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Topological properties of some sequences defined over n−normed spaces

Tanweer Jalal[1] orcid.org/0000-0003-0676-1356

[1]National Institute of Technology, Dept. of Mathematics, Srinagar, JK, India.
tjalal@nitsri.net

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Abstract:

Some classes of real number sequences over n−normed spaces defined by means of Orlicz functions, a bounded sequence of strictly positive real numbers, a multiplier and a normal paranormed sequence space are investigated. Relevant properties of such classes have been investigated. Moreover, relationships among different such classes of sequences have also been studied under various parameters and conditions. Finally, the spaces are investigated for some other useful properties.

Keywords: Orlicz function; n−norm; Paranormed spaces; Completeness; Solidity.

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1. Preliminiries and Definitions

The concept of 2-normed spaces was initially introduced by Gahler [9] , in the mid of 1960's, while that of n-normed spaces can be found in Misiak [27] . Since then, many others have studied this concept and obtained various results, Gunawan [11, 12], Gunawan and Mashadi [13] and many others.

Let n be a non-negative integer and X be a real vector space of dimension d where $d \geq n$. A real-valued function $\|.,\ldots,.\|$ on X^n satisfying the following conditions :
$(N1)$ $\|(x_1, x_2, \ldots, x_n)\| = 0$ if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent,
$(N2)$ $\|(x_1, x_2, \ldots, x_n)\|$ is invariant under permutation,
$(N3)$ $\|\alpha(x_1, x_2, \ldots, x_n)\| = |\alpha|\|(x_1, x_2, \ldots, x_n)\|$, for any $\alpha \in \mathbf{R}$,
$(N4)$ $\|(x_1 + x, x_2, \ldots, x_n)\| \leq \|(x_1, x_2, \ldots, x_n)\| + \|(x, x_2, \ldots, x_n)\|$
is called an $n-$ norm on X and the pair $(X, \|.,\ldots,.\|)$ is called an $n-$ normed space.

A trivial example of an $n-$ normed space is $X = \mathbf{R}^n$ equipped with the Euclidean $n-$ norm $\|(x_1, x_2, \ldots, x_n)\|_E =$ volume of the n-dimensional parallelepiped spanned by the vectors $x_1, x_2, \ldots, x_n$ which may be given explicitly by the formula

$$\|(x_1, x_2, \ldots, x_n)\|_E = |\det(x_{ij})| = abs\,(\det(\langle x_i, x_j\rangle))$$

where $x_i = (x_{i1}, x_{i2}, \ldots, x_{in}) \in \mathbf{R}^n$ for each $i = 1, 2, 3, ..., n$.

The standard n -norm on X a real inner product space of dimension $d \geq n$ is defined as follows:

$$\|(x_1, x_2, \ldots, x_n\|_S = [\det(\langle x_i, x_j\rangle)]^{1/2},$$

where $\langle .,. \rangle$ denotes the inner product on X. If we take $X = \mathbf{R}^n$ then this n -norm is exactly the same as the Euclidean n -norm mentioned earlier. For $n = 1$ this n-norm is the usual norm $\|x_1\| = \sqrt{\langle x_1, x_1\rangle}$ for further details refer to Gunawan [11].
We first introduce the following definitions :
A sequence (x_k) in an n-normed space $(X, \|.,\ldots,.\|)$ is said to be convergent to some $L \in X$ if

$$\lim_{k\to\infty} \|x_k - L, z_1, z_2, \ldots, z_{n-1}\| = 0,\ for\ every\ z_1, z_2, \ldots, z_{n-1} \in X.$$

A sequence (x_k) in an n -normed space $(X, \|.,\ldots,.\|)$ is said to be Cauchy if

$$\lim_{k,p\to\infty} \|x_k - x_p, z_1, z_2, \ldots, z_{n-1}\| = 0,\ for\ every\ z_1, z_2, \ldots, z_{n-1} \in X.$$

If every Cauchy sequence space in X converges to some $L \in X$ then X is said to be complete with respect to the n-norm. A complete n -normed space is said to be a n- Banach space.

The details about above and associated notions and results, we refer to Gurdal and Sahiner [14], Savas [33] , Jalal [17, 18, 19, 20] and Dutta [5].

The work of this paper is related to functional analytic study of Orlicz sequence space as well as composite Orlicz sequence spaces of real number over $n-$normed spaces. From functional analytic point of view, the Orlicz sequence spaces are the special cases of Orlicz spaces studied in Krasnoselskii and Rutisky [23]. Lindenstrauss and Tzafriri [24] first investigated Orlicz sequence spaces in detail with certain aims in Banach space theory.

A function $M : [0, \infty) \to [0, \infty)$ which is continuous, non-decreasing and convex with $M(0) = 0,\ M(x) > 0$ as $x > 0$ and $M(x) \to \infty$ as $x \to \infty$ is called an Orlicz function.

A function M is said to satisfy Δ_2- condition for all values of x, if there exists constant K such that $M(2x) \leq KM(x), x \geq 0$. The Δ_2- condition implies $M(2x) \leq Kl^{\log_2 L}M(x), x \geq 0, l > 1$. Also an Orlicz function satisfies the inequality $M(\lambda x) \leq \lambda M(x)$ for all λ with $0 < \lambda < 1$.

