Mappings and Decompositions of Pairwise Continuity on (i, j) -almost Lindelöf and (i, j) -weakly Lindelöf Spaces

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Received: 2018/11/11 | Accepted: 2021/01/13

Abstract:

The purpose of this paper is to study the effect of mappings and some decompositions of pairwise continuity on (i, j) -almost Lindelöf spaces and (i, j) -weakly Lindelöf spaces. The main results are that an (i, j) - θ -continuous image of an (i, j) -almost Lindelöf space is (i, j) -almost Lindelöf and a pairwise almost continuous image of an (i, j) -weakly Lindelöf space is (i, j) -weakly Lindelöf. We also show that (i, j) -almost Lindelöf, pairwise almost Lindelöf, (i, j) -weakly Lindelöf and pairwise weakly Lindelöf properties are bitopological properties.

Keywords: Bitopological space; (i, j) -almost Lindelöf; (i, j) -weakly Lindelöf; (i, j) – θ -continuous; Pairwise almost continuous.

MSC (2020): 54E55, 54B05, 54C08, 54D20.

Cite this article as (IEEE citation style):

Z. Salleh and A. Kılıçman, "Mappings and Decompositions of Pairwise Continuity on (i, j) -almost Lindelöf and (i, j) -weakly Lindelöf Spaces", *Proyecciones (Antofagasta, On line)*, vol. 40, no. 4, pp. 815-836, 2021, doi: 10.22199/issn.0717-6279-3253

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1. Introduction

In literature there are several generalizations of the notion of Lindelöf spaces and these are studied separately for different reasons and purposes. In 1984, Willard and Dissanayake [17] introduced the notion of almost Lindelöf spaces and in 1959, Frolik [9] introduced the notion of weakly Lindelöf spaces that, afterward, was studied by several authors. Thereafter in 1996, Cammaroto and Santoro [3] studied and gave further new results about these spaces. The authors [11, 13] introduced and studied the notion of (i, j) -almost Lindelöf and (i, j) -weakly Lindelöf in bitopological spaces and extended some results due to Cammaroto and Santoro [3] and Fawakhreh and Kılıçman [7, 8].

Our purpose in this paper is to study some decompositions of pairwise continuity concepts, openness and closedness functions and its generalizations concepts, and mappings on (i, j) -almost Lindelöf and (i, j) -weakly Lindelöf spaces in a suitable way of bitopological spaces after the manner of Fawakhreh and Kılıçman [7, 8]. We extend most of their results that in topological spaces to bitopological spaces. The concepts of continuous functions and its generalizations have been introduced and studied in topological spaces. Some of the decompositions of pairwise continuity concepts are studied by the authors [12]. In this paper we continue the study by extend the other types of continuity in topological spaces to bitopological spaces and investigate their relationships. Some examples and counterexamples will be given in order to establish their relationships.

In section 3 and section 4, we shall study the effect of mappings and some decompositions of pairwise continuity on (i, j) -almost Lindelöf and (i, j) -weakly Lindelöf spaces, respectively. We also show that some mappings preserve these properties. The main results in our study is that the image of an (i, j) -almost Lindelöf space under an (i, j) - θ -continuous function is (i, j) -almost Lindelöf and the image of a (i, j) -weakly Lindelöf space under a pairwise almost continuous function is (i, j) -weakly Lindelöf. We also show that (i, j) -almost Lindelöf, pairwise almost Lindelöf, (i, j) -weakly Lindelöf and pairwise weakly Lindelöf properties are bitopological properties.

2. Preliminaries

Throughout this paper, all spaces (X, τ) and (X, τ_1, τ_2) (or simply X) are always mean topological spaces and bitopological spaces, respectively unless

