$ *for all* $x \in \mathcal{R}$*;*

(ii) $[S(x), T(x^*)] + x \circ x^* \in Z(\mathcal{R})$ *for all* $x \in \mathcal{R}$*;*

(iii) $\mathcal{R}$ *is commutative.*

Corollary 3.5. *Let $\mathcal{R}$ be a noncommutative prime ring with extended centroid C and involution $'*'$ of the second kind such that the characteristic of $\mathcal{R}$ is different from two and let S, T be nonzero left multipliers on $\mathcal{R}$. Then there exists $\lambda \in C$ such that $S = \lambda T$ if $\mathcal{R}$ has one of the following properties:*

(i) $[S(x), T(x^*)] + [x, x^*] \in Z(\mathcal{R})$ *for all* $x \in \mathcal{R}$*;*

(ii) $S(x) \circ T(x^*) + x \circ x^* \in Z(\mathcal{R})$ *for all* $x \in \mathcal{R}$*;*

The following example demonstrates that the condition "primeness of $\mathcal{R}$" in various Theorems is crucial.

Example 3. Let $\mathcal{R} = \left\{ \begin{pmatrix} 0 & \alpha & \beta \\ 0 & 0 & \gamma \\ 0 & 0 & 0 \end{pmatrix} \mid \alpha, \beta, \gamma \in \mathcal{S} \right\}$ where $\mathcal{S}$ is a non-commutative ring of characteristic different from 2 such that $s^2 = 0$ for all $s \in \mathcal{S}$. It is obvious that $\mathcal{R}$ is not prime ring. Next, we define the maps $T, S, * : \mathcal{R} \to \mathcal{R}$ by $T \begin{pmatrix} 0 & \alpha & \beta \\ 0 & 0 & \gamma \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \gamma \\ 0 & 0 & 0 \end{pmatrix}$, $S \begin{pmatrix} 0 & \alpha & \beta \\ 0 & 0 & \gamma \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & \alpha & \beta \\ 0 & 0 & \gamma \\ 0 & 0 & 0 \end{pmatrix}^* = \begin{pmatrix} 0 & \gamma & -\beta \\ 0 & 0 & \alpha \\ 0 & 0 & 0 \end{pmatrix}$. It is easy to see that T is a nonzero left multiplier and $'*'$ an involution of the second kind on $\mathcal{R}$ which satisfies the conditions:

(i) $T^n([x, x^*]) \in Z(\mathcal{R})$;

(ii) $T^n(x \circ x^*) \in Z(\mathcal{R})$;

(ii) $T^n([x, x^*]) \pm x \circ x^* \in Z(\mathcal{R})$;

(iv) $T^n(x \circ x^*) \pm [x, x^*] \in Z(\mathcal{R})$

(v) $[S(x), T(x^*)] \in Z(\mathcal{R})$

(vi) $[S(x), T(x^*)] \pm (x \circ x^*) \in Z(\mathcal{R})$;

(vii) $S(x) \circ T(x^*) \in Z(\mathcal{R})$;

(viii) $S(x) \circ T(x^*) \pm [x, x^*] \in Z(\mathcal{R})$;

for all $x \in \mathcal{R}$. However, $\mathcal{R}$ is not commutative.

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