If convexity of Orlicz function is replaced by $M(x+y) \leq M(x)+M(y)$ then the function reduces to a modulus function. For more details about this function and its subsequent use, one may refer to Krasnoselskii and Rutisky [23], Kamthan and Gupta [21] , Rao and Ren [30] , Ruckle [31], Maddox [26], Ghosh and Srivastava [10] , Jalal and Rather [16] , Altin [2] , Debnath and Saha [4] and many others.

Lindenstrauss and Tzafriri [24] studied some Orlicz type sequence spaces defined as follows:

$$\ell_M = \left\{ (x_k) \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty,\ for\ some\ \ \rho > 0 \right\}.$$

The space ℓ_M with the norm

$$\|x\| = \inf\left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1\right\},$$

becomes a Banach space which is called an Orlicz sequence space, where ω is the family of real or complex sequences. The space ℓ_M is closely related to the space ℓ_p which is an Orlicz sequence space with $M(t) = |t|^p$, for $1 \leq p < \infty$. Esi et al. [8], Nuray and Glc [29], Mursaleen et al. [28],Ahmad and Bataineh [1] , Bektas and Altin [3], Savas [32], Isik [15], Dutta and Basar [6], Karakaya and Dutta [21], Dutta and Jebril [7] and many others have used Orlicz functions to construct several new sequence spaces.

Let P be a subset of the set of all scalar valued sequences ω. Now we recall the following notions.

A scalar valued paranormed (Maddox [25]) sequence space $(P,\ g_p)$ where g_p is a paranorm on P is called monotone paranormed space if $x = (x_k) \in P, y = (y_k) \in P$ and $|x_k| \leq |y_k|$ for all k implies $g_p(x) \leq g_p(y)$. P is called normal or solid if $y = (y_k) \in P$
$i \geq 1$ for some $x = (x_k) \in P$.
whenever $|y_i| \leq |x_i|$, A sequence space P with linear topology is called a

K -space provided each of the maps $p_i : P \to \mathbf{N}, p_i(x) = x_i$ continuous for

$i \geq 1$.

A sequence space P is said to be symmetric if $(X_{\pi(k)}) \in P$ whenever $(X_k) \in P$ where π is permutation of $\mathbf{N}$.

A sequence space P is said to be convergence free if $(X_k) \in P$ whenever $(Y_k) \in P$ and $Y_k = 0$ implies $X_k = 0$.

Let (P, g_p) be a paranormed space and $(a_n) \subset P$ where $a_n = (a_k^n)$. If $a_k^n \to 0$ as $n \to \infty$ for each k implies $g_p(a_n) \to 0$ as $n \to \infty$, then we say then we say that co-ordinate wise convergence implies convergence in g_p e.g., c_0, ℓ_1, ℓ_∞, etc.

The following inequalities (Maddox [25]) will be used throughout the paper.

Proposition Let $p = (p_k)$ be a bounded sequence of strictly positive real numbers with $0 \leq p_k \leq \sup_k p_k = H, D = max\{1, 2^{H-1}\}$. Then
$(i)|a_k + b_k|^{p_k} \leq S\left(|a_k|^{p_k} + |b_k|^{p_k}\right);$

$(ii)|\lambda|^{p_k} \leq max(1, [\lambda]^H)$.

2. The new class $Z(\|.,\ldots,.\|,M,p,s)$ and some other classes

In this section, we construct the new sets to be investigated and give a few descriptions of such sets along with intended aims for results concerning the sets and their possible extensions and derivatives

Let (Z,g_z) be a normal paranormed sequence space with paranorm g_z which satisfies the following properties:
(i) g_z is a monotone paranorm;
(ii) coordinate wise convergence implies convergence in paranorm g_z , which implies that for each $(X^n)=(X_k^n)\in\mathbf{Z},n,k\in\mathbf{N}$,

$$X_k^n\to 0\ as\ n\to\infty\ (for\ each\ k)g_Z(X^n)\to 0,\ as\ n\to\infty.$$

Let M be a Orlicz function and $(T,\|.,\ldots,.\|)$ be a n-normed space. We now define the new class of sequences as follows for every $z_1,z_2,\ldots,z_{n-1}\in T$:

$$Z(\|.,\ldots,.\|,M,p,s)$$

$$=\left\{X=(X_k):X_k\in\left(k^{-s}\left[M\left(\frac{\|X_k,z_1,z_2,\ldots,z_{n-1}\|}{\rho}\right)\right]\right)^{p_k}\in Z, for\ some\ \rho>0\right\},$$

where $s\geq 0$ and p_k is a bounded sequence of strictly positive real numbers with $\inf p_k>0$.
This class give rises different other classes of sequences as follows:

$$\mathrm{Z}(\|.,\ldots,.\|,\mathrm{M^r},\mathrm{p},\mathrm{s})$$

$$=\left\{X=(X_k):X_k\in\left(k^{-s}\left[M^r\left(\frac{\|X_k,z_1,z_2,\ldots,z_{n-1}\|}{\rho}\right)\right]\right)^{p_k}\in Z,\right.$$

$$\left. for\ some\ \rho>0\right\},$$

where r is any positive integer.

$$Z(\|.,\ldots,.\|,M,s)$$

$$=\left\{X=(X_k):X_k\in\left(k^{-s}\left[M\left(\frac{\|X_k,z_1,z_2,\ldots,z_{n-1}\|}{\rho}\right)\right]\right)\in\mathbf{Z}, for\ some\ \rho>0\right\},$$

$Z(\|.,\ldots,.\|,p,s)$

$$=\left\{X=(X_k): X_k \in \left(k^{-s}\left[M\left(\|X_k,z_1,z_2,\ldots,z_{n-1}\|\right)\right]\right)^{p_k} \in Z\right\}$$

and so on.