explicitly stated. If $\mathcal{P}$ is a topological property, then $(\tau_i, \tau_j)\text{-}\mathcal{P}$ denotes an analogue of this property for τ_i has property $\mathcal{P}$ with respect to τ_j , and pairwise- $\mathcal{P}$ denotes the conjunction $(\tau_1, \tau_2)\text{-}\mathcal{P} \wedge (\tau_2, \tau_1)\text{-}\mathcal{P}$, i.e., pairwise- $\mathcal{P}$ denotes an absolute bitopological analogue of $\mathcal{P}$. As we shall see below, sometimes $(\tau_1, \tau_2)\text{-}\mathcal{P} \iff (\tau_2, \tau_1)\text{-}\mathcal{P}$ (and thus $\iff$ pairwise- $\mathcal{P}$) so that sometime it suffices to consider one of these three bitopological analogue. Sometimes we shall use $p\text{-}$ to denote pairwise. Also sometimes $\tau_1\text{-}\mathcal{P} \iff \tau_2\text{-}\mathcal{P}$ and thus $\mathcal{P} \iff \tau_1\text{-}\mathcal{P} \wedge \tau_2\text{-}\mathcal{P}$, i.e., (X, τ_i) has property $\mathcal{P}$ for each $i = 1, 2$. Also note that (X, τ_i) has a property $\mathcal{P}$ if and only if (X, τ_1, τ_2) has a property $\tau_i\text{-}\mathcal{P}$. The prefixes $(\tau_i, \tau_j)\text{-}$ or $\tau_i\text{-}$ will be replaced by $(i, j)\text{-}$ or $i\text{-}$ respectively, if there is no chance for confusion. By $i\text{-int}(A)$ and $i\text{-cl}(A)$, we shall mean the interior and the closure of a subset A of X with respect to topology τ_i , respectively. By $i\text{-open}$ cover of X , we mean that the cover of X by $i\text{-open}$ sets in X ; similar for the $(i, j)\text{-regular open}$ cover of X , etc. In this paper always $i, j \in \{1, 2\}$ and $i \neq j$. The reader may consult [2] for the detail notations.

Definition 2.1. A subset S of a bitopological space (X, τ_1, τ_2) is said to be

- (a) $(i, j)\text{-regular open}$ [12] if $S = i\text{-int}(j\text{-cl}(S))$, S is called pairwise regular open if it is both $(1, 2)\text{-regular open}$ and $(2, 1)\text{-regular open}$;
- (b) $(i, j)\text{-regular closed}$ [12] if $S = i\text{-cl}(j\text{-int}(S))$, S is called pairwise regular closed if it is both $(1, 2)\text{-regular closed}$ and $(2, 1)\text{-regular closed}$;
- (c) $(i, j)\text{-preopen}$ [12] if $S \subseteq i\text{-int}(j\text{-cl}(S))$, S is called pairwise preopen if it is both $(1, 2)\text{-preopen}$ and $(2, 1)\text{-preopen}$;
- (d) $(i, j)\text{-clopen}$ if S is $i\text{-closed}$ and $j\text{-open}$, S is called pairwise clopen if it is both $(1, 2)\text{-clopen}$ and $(2, 1)\text{-clopen}$.

Definition 2.2. [12] A function $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is said to be

- (a) $i\text{-continuous}$ if the functions $f : (X, \tau_i) \rightarrow (Y, \sigma_i)$ is continuous, f is called continuous if it is $i\text{-continuous}$ for each $i = 1, 2$;
- (b) $(i, j)\text{-almost continuous}$ if $f^{-1}(V)$ is $\tau_i\text{-open}$ set in X for every $(\sigma_i, \sigma_j)\text{-regular open}$ set V in Y , f is called pairwise almost continuous if it is both $(1, 2)\text{-almost continuous}$ and $(2, 1)\text{-almost continuous}$;
- (c) $(i, j)\text{-precontinuous}$ if $f^{-1}(V)$ is $(\tau_i, \tau_j)\text{-preopen}$ set in X for every $\sigma_i\text{-open}$ set V in Y , f is called pairwise precontinuous if it is both $(1, 2)\text{-precontinuous}$ and $(2, 1)\text{-precontinuous}$;
- (d) $(i, j)\text{-}\delta\text{-continuous}$ (resp. $(i, j)\text{-almost } \delta\text{-continuous}$) if for each $x \in X$ and each $(\sigma_i, \sigma_j)\text{-regular open}$ subset V of Y containing $f(x)$, there exists a $(\tau_i, \tau_j)\text{-regular open}$ subset U of X containing x such that $f(U) \subseteq V$

(resp. $f(U) \subseteq \sigma_j\text{-cl}(V)$), f is called *pairwise δ -continuous* (resp. *pairwise almost δ -continuous*) if it is both $(1, 2)$ - δ -continuous (resp. $(1, 2)$ -almost δ -continuous) and $(2, 1)$ - δ -continuous (resp. $(2, 1)$ -almost δ -continuous).

3. Decomposition of Pairwise Continuity

The concepts of θ -continuous, contra-continuous and subcontra-continuous functions have been introduced in a topological space (see [1, 4]). These concepts are extended to bitopological setting as follows.