We define a function on $Z(\|.,\ldots,.\|,M,p,s)$ as follows which is proved to be a paranorm in the next section:

$$X=(X_k)\in Z(\|.,\ldots,.\|,M,p,s) \text{ and } z_1,z_2,\ldots,z_{n-1}\in T,$$

$$g(X)=\inf\left\{\rho^{\frac{p_k}{D}}>0:\left[g_z\left(k^{-s}\left[M^r\left(\frac{\|X_k,z_1,z_2,\ldots,z_{n-1}\|}{p}\right)\right]\right)^{p_k}\right]^{\frac{1}{D}} \leq 1, k=1,2,..\right\} \tag{2.1}$$

Where $D=max(1,H), H=\sup_k p_k<\infty$ and $\inf p_k>0$.

The above classes of sequences of real numbers give rise to many well known sequence spaces on specifying the space Z, the Orlicz function M, the bounded sequence p_k of positive real numbers,$s\geq 0$ and the base space $(T,\|,...,.\|)$. Further, we can derive several other similar classes for study. The main results of the paper are obtained using the properties of Orlicz functions, n-norm spaces and most importantly that are of normal paranormed spaces with monotone paranorm and coordinate wise convergence property. One may find it interesting and useful to study further the sets for several other algebraic and topological properties as well as convergence and completeness related and geometric properties.

3. Main results

In this section, we first examine the linearity of the sets defined above. Then the sets will be investigated for completeness under a suitably defined paranorm. Further, the sets will be examined for K-space property. The next few results will be given for the set $Z(\|.,\ldots,.\|,M,p,s)$ only as for other sets the proofs can be obtained applying similar arguments.

Theorem 3.1 The set $Z(\|.,\ldots,.\|,M,p,s)$ is linear over the set $\mathbf{R}$ of real numbers.

Proof. Let $X=(X_k),Y=(Y_k)\in Z(\|.,\ldots,.\|,M,p,s)$ and $\alpha,\beta\in\mathbf{R}$. Then there exists some positive numbers ρ_1 and ρ_2 such that for every $z_1,z_2,\ldots,z_n\in T$

$$\left(k^{-s}\left[M\left(\frac{\|X_k,z_1,z_2,\ldots,z_{n-1}\|}{\rho_1}\right)\right]^{p_k}\right)\in Z$$

and

$$\left(k^{-s}\left[M\left(\frac{\|Y_k,z_1,z_2,\ldots,z_{n-1}\|}{\rho_2}\right)\right]^{p_k}\right)\in Z.$$

Let us choose $\rho=max\{2|\alpha|\rho_1,2|\beta|\rho_2\}$ so that

$$\begin{aligned}
&\mathrm{k}^{-s}\left[M\left(\frac{\|\alpha X_k+\beta Y_k,z_1,z_2,\ldots,z_{n-1}\|}{\rho}\right)\right]^{p_k}\\
&\leq [k^{-s}\left[M\left(\frac{\|\alpha X_k,z_1,z_2,\ldots,z_{n-1}\|+\|\beta Y_k,z_1,z_2,\ldots,z_{n-1}\|}{\rho_1}\right)\right]^{p_k}\\
&\leq k^{-s}\left[M\left(|\alpha|\frac{\|X_k,z_1,z_2,\ldots,z_{n-1}\||}{\rho_1}+|\beta|\frac{\|Y_k,z_1,z_2,\ldots,z_{n-1}\||}{\rho_2}\right)\right]^{p_k}\\
&\leq k^{-s}\frac{1}{2^{p_k}}\left[M\left(\frac{\|X_k,z_1,z_2,\ldots,z_{n-1}\||}{\rho_1}+\frac{\|Y_k,z_1,z_2,\ldots,z_{n-1}\||}{\rho_2}\right)\right]^{p_k}\\
&< k^{-s}\left[M\left(\frac{\|X_k,z_1,z_2,\ldots,z_{n-1}\||}{\rho_1}+\frac{\|Y_k,z_1,z_2,\ldots,z_{n-1}\||}{\rho_2}\right)\right]^{p_k}\\
&\leq Ck^{-s}\left[M\left(\frac{\|X_k,z_1,z_2,\ldots,z_{n-1}\||}{\rho_1}\right)\right]^{p_k}\\
&+Ck^{-s}\left[M\left(\frac{\|Y_k,z_1,z_2,\ldots,z_{n-1}\||}{\rho_2}\right)\right]^{p_k}\in Z
\end{aligned}$$

where $C=max\{1,2^{H-1}\}$. Thus $\alpha X+\beta Y\in Z(\|.,\ldots,.\|,M,p,s)$.

Theorem 3.2 $Z(\|.,\ldots,.\|,M,p,s)$ is a paranormed space under the function g given by Eq. (2.1).

Proof. Since g_z is a paranorm on Z, by definition $g(X)\geq 0$, $\forall X\in Z(\|.,\ldots,.\|,M,p,s)$. Clearly, $g(\theta)=0$. Again, by property (N3) in the definition, $g(-X)=g(X)$ holds for all $X\in Z(\|.,\ldots,.\|,M,p,s)$. Also, by taking $\alpha = \beta = 1$ in Theorem 3.1 and using the fact that $g_z$is monotone, we get $g(X+Y) \leq g(X)+g(Y)$ for $X = (X_k), Y = (Y_k) \in Z(\|.,\ldots,.\|, M, p, s)$. We are only left to show that g is continuous under scalar multiplication. Let λ be any number. Then for some $\rho > 0$, we have

$$g(\lambda X) = \inf\left\{\rho^{\frac{p_k}{D}} > 0 : \left[g_z\left(k^{-s}\left[M\left(\frac{\|\lambda X_k, z_1, z_2, \ldots, z_{n-1}\|}{\rho}\right)\right]\right)^{p_k}\right]^{\frac{1}{D}} \leq 1, k = 1, 2, ..\right\}$$

$$= \inf\left\{\rho^{\frac{p_k}{D}} > 0 : \left[g_z\left(k^{-s}\left[M\left(\frac{|\lambda|\|X_k, z_1, z_2, \ldots, z_{n-1}\|}{\rho}\right)\right]\right)^{p_k}\right]^{\frac{1}{D}} \leq 1, k = 1, 2, ..\right\}.$$

Let $r = \rho/|\lambda|$. Then

$$g(\lambda X) = \inf\left\{(|\lambda| r)^{\frac{p_k}{D}} > 0 : \left[g_z\left(k^{-s}\left[M\left(\frac{\|X_k, z_1, z_2, \ldots, z_{n-1}\|}{\rho}\right)\right]\right)^{p_k}\right]^{\frac{1}{D}} \leq 1, k = 1, 2, ..\right\}$$

Since $|\lambda|^{p_k} \leq max(1, |\lambda|^H)$. So $|\lambda|^{\frac{p_k}{D}} \leq \left(max(1, |\lambda|^H)\right)^{\frac{1}{D}}$. Therefore, it converges to zero if $g(X)$ converges to zero in $Z(\|.,\ldots,.\|, M, p, s)$.

Now suppose $\lambda_n \to 0$ as $n \to \infty$ and let $X = (X_k) \in Z(\|.,\ldots,.\|, M, p, s)$. Let $\epsilon > 0$ be arbitrarily chosen and let K be a positive integer such that for some $\rho > 0$,

$$g_z\left(k^{-s}\left[M\left(\frac{\|X_k, z_1, z_2, \ldots, z_{n-1}\|}{\rho}\right)\right]^{p_k}\right) < \frac{\epsilon}{2}, \ for\ k > K$$

which implies for $k > K$

$$\left[g_z\left(k^{-s}\left[M\left(\frac{\|X_k, z_1, z_2, \ldots, z_{n-1}\|}{\rho}\right)\right]^{p_k}\right)\right]^{\frac{1}{D}} \leq \frac{\epsilon}{2}.$$

Let $0 < |\lambda| < 1$, using convexity of M and the property (N3) of n-norm, for $k > K$ we get

$$\begin{aligned} &g_z\left(k^{-s}\left[M\left(\frac{\|\lambda X_k, z_1, z_2, \ldots, z_{n-1}\|}{\rho}\right)\right]^{p_k}\right) \\ &= g_z\left(k^{-s}\left[M\left(\frac{|\lambda|\|X_k, z_1, z_2, \ldots, z_{n-1}\|}{\rho}\right)\right]^{p_k}\right) \\ &< g_z\left(k^{-s}\left[|\lambda|M\left(\frac{\|X_k, z_1, z_2, \ldots, z_{n-1}\|}{\rho}\right)\right]^{p_k}\right) \\ &< g_z\left(k^{-s}\left[M\left(\frac{\|X_k, z_1, z_2, \ldots, z_{n-1}\|}{\rho}\right)\right]^{p_k}\right) \\ &< \left(\frac{\epsilon}{2}\right)^{D}. \end{aligned}$$

Since M is continuous everywhere in $[0, \infty)$ and by definition of g_z, it follows that for $k \leq K$

$$\phi(t) = g_z\left(k^{-s}\left[M\left(\frac{\|tX_k, z_1, z_2, \ldots, z_{n-1}\|}{\rho}\right)\right]^{p_k}\right)$$

is continuous at 0. So there is $0 < \delta < 1$ such that $|\phi(t)| < \epsilon/2$ for $0 < t < \delta$. let L be such that $|\lambda_n| < \delta$ for $n > L$, then

$$\left[g_z\left(k^{-s}\left[M\left(\frac{\|\lambda_n X_k, z_1, z_2, \ldots, z_{n-1}\|}{\rho}\right)\right]^{p_k}\right)\right]^{\frac{1}{D}} < \frac{\epsilon}{2}.$$

for $n > L$ and $k \leq K$. hence

$$\left[g_z\left(k^{-s}\left[M\left(\frac{\|\lambda_n X_k, z_1, z_2, \ldots, z_{n-1}\|}{\rho}\right)\right]^{p_k}\right)\right]^{\frac{1}{D}} < \epsilon,$$

for $n > L$ and for all k. Hence $\lambda_n X \to \theta$ as $n \to \infty$.