Definition 3.1. A function $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is said to be

- (a) (i, j) - θ -continuous (resp. (i, j) -strongly θ -continuous) if for each $x \in X$ and each σ_i -open subset V of Y containing $f(x)$, there exists a τ_i -open subset U of X containing x such that $f(\tau_j\text{-cl}(U)) \subseteq \sigma_j\text{-cl}(V)$ (resp. $f(\tau_j\text{-cl}(U)) \subseteq V$), f is called *pairwise θ -continuous* (resp. *pairwise strongly θ -continuous*) if it is both $(1, 2)$ - θ -continuous (resp. $(1, 2)$ -strongly θ -continuous) and $(2, 1)$ - θ -continuous (resp. $(2, 1)$ -strongly θ -continuous);
- (b) (i, j) -contra-continuous if $f^{-1}(V)$ is τ_j -closed set in X for every σ_i -open set V in Y , f is called *pairwise contra-continuous* if it is both $(1, 2)$ -contra-continuous and $(2, 1)$ -contra-continuous;
- (c) (i, j) -subcontra-continuous if there exists a σ_i -open base $\mathcal{B}_i$ for the topology σ_i on Y such that $f^{-1}(V)$ is τ_j -closed set in X for every $V \in \mathcal{B}_i$, f is called *pairwise subcontra-continuous* if it is both $(1, 2)$ -subcontra-continuous and $(2, 1)$ -subcontra-continuous.

Observe that, every (i, j) -strongly θ -continuous function is (i, j) - θ -continuous and every (i, j) -strongly θ -continuous function is i -continuous, but the converses are not true in general as Examples 3.1 and 3.2 below show. Proposition 3.1 shows that (i, j) -strongly θ -continuity implies (i, j) - δ -continuity, but the converse is not true as Example 3.1 below shows. Moreover, (i, j) - θ -continuity implies (i, j) -almost δ -continuity by Lemma 4.2 but the converse is not true as Example 4.2 below shows. Example 3.1 and Example 4.2 shows that i -continuity and (i, j) - θ -continuity are independent concepts. Furthermore, i -continuity and (i, j) - δ -continuity are independent concepts (see [12]).

It is very clear that (i, j) - δ -continuity implies (i, j) -almost δ -continuity but the converse is not true (see [12]). While (i, j) - δ -continuity and (i, j) - θ -continuity are independent concepts as Example 3.3 and Example 3.4 below show. Examples 3.3 and 3.4 also show that (i, j) - θ -continuity and (i, j) -almost continuity are independent concepts. It is easy to prove that (i, j) -

contra-continuity implies (i, j) -subcontra-continuity but the converse is not true as Example 3.5 below shows. Moreover, (i, j) -subcontra-continuity and i -continuity properties are independent notions. The function in Example 3.5 shows that (i, j) -subcontra-continuity does not imply i -continuity. Example 3.6 below shows that i -continuity does not imply (i, j) -subcontra-continuity. In fact (i, j) -contra-continuity and i -continuity are independent notion as Example 3.7 and Example 3.8 below show.

Proposition 3.1. *If $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is (i, j) -strongly θ -continuous function, then f is (i, j) - δ -continuous.*

Proof. Let $x \in X$ and let V be a (σ_i, σ_j) -regular open set in Y containing $f(x)$. Then V is also a σ_i -open set in Y containing $f(x)$. Since f is (i, j) -strongly θ -continuous, there exists a τ_i -open subset U of X containing x such that $f(\tau_j\text{-cl}(U)) \subseteq V$. So $\tau_i\text{-int}(\tau_j\text{-cl}(U))$ is a (τ_i, τ_j) -regular open subset of X containing x . Take $W = \tau_i\text{-int}(\tau_j\text{-cl}(U))$ and then W is a (τ_i, τ_j) -regular open subset of X containing x such that $f(W) = f(\tau_i\text{-int}(\tau_j\text{-cl}(U))) \subseteq f(\tau_j\text{-cl}(U)) \subseteq V$. This shows that f is (i, j) - δ -continuous. $\square$

Corollary 3.1. *If $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is pairwise strongly θ -continuous function, then f is pairwise δ -continuous.*

The converse of Proposition 3.1 is not true as the following examples show.

Example 3.1. Let $X = \{a, b, c, d\}$ with topologies

$$\tau_1 = \{\emptyset, \{c\}, \{d\}, \{a, c\}, \{c, d\}, \{a, c, d\}, X\}, \tau_2 = \{\emptyset, \{c\}, \{a, c\}, X\}$$

and $Y = \{x, y, z\}$ with topologies

$$\sigma_1 = \{\emptyset, \{x\}, \{x, y\}, Y\}, \sigma_2 \text{ is indiscrete topology.}$$

Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be a function defined by $f(a) = f(d) = x$, $f(b) = y$ and $f(c) = z$. Then f is $(1, 2)$ - δ -continuous since the (σ_1, σ_2) -regular open sets in (Y, σ_1, σ_2) are $\emptyset$ and Y . It is also $(1, 2)$ - θ -continuous since σ_2 -closure of any subset in (Y, σ_1, σ_2) is Y . However f is not $(1, 2)$ -strongly θ -continuous since $\{x\}$ is σ_1 -open set in (Y, σ_1, σ_2) containing $f(\{a\}) = \{x\}$, but there is no τ_1 -open set U in (X, τ_1, τ_2) containing $\{a\}$ such that $f(\tau_2\text{-cl}(U)) \subseteq \{x\}$. Even f is $(1, 2)$ - θ -continuous, it is not 1-continuous since $f^{-1}(\{x\}) = \{a, d\}$ is not τ_1 -open set in (X, τ_1, τ_2) for $\{x\}$ is σ_1 -open set in (Y, σ_1, σ_2) .