Theorem 3.3. Let the base space $(T, \|., ..., .\|)$ be a n- Banach Space. Then $Z(\|., \ldots, .\|, M, p, s)$ is a complete paranormed space under the paranorm g given by (2.1). where Z is a K-space.

Proof. Let (X^i) be a Cauchy sequence in $Z(\|., \ldots, .\|, M, p, s)$. Then $g(X^i - X^j) \to 0$ as $i, j \to \infty$. For any given $\epsilon > 0$, let r and x_0 be such that $\dfrac{\epsilon}{rx_0} > 0$ and $M\left(\dfrac{\epsilon}{rx_0}\right) \geq \sup_{k\geq 1} k^{s/p_k}$.
Now $g(X^i - X^j) \to 0$ as $i, j \to \infty$ implies that there exist $N_0 \in \mathbf{N}$ such that

$$g\left(X^i - X^j\right) < \frac{\epsilon}{rx_0} \; for\ all\ i, j \geq N_0.$$

Then we have for all $i, j \geq N_0$ such that for every $z_1, z_2, \ldots, z_{n-1} \in T$,

$$\inf\left\{\rho^{\frac{p_k}{D}} > 0 : \left[g_z\left(k^{-s}\left[M\left(\frac{\|X_k^i - X_k^j, z_1, z_2, \ldots, z_{n-1}\|}{\rho}\right)\right]\right)^{p_k}\right]^{\frac{1}{D}} \leq 1,\right.$$

$$\left. k = 1, 2, ..\right\} < \frac{\epsilon}{rx_0}.$$

Hence we have for every $z_1, z_2, \ldots, z_{n-1} \in T$,

$$g_z\left(k^{-s}\left[M\left(\frac{\|X_k^i - X_k^j, z_1, z_2, \ldots, z_{n-1}\|}{g(X^i - X^j)}\right)\right]^{p_k}\right) \leq 1 \; for\ i, j \geq N_0.$$

Since Z is a K-space, $p_k \geq 0$ and we can choose s suitably so that

$$k^{-s}\left[M\left(\frac{\|X_k^i - X_k^j, z_1, z_2, \ldots, z_{n-1}\|}{g(X^i - X^j)}\right)\right]^{p_k} \leq 1$$

for each k and for $i, j \geq N_0$ and $z_1.z_2, \ldots, z_{n-1} \in T$. Therefore,

$$M\left(\frac{\|X_k^i - X_k^j, z_1, z_2, \ldots, z_{n-1}\|}{g(X^i - X^j)}\right) \leq k^{s/p_k} \leq M\left(\frac{rx_0}{2}\right).$$

Thus we get

$$\|X_k^i - X_k^j, z_1, z_2, \ldots, z_{n-1}\| < \frac{\epsilon}{rx_0}\frac{rx_0}{2} = \frac{\epsilon}{2}$$

for each k and for $i, j \geq N_0$ and for every $z_1.z_2, \ldots, z_{n-1} \in T$. Therefore (X_k^i) becomes a Cauchy sequence in T. Since $(T, \|., \ldots, .\|)$ is complete, there exist $X = (X_k) \in T$ such that $X_k^i \to X_k$ as $i \to \infty$ for each k. Since M is continuous it follows that

$$M\left(\frac{\|X_k - X_k^j, z_1, z_2, \ldots, z_{n-1}\|}{\rho}\right) \to 0 \; as \; i \to \infty.$$

for each $z_1, z_2, \ldots, z_{n-1} \in T$ and for some $\rho > 0$. Consequently,

$$k^{-s}\left[M\left(\frac{\|X_k - X_k^j, z_1, z_2, \ldots, z_{n-1}\|}{\rho}\right)\right]^{p_k} \to 0 \; as \; i \to \infty$$

for each $k, z_1, z_2, \ldots, z_{n-1} \in T$ and for some $\rho > 0$.

Let

$$\alpha_k^j = k^{-s}\left[M\left(\frac{\|X_k - X_k^j, z_1, z_2, \ldots, z_{n-1}\|}{\rho}\right)\right]^{p_k}.$$

Then since M is non-decreasing, by suitable choice of δ (depending on j and k),

$$\alpha_k^j < \delta k^{-s}\left[M\left(\frac{\|X_k^j, z_1, z_2, \ldots, z_{n-1}\|}{\rho}\right)\right]^{p_k}$$

where $0 < \delta < 1$. Since Z is normal, it follows that $(\alpha^i) \in Z$ for each i. Also $\alpha_k^i \to 0$ as $i \to \infty$ implies that $g_Z(\alpha^i) \to 0$ as $i \to \infty$. Hence $X^i \to {}^g X$ as $i \to \infty$ in $Z(\|., \ldots, .\|, M, p, s)$.