Example 3.2. Let $X = \{a, b, c\}$ with topologies

$$\tau_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, X\}, \tau_2 = \{\emptyset, \{a\}, \{c\}, \{a, c\}, X\}$$

and

$$\sigma_1 = \{\emptyset, \{a\}, \{c\}, \{a, c\}, X\}, \sigma_2 = \{\emptyset, \{b, c\}, X\}.$$

Then the function $f : (X, \tau_1, \tau_2) \rightarrow (X, \sigma_1, \sigma_2)$ defined by $f(a) = a$, $f(b) = c$ and $f(c) = b$ is 1-continuous but it is not $(1, 2)$ -strongly θ -continuous since $\{a\}$ is σ_1 -open set in (X, σ_1, σ_2) containing $f(a) = a$ but there is no τ_1 -open set U in (X, τ_1, τ_2) containing a such that $f(\tau_2\text{-cl}(U)) \subseteq \{a\}$.

Example 3.3. Let $X = \{a, b, c\}$ with topologies

$$\tau_1 = \{\emptyset, \{b\}, \{a, b\}, \{b, c\}, X\}, \tau_2 = \{\emptyset, \{c\}, X\}$$

and

$$\sigma_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}, \sigma_2 = \{\emptyset, \{a\}, X\}.$$

Let $f : (X, \tau_1, \tau_2) \rightarrow (X, \sigma_1, \sigma_2)$ be a function defined by $f(a) = b$ and $f(b) = f(c) = c$. Then f is $(1, 2)$ - θ -continuous but it is not $(1, 2)$ - δ -continuous since $\{b\}$ is a (σ_1, σ_2) -regular open set in (X, σ_1, σ_2) containing $f(a) = b$ but there is no (τ_1, τ_2) -regular open set U in (X, τ_1, τ_2) containing a such that $f(U) \subseteq \{b\}$. Even f is $(1, 2)$ - θ -continuous but it is not $(1, 2)$ -almost continuous since $f^{-1}(\{b\}) = \{a\}$ is not τ_1 -open set in (X, τ_1, τ_2) for $\{b\}$ is (σ_1, σ_2) -regular open set in (X, σ_1, σ_2) .

Example 3.4. Let $X = \{a, b, c\}$ with topologies

$$\tau_1 = \{\emptyset, \{a\}, \{b, c\}, X\}, \tau_2 = \{\emptyset, \{a\}, \{c\}, \{a, c\}, X\}$$

and

$$\sigma_1 = \{\emptyset, \{a\}, \{c\}, \{a, c\}, X\}, \sigma_2 = \{\emptyset, \{b, c\}, X\}.$$

Then the function $f : (X, \tau_1, \tau_2) \rightarrow (X, \sigma_1, \sigma_2)$ defined by $f(a) = a$, $f(b) = c$ and $f(c) = b$ is $(1, 2)$ - δ -continuous since (τ_1, τ_2) -regular open sets in (X, τ_1, τ_2) are $\emptyset, \{a\}, \{b, c\}$ and X , and (σ_1, σ_2) -regular open sets in (X, σ_1, σ_2) are $\emptyset, \{a\}$ and X . Thus f is $(1, 2)$ -almost continuous (see [12]).

But f is not $(1, 2)$ - θ -continuous since $\{a\}$ is σ_1 -open set in (X, σ_1, σ_2) containing $f(a) = a$ but there is no τ_1 -open set U in (X, τ_1, τ_2) containing a such that $f(\tau_2\text{-cl}(U)) \subseteq \sigma_2\text{-cl}(\{a\})$.

Example 3.5. Let $\mathbf{R}$ be a real line with topologies

τ_1 is usual topology, τ_2 is cofinite topology

and

σ_1 is discrete topology, σ_2 is cocountable topology.

If $\mathcal{B}_1$ is the collection of all singleton subsets of $\mathbf{R}$, then $\mathcal{B}_1$ is a σ_1 -open base for the topology σ_1 on $\mathbf{R}$. Let $f : (\mathbf{R}, \tau_1, \tau_2) \rightarrow (\mathbf{R}, \sigma_1, \sigma_2)$ be an identity function. Then f is $(1, 2)$ -subcontra-continuous but it is not $(1, 2)$ -contra-continuous since there exists a σ_1 -open set $(1, 5]$ in $(\mathbf{R}, \sigma_1, \sigma_2)$ such that $f^{-1}((1, 5]) = (1, 5]$ is not τ_2 -closed set in $(\mathbf{R}, \tau_1, \tau_2)$. It is also not 1-continuous since there exists a σ_1 -open set $[1, 3]$ in $(\mathbf{R}, \sigma_1, \sigma_2)$ such that $f^{-1}([1, 3]) = [1, 3]$ is not τ_1 -open set in $(\mathbf{R}, \tau_1, \tau_2)$.

Example 3.6. Let $X = \{a, b\}$ with topologies τ_1 is discrete topology $\{\emptyset, \{a\}, \{b\}, X\}$ and $\tau_2 = \{\emptyset, \{b\}, X\}$. Let $g : (X, \tau_1, \tau_2) \rightarrow (X, \tau_1, \tau_2)$ be an identity function. Obviously g is 1-continuous. However, any τ_1 -open base for the topology τ_1 on X must contain $\{b\}$ but $g^{-1}(\{b\}) = \{b\}$ is not τ_2 -closed set in X . It follows that g is not $(1, 2)$ -subcontra-continuous.

Example 3.7. Let $X = \{a, b\}$ with topologies $\tau_1 = \{\emptyset, X\}$, $\tau_2 = \{\emptyset, \{a\}, X\}$ and $\sigma_1 = \{\emptyset, \{b\}, X\}$, σ_2 is discrete topology. Then the identity function $f : (X, \tau_1, \tau_2) \rightarrow (X, \sigma_1, \sigma_2)$ is $(1, 2)$ -contra-continuous but it is not 1-continuous since $f^{-1}(\{b\}) = \{b\}$ is not τ_1 -open set in (X, τ_1, τ_2) for $\{b\}$ σ_1 -open set in (X, σ_1, σ_2) .

Example 3.8. Let $\mathbf{R}$ be a real line with topologies

τ_1 is discrete topology, τ_2 is cofinite topology

and

σ_1 is discrete topology, σ_2 is usual topology.

If $\mathcal{B}_1$ is the collection of all singleton subsets of $\mathbf{R}$, then $\mathcal{B}_1$ is a σ_1 -open base for the topology σ_1 on $\mathbf{R}$. Let $f : (\mathbf{R}, \tau_1, \tau_2) \rightarrow (\mathbf{R}, \sigma_1, \sigma_2)$ be an identity function. Then f is 1-continuous but it is not $(1, 2)$ -contra-continuous since there exists a σ_1 -open set $(1, 5] \in (\mathbf{R}, \sigma_1, \sigma_2)$ such that $f^{-1}((1, 5]) = (1, 5]$ is not τ_2 -closed set in $(\mathbf{R}, \tau_1, \tau_2)$.

A pairwise almost continuous (briefly p -almost continuous) function is (i, j) - θ -continuous by Lemma 4.1 below but the converse is not true as Example 4.1 below shows. Therefore, we obtain the following diagram in which none of these implications are reversible.

$$\begin{array}{ccccc}
 (i, j)\text{-}\delta\text{-continuous} & \Longleftarrow & (i, j)\text{-strongly } \theta\text{-continuous} & \Longrightarrow & i\text{-continuous} \\
 \Downarrow & & \Downarrow & & \\
 (i, j)\text{-almost } \delta\text{-continuous} & \Longleftarrow & (i, j)\text{-}\theta\text{-continuous} & \Longleftarrow & p\text{-almost continuous}
 \end{array}$$

In terms of pairwise topology, we have the following diagram in which none of these implications is reversible. We shall use abbreviation p - to denote pairwise. Note that it is obvious, continuity implies pairwise almost continuity but the converse is not true (see [12]).

$$\begin{array}{ccccc}
 p\text{-}\delta\text{-continuous} & \Longleftarrow & p\text{-strongly } \theta\text{-continuous} & \Longrightarrow & \text{continuous} \\
 \Downarrow & & \Downarrow & & \Downarrow \\
 p\text{-almost } \delta\text{-continuous} & \Longleftarrow & p\text{-}\theta\text{-continuous} & \Longleftarrow & p\text{-almost continuous}
 \end{array}$$

4. Mappings on (i, j) -almost Lindelöf Spaces

Definition 4.1. [11, 15] A bitopological space (X, τ_1, τ_2) is said to be (i, j) -almost Lindelöf if for every i -open cover $\{U_\alpha : \alpha \in \Delta\}$ of X , there exists a countable subset $\{\alpha_n : n \in \mathbf{N}\}$ of Δ such that $X = \bigcup_{n \in \mathbf{N}} j\text{-cl}(U_{\alpha_n})$. X is called pairwise almost Lindelöf if it is both $(1, 2)$ -almost Lindelöf and $(2, 1)$ -almost Lindelöf.