Again

$$k^{-s}\left[M\left(\frac{\|X_k, z_1, z_2, \ldots, z_{n-1}\|}{\rho}\right)\right]^{p_k}$$

$$= k^{-s}\left[M\left(\frac{\|X_k^i + (X_k - X_k^i), z_1, z_2, \ldots, z_{n-1}\|}{\rho}\right)\right]^{p_k}$$

$$\leq Ck^{-s}\left[M\left(\frac{\|X_k^i, z_1, z_2, \ldots, z_{n-1}\|}{\rho}\right)\right]^{p_k} + C\alpha_k^i, where \; C = max\{1, 2^{H-1}\}$$

$$\leq C(1+\delta)k^{-s}\left[M\left(\frac{\|X_k^i, z_1, z_2, \ldots, z_{n-1}\|}{\rho}\right)\right]^{p_k}.$$

Since $(X^i) \in Z(\|., \ldots, .\|, M, p, s)$ and Z is a normal space, it seems that $X = (X_k) \in Z(\|., \ldots, .\|, M, p, s)$.

Hence the proof is complete.

Theorem 3.4. $Z(\|.,\ldots,.\|,M,p,s)$ is a K-space if Z is a K-space.

Proof. Let us define a mapping

$$P_n : Z(\|.,\ldots,.\|,M,p,s) \to T$$

by $P_n(X) = X_n$, for all $n \in \mathbf{N}$. To show P_n is continuous.
Let (X^m) be a sequence in $Z(\|.,\ldots,.\|,M,p,s)$ such that $X^m \to {}^g0$ as $m \to \infty$. Then for some suitable choice of $\rho > 0$,

$$\left[g_z\left(k^{-s}\left[M\left(\frac{\|X_k^m, z_1, z_2, \ldots, z_{n-1}\|}{\rho}\right)\right]^{p_k}\right)\right]^{1/D} \to 0 \ as \ m \to \infty.$$

Since Z is a K-space, this implies that for each k and as m tending to ∞

$$k^{-s}\left[M\left(\frac{\|X_k^m, z_1, z_2, \ldots, z_{n-1}\|}{\rho}\right)\right]^{p_k} \to \infty$$

for some $\rho > 0$. Since M is an Orlicz function, it follows that

$$\|X_k^m, z_1, z_2, \ldots, z_{n-1}\| \to 0 \ as \ m \to \infty.$$

Consequently, $X^m \to 0$ in T. Hence the Proof.

4. Relationship Results

In this section, we shall investigate the relationship among the spaces defined in second section and their possible variants under different conditions.

Theorem 4.1. Let M_1 and M_2 be two Orlicz functions. Then

$$Z(\|.,\ldots,.\|,M_1,p,s) \cap Z(\|.,\ldots,.\|,M_2,p,s) \subseteq Z(\|.,\ldots,.\|,M_1+M_2,p,s)$$

where Z is a normal sequence space.

Proof. Let $X = (X_k) \in Z(\|.,\ldots,.\|,M_1,p,s) \cap Z(\|.,\ldots,.\|,M_2,p,s)$. Then we can choose $\rho_1, \rho_2 > 0$, such that

$$k^{-s}\left[M_1\left(\frac{\|X_k, z_1, z_2, \ldots, z_{n-1}\|}{\rho}\right)\right]^{p_k} \in Z$$

and

$$k^{-s}\left[M_2\left(\frac{\|X_k,z_1,z_2,\ldots,z_{n-1}\|}{\rho}\right)\right]^{p_k}\in Z.$$

Let us choose $\rho = max(\rho_1,\rho_2)$. Then

$$\text{k}^{-s}\left[(M_1+M_2)\left(\frac{\|X_k,z_1,z_2,\ldots,z_{n-1}\|}{\rho}\right)\right]^{p_k}$$
$$\leq k^{-s}C\left\{\left[M_1\left(\frac{\|X_k,z_1,z_2,\ldots,z_{n-1}\|}{\rho_1}\right)\right]^{p_k}+\left[M_2\left(\frac{\|X_k,z_1,z_2,\ldots,z_{n-1}\|}{\rho_2}\right)\right]^{p_k}\right\}\in$$
Z,
where $C = max(1,2^{H-1})$.
Now the proof follows immediately as Z being normal.

Theorem 4.2. Let M_1 and M_2 be Orlicz functions satisfying Δ_2-condition. then we have the following inclusion

$$Z(\|.,\ldots,.\|,M_1,p,s)\subseteq Z(\|.,\ldots,.\|,M_2oM_1,p,s)\ for\ s>1.$$

Proof. Let $X = (X_k) \in Z(\|.,\ldots,.\|,M_1,p,s)$. Since M_2 is continuous from the right at 0, there exists $0<\xi<1$ such that for any arbitrary $\epsilon>0, M_2(t)<\epsilon$ whenever $0\leq t\leq\xi$. Let us define the sets

$$A_1=\left\{k\in\mathbf{N}:\left[M_1\left(\frac{\|X_k,z_1,z_2,\ldots,z_{n-1}\|}{\rho}\right)\right]\leq\xi\right\}$$

$$A_2=\left\{k\in\mathbf{N}:\left[M_2\left(\frac{\|X_k,z_1,z_2,\ldots,z_{n-1}\|}{\rho}\right)\right]>\xi\right\}$$

for some $\rho>0$.

If $k\in A_2$,

$$M_1\left(\frac{\|X_k,z_1,z_2,\ldots,z_{n-1}\|}{\rho}\right)<\frac{1}{\xi}M_1\left(\frac{\|X_k,z_1,z_2,\ldots,z_{n-1}\|}{\rho}\right)$$
$$<1+\left[\frac{1}{\xi}M_1\left(\frac{\|X_k,z_1,z_2,\ldots,z_{n-1}\|}{\rho}\right)\right].$$

Since M_2 is non-decreasing and convex it follows that

$$\text{M}_2\left[M_1\left(\frac{\|X_k,z_1,z_2,\ldots,z_{n-1}\|}{\rho}\right)\right]<M_2\left[1+\frac{1}{\xi}M_1\left(\frac{\|X_k,z_1,z_2,\ldots,z_{n-1}\|}{\rho}\right)\right]$$
$$<\frac{1}{2}M_2(2)+\frac{1}{2}M_2\left[2\frac{1}{\xi}M_1\left(\frac{\|X_k,z_1,z_2,\ldots,z_{n-1}\|}{\rho}\right)\right].$$

Again since M_2 satisfies Δ_2-condition, we have
$$\mathrm{M}_2\left[M_1\left(\frac{\|X_k,z_1,z_2,\dots,z_{n-1}\|}{\rho}\right)\right]$$
$$<\frac{1}{2}L\left[\frac{1}{\xi}M_1\left(\frac{\|X_k,z_1,z_2,\dots,z_{n-1}\|}{\rho}\right)\right]M_2(2)$$
$$+\frac{1}{2}L\left[\frac{1}{\xi}M_1\left(\frac{\|X_k,z_1,z_2,\dots,z_{n-1}\|}{\rho}\right)\right]M_2(2)$$
$$=L\xi^{-1}M_2(2)M_1\left(\frac{\|X_k,z_1,z_2,\dots,z_{n-1}\|}{\rho}\right).$$
So,
$$k^{-s}\left[M_2\left[M_1\left(\frac{\|X_k,z_1,z_2,\dots,z_{n-1}\|}{\rho}\right)\right]\right]^{p_k}$$
$$\leq k^{-s}C_1\left[M_1\left(\frac{\|X_k,z_1,z_2,\dots,z_{n-1}\|}{\rho}\right)\right]^{p_k} \tag{4.1}$$
where $C_1=max\{1,[L\xi^{-1}M_2(2)]^H\}$.
For $k\in A_1$,
$$M_1\left(\frac{\|X_k,z_1,z_2,\dots,z_{n-1}\|}{\rho}\right)\leq\xi M_2\left[M_1\left(\frac{\|X_k,z_1,z_2,\dots,z_{n-1}\|}{\rho}\right)\right]<\epsilon,$$
and therefore,
$$k^{-s}\left[M_2\left[M_1\left(\frac{\|X_k,z_1,z_2,\dots,z_{n-1}\|}{\rho}\right)\right]\right]^{p_k}\leq k^{-s}[\epsilon]^H. \tag{4.2}$$
Hence from (4.1) and (4.2) we have
$$k^{-s}\left[M_2\left[M_1\left(\frac{\|X_k,z_1,z_2,\dots,z_{n-1}\|}{\rho}\right)\right]\right]^{p_k}$$
$$\leq k^{-s}[\epsilon]^H+k^{-s}C_1\left[M_1\left(\frac{\|X_k;z_1,z_2,\dots,z_{n-1}\|}{\rho}\right)\right]^{p_k}\in Z$$
for all k. Then the proof follows by the normality of Z.

We have the well known inclusion $c_0\subset c\subset\ell_\infty$. The following result shows that if Z is replaced by these three spaces, the corresponding extended versions also preserve this inclusion.

Theorem 4.3. Let M be an Orlicz function. Then
$$c_0(\|.,\dots,.\|,M,p,s)\subset c(\|.,\dots,.\|,M,p,s)\subset\ell_\infty(\|.,\dots,.\|,M,p,s).$$

Proof. The first inclusion follows immediately from the definitions. For second inclusion, let $X = (X_k) \in c(\|.,\ldots,.\|, M, p, s)$. Then for some $\rho = 2\xi > 0$, we have
$$k^{-s}\left[M\left(\frac{\|X_k, z_1, z_2, \ldots, z_{n-1}\|}{\rho}\right)\right]^{p_k}$$
$$= k^{-s}\left[M\left(\frac{\|X_k - L + L, z_1, z_2, \ldots, z_{n-1}\|}{\rho}\right)\right]^{p_k}$$
$$\leq k^{-s}\left[M\left(\frac{\|X_k - L, z_1, z_2, \ldots, z_{n-1}\| + \|L, z_1, z_2, \ldots, z_{n-1}\|}{\rho}\right)\right]^{p_k}$$
$$\leq k^{-s}C\left[M\left(\frac{\|X_k - L, z_1, z_2, \ldots, z_{n-1}\|}{\xi}\right)\right]^{p_k} + k^{-s}C\left[M\left(\frac{\|L, z_1, z_2, \ldots, z_{n-1}\|}{\xi}\right)\right]^{p_k}$$
$$\leq k^{-s}C\left[M\left(\frac{\|X_k - L, z_1, z_2, \ldots, z_{n-1}\|}{\xi}\right)\right]^{p_k}$$
$$+ k^{-s}Cmax\left\{1, \left[M\left(\frac{\|L, z_1, z_2, \ldots, z_{n-1}\|}{\xi}\right)\right]^{H}\right\}.$$
Thus $X = (X_k) \in \ell_\infty(\|.,\ldots,.\|, M, p, s)$.
Our next result is to examine the effect of the parameter p on the relationships of some spaces.