The following theorem give some characterizations of (i, j) -almost Lindelöf spaces.

Theorem 4.1. [11, 15] Let X be a bitopological space. The following conditions are equivalent:

- (i) X is (i, j) -almost Lindelöf;
- (ii) every family $\{C_\alpha : \alpha \in \Delta\}$ of i -closed subsets of X such that $\bigcap_{\alpha \in \Delta} C_\alpha = \emptyset$ admits a countable subfamily $\{C_{\alpha_n} : n \in \mathbf{N}\}$ such that $\bigcap_{n \in \mathbf{N}} j\text{-int}(C_{\alpha_n}) = \emptyset$;

(iii) for every family $\{C_\alpha : \alpha \in \Delta\}$ of i -closed subsets of X for which every countable subfamily $\{C_{\alpha_n} : n \in \mathbf{N}\}$ satisfies $\bigcap_{n \in \mathbf{N}} j\text{-int}(C_{\alpha_n}) \neq \emptyset$, the intersection $\bigcap_{\alpha \in \Delta} C_\alpha \neq \emptyset$.

It is well known that in a topological space and a bitopological space, the continuous image of a Lindelöf space is Lindelöf. While Fawakhreh and Kılıçman [8] proved that the θ -continuous image of an almost Lindelöf space is almost Lindelöf. For the (i, j) -almost Lindelöf spaces we give the following theorem.

Theorem 4.2. *Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be a surjective and (i, j) - θ -continuous function. If X is (τ_i, τ_j) -almost Lindelöf, then Y is (σ_i, σ_j) -almost Lindelöf.*

Proof. Let $\{V_\alpha : \alpha \in \Delta\}$ be a σ_i -open cover of Y . Let $x \in X$ and let $\alpha_x \in \Delta$ such that $f(x) \in V_{\alpha_x}$. Since f is (i, j) - θ -continuous, there exists a τ_i -open subset U_{α_x} of X containing x such that $f(\tau_j\text{-cl}(U_{\alpha_x})) \subseteq \sigma_j\text{-cl}(V_{\alpha_x})$. So $\{U_{\alpha_x} : x \in X\}$ forms a τ_i -open cover of X . Since X is (τ_i, τ_j) -almost Lindelöf, there exists a countable subset of points $x_1, \dots, x_n, \dots$ of X such that $X = \bigcup_{n \in \mathbf{N}} \tau_j\text{-cl}(U_{\alpha_{x_n}})$. Since f is surjective, we have $Y =$

$$f(X) = f\left(\bigcup_{n \in \mathbf{N}} \tau_j\text{-cl}(U_{\alpha_{x_n}})\right) = \bigcup_{n \in \mathbf{N}} f(\tau_j\text{-cl}(U_{\alpha_{x_n}})) \subseteq \bigcup_{n \in \mathbf{N}} \sigma_j\text{-cl}(V_{\alpha_{x_n}}).$$

This shows that Y is (σ_i, σ_j) -almost Lindelöf. $\square$

Corollary 4.1. *Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be a surjective and pairwise θ -continuous function. If X is pairwise almost Lindelöf, then Y is pairwise almost Lindelöf.*

Lemma 4.1. *If $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is pairwise almost continuous function, then f is (i, j) - θ -continuous.*

Proof. Let $x \in X$ and let V be a σ_i -open set in Y containing $f(x)$. Then $x \in f^{-1}(V)$ and $\sigma_i - \text{int}(\sigma_j\text{-cl}(V))$ is a (σ_i, σ_j) -regular open set in Y containing $f(x)$. Since f is (i, j) -almost continuous, $f^{-1}(\sigma_i - \text{int}(\sigma_j\text{-cl}(V)))$ is τ_i -open subset of X containing x . Since f is (j, i) -almost continuous and $\sigma_j\text{-cl}(\sigma_i - \text{int}(\sigma_j\text{-cl}(V)))$ is a (σ_j, σ_i) -regular closed set in Y ,

$$\tau_j\text{-cl}\left(f^{-1}(\sigma_i\text{-int}(\sigma_j\text{-cl}(V)))\right) \subseteq f^{-1}(\sigma_j\text{-cl}(\sigma_i\text{-int}(\sigma_j\text{-cl}(V))))$$