Theorem 4.4 Let M be a Orlicz function. Then
(i) If $0 < \inf p_k \leq p_k < 1$, then $c_0(\|.,\ldots,.\|, M, s) \subset c_0(\|.,\ldots,.\|, M, p, s)$.
(ii) If $1 \leq p_k \leq \sup p_k < \infty$, then $c_0(\|.,\ldots,.\|, M, p, s) \subset c_0(\|.,\ldots,.\|, M, s)$.

Proof. (i) Let $X = (X_k) \in c_0(\|.,\ldots,.\|, M, s)$. Since $0 < \inf p_k \leq p_k < 1$,, the proof follows from the following inequality
$$\left[M\left(\frac{\|X_k, z_1, z_2, \ldots, z_{n-1}\|}{\rho}\right)\right]^{p_k} \leq \left[M\left(\frac{\|X_k, z_1, z_2, \ldots, z_{n-1}\|}{\rho}\right)\right]$$
(ii) Let $1 \leq p_k \leq \sup p_k < \infty$, and $X = (X_k) \in c_0(\|.,\ldots,.\|, M, p, s)$. Then for each $0 < \epsilon < 1$ there exists a positive integer L such that
$$k^{-s}\left[M\left(\frac{\|X_k, z_1, z_2, \ldots, z_{n-1}\|}{\rho}\right)\right]^{p_k} \leq \epsilon < 1 \; for \; all \; k \geq L.$$
Since $1 \leq p_k \leq \sup p_k < \infty$,, the proof follows from the following inequality
$$k^{-s}\left[M\left(\frac{\|X_k, z_1, z_2, \ldots, z_{n-1}\|}{\rho}\right)\right] \leq k^{-s}\left[M\left(\frac{\|X_k, z_1, z_2, \ldots, z_{n-1}\|}{\rho}\right)\right]^{p_k}.$$

Theorem 4.5. The space $Z(\|.,\ldots,.\|, M, p, s)$ is not convergence free in general.

Proof. Consider $Z = \ell_\infty, s = 0, p_k = 1$, for eeach $k \in \mathbf{N}, M(x) = x^2$, and for all $x \in [0, \infty)$. Let $X = (X_k) \in Z(\|.,\ldots,.\|, M, p, s)$ as follows:

$$\mathrm{X}_k = \begin{cases} \dfrac{1}{k+1}, & if\ k\ is\ even \\ \\ 0, & if\ k\ is\ odd. \end{cases}$$

Let us define a sequence (Y_k) as follows:

$$\mathrm{Y}_k = \begin{cases} k+1, & if\ k\ is\ even \\ \\ 0, & if\ k\ is\ odd. \end{cases}$$

Then $X_k = 0$ implies $Y_k = 0$, but $(Y_k) \notin Z(\|.,\ldots,.\|, M, p, s)$. However, the space $Z(\|.,\ldots,.\|, M, p, s)$ is solid and symmetric in general. The following two results establish our claim with proof.

Theorem 4.6. The space $Z(\|.,\ldots,.\|, M, p, s)$ is solid (normal) in general.

Proof. Let $X = (X_k) \in Z(\|.,\ldots,.\|, M, p, s)$, and $Y = (Y_k)$ be such that

$$\|Y_k, z_1, z_2, \ldots, z_{n-1}\| \leq \|X_k, z_1, z_2, \ldots, z_{n-1}\|\ for\ every\ z_1, z_2, \ldots, z_{n-1} \in T.$$

Since M is non-decreasing,

$$k^{-s}\left[M\left(\frac{\|Y_k, z_1, z_2, \ldots, z_{n-1}\|}{\rho}\right)\right]^{p_k} \leq k^{-s}\left[M\left(\frac{\|X_k, z_1, z_2, \ldots, z_{n-1}\|}{\rho}\right)\right]^{p_k} \in Z.$$

for some $\rho > 0$. Hence $Y = (Y_k) \in Z(\|.,\ldots,.\|, M, p, s)$, since Z is normal and the space is solid.

Theorem 4.7. The space $Z(\|.,\ldots,.\|, M, p, s)$ is symmetric in general.

Proof. Let $X = (X_k) \in Z(\|.,\ldots,.\|, M, p, s)$, and $Y = (Y_{m_k})$ be an arrangement of the sequence (X_k) such that $(X_k) = (Y_{m_k})$ for each $k \in \mathbf{N}$. Then

$$k^{-s}\left[M\left(\frac{\|Y_{m_k}, z_1, z_2, \ldots, z_{n-1}\|}{\rho}\right)\right]^{p_k} \leq k^{-s}\left[M\left(\frac{\|X_k, z_1, z_2, \ldots, z_{n-1}\|}{\rho}\right)\right]^{p_k} \in Z.$$

Hence these spaces are symmetric in general.

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