because $f^{-1}(\sigma_j\text{-cl}(\sigma_i\text{-int}(\sigma_j\text{-cl}(V))))$ is a τ_j -closed subset of X containing $f^{-1}(V)$. Take $U = f^{-1}(\sigma_i\text{-int}(\sigma_j\text{-cl}(V)))$, then U is a τ_i -open subset of X containing x such that

$$\begin{aligned} f(\tau_j\text{-cl}(U)) &= f\left(\tau_j\text{-cl}\left(f^{-1}(\sigma_i\text{-int}(\sigma_j\text{-cl}(V)))\right)\right) \\ &\subseteq f\left(f^{-1}(\sigma_j\text{-cl}(\sigma_i\text{-int}(\sigma_j\text{-cl}(V))))\right) \\ &\subseteq \sigma_j\text{-cl}(V). \end{aligned}$$

This shows that f is (i, j) - θ -continuous. $\square$

Corollary 4.2. *If $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is a pairwise almost continuous function, then f is pairwise θ -continuous.*

The converse of Lemma 4.1 is not true as the following example shows.

Example 4.1. *Let $X = \{a, b, c\}$ with topologies*

$$\tau_1 = \{\emptyset, \{b\}, \{a, b\}, \{b, c\}, X\}, \tau_2 = \{\emptyset, \{c\}, X\}$$

and

$$\sigma_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}, \sigma_2 = \{\emptyset, \{a\}, X\}.$$

Let $f : (X, \tau_1, \tau_2) \rightarrow (X, \sigma_1, \sigma_2)$ be a function defined by $f(a) = b$ and $f(b) = f(c) = c$. Then f is $(1, 2)$ - θ -continuous but it is not $(1, 2)$ -almost continuous since there exists a (σ_1, σ_2) -regular open set $\{b\}$ in (X, σ_1, σ_2) such that $f^{-1}(\{b\}) = \{a\}$ is not τ_1 -open set in (X, τ_1, τ_2) . Thus f is not pairwise almost continuous.

Corollary 4.3. *Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be a surjective and pairwise almost continuous function. If X is (τ_i, τ_j) -almost Lindelöf, then Y is (σ_i, σ_j) -almost Lindelöf.*

Proof. This is a direct consequence of Lemma 4.1 and Theorem 4.2 above. $\square$

Corollary 4.4. *Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be a surjective and pairwise almost continuous function. If X is pairwise almost Lindelöf, then Y is pairwise almost Lindelöf.*

The concept of homeomorphism is well known in topological spaces. Now we extend this concept to bitopological spaces in the sense of Tallafha et al. [16] as follows.

Definition 4.2. *Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) are two bitopological spaces. Then a function $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is called i -homeomorphism if the function $f : (X, \tau_i) \rightarrow (Y, \sigma_i)$ is homeomorphism, or equivalently, if $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is bijection, i -continuous and $f^{-1} : (Y, \sigma_1, \sigma_2) \rightarrow (X, \tau_1, \tau_2)$ is i -continuous. The bitopological spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) are then called i -homeomorphic. A function $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is called homeomorphism if the function $f : (X, \tau_i) \rightarrow (Y, \sigma_i)$ is homeomorphism for each $i = 1, 2$, or equivalently, if $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is bijection, continuous and $f^{-1} : (Y, \sigma_1, \sigma_2) \rightarrow (X, \tau_1, \tau_2)$ is continuous. The bitopological spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) are then called homeomorphic.*

Recall that, a property $\mathcal{P}$ of sets is called topological property if whenever a topological space (X, τ) has property $\mathcal{P}$, then every space homeomorphic to (X, τ) also has property $\mathcal{P}$. In the case of bitopological spaces, a property $\mathcal{P}$ will be called i -bitopological property if whenever (X, τ_1, τ_2) has property $\mathcal{P}$, then every space i -homeomorphic to (X, τ_1, τ_2) also has property $\mathcal{P}$. If $\mathcal{P}$ is an i -bitopological property for each $i = 1, 2$, then it is called a bitopological property.

(Catatan: gunakan nama i -topological property)

Since an (i, j) -property depends on τ_i and τ_j , it cannot be expected to be an i -bitopological property, but it can be a bitopological property as we will see below. We cannot directly say that (i, j) -almost Lindelöf is an i -bitopological property since i -continuity and (i, j) - θ -continuity are independent concepts. Although i -continuity and (i, j) - θ -continuity are independent concepts but every continuous function is pairwise almost continuous and so pairwise θ -continuous. So by using Corollary 4.3 and Corollary 4.4 or Corollary 4.1, we obtain the following result.

Corollary 4.5. *The (i, j) -almost Lindelöf property and the pairwise almost Lindelöf property are bitopological properties.*

In [11], it was stated that an (i, j) -almost regular space is (i, j) -almost Lindelöf if and only if it is (i, j) -nearly Lindelöf. Moreover, an (i, j) -almost regular and (i, j) -nearly Lindelöf space is (i, j) -nearly paracompact (see [12]). Thus by using these facts and Theorem 4.2 above, we obtain the following corollary.

Corollary 4.6. *Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be an (i, j) - θ -continuous and surjective function. If X is (τ_i, τ_j) -almost Lindelöf and Y is a (σ_i, σ_j) -almost regular space, then Y is (σ_i, σ_j) -nearly paracompact.*

Corollary 4.7. *Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be a pairwise θ -continuous and surjective function. If X is pairwise almost Lindelöf and Y is a pairwise almost regular space, then Y is pairwise nearly paracompact.*

Proposition 4.1. *Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be an (i, j) -strongly- θ -continuous and surjective function. If X is (τ_i, τ_j) -almost Lindelöf, then Y is σ_i -Lindelöf.*

Proof. Let $\{V_\alpha : \alpha \in \Delta\}$ be a σ_i -open cover of Y . Let $x \in X$ and take $\alpha_x \in \Delta$ such that $f(x) \in V_{\alpha_x}$. Since f is (i, j) -strongly- θ -continuous, there exists a τ_i -open subset U_{α_x} of X containing x such that $f(\tau_j\text{-cl}(U_{\alpha_x})) \subseteq V_{\alpha_x}$. Now $\{U_{\alpha_x} : x \in X\}$ forms a τ_i -open cover of X . Since X is (τ_i, τ_j) -almost Lindelöf, there exists a countable subset of points $x_1, \dots, x_n, \dots$ of X such that $X = \bigcup_{n \in \mathbf{N}} \tau_j\text{-cl}(U_{\alpha_{x_n}})$. Since f is surjective, $Y = f(X) = f\left(\bigcup_{n \in \mathbf{N}} \tau_j\text{-cl}(U_{\alpha_{x_n}})\right) = \bigcup_{n \in \mathbf{N}} f(\tau_j\text{-cl}(U_{\alpha_{x_n}})) \subseteq \bigcup_{n \in \mathbf{N}} V_{\alpha_{x_n}}$. Therefore Y is σ_i -Lindelöf. $\square$

Corollary 4.8. *Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be a pairwise strongly- θ -continuous and surjective function. If X is pairwise almost Lindelöf, then Y is Lindelöf.*

Proposition 4.2. *Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be an (i, j) -almost δ -continuous and surjective function. If X is (τ_i, τ_j) -nearly Lindelöf, then Y is (σ_i, σ_j) -almost Lindelöf.*

Proof. Let $\{V_\alpha : \alpha \in \Delta\}$ be a σ_i -open cover of Y . Let $x \in X$ and take $\alpha_x \in \Delta$ such that $f(x) \in V_{\alpha_x} \subseteq \sigma_i\text{-int}(\sigma_j\text{-cl}(V_{\alpha_x}))$. Since f is (i, j) -almost

δ -continuous and σ_i -int(σ_j -cl(V_{α_x})) is (σ_i, σ_j) -regular open set in Y containing $f(x)$, then for each $x \in X$ there exists a (τ_i, τ_j) -regular open subset U_{α_x} of X containing x such that $f(U_{\alpha_x}) \subseteq \sigma_j$ -cl(σ_i -int(σ_j -cl(V_{α_x}))) $\subseteq \sigma_j$ -cl(V_{α_x}). So $\{U_{\alpha_x} : x \in X\}$ forms a (τ_i, τ_j) -regular open cover of X . Since X is (τ_i, τ_j) -nearly Lindelöf, there exists a countable subset of points $x_1, \dots, x_n, \dots$ of X such that $X = \bigcup_{n \in \mathbf{N}} U_{\alpha_{x_n}}$. Since f is surjective, $Y = f(X) = \bigcup_{n \in \mathbf{N}} f(U_{\alpha_{x_n}}) \subseteq \bigcup_{n \in \mathbf{N}} \sigma_j$ -cl($V_{\alpha_{x_n}}$). This shows that Y is (σ_i, σ_j) -almost Lindelöf. $\square$

Corollary 4.9. *Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be a pairwise almost δ -continuous and surjective function. If X is pairwise nearly Lindelöf, then Y is pairwise almost Lindelöf.*

Lemma 4.2. *If $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is (i, j) - θ -continuous, then f is (i, j) -almost δ -continuous.*

Proof. Let $x \in X$ and let V be a (σ_i, σ_j) -regular open subset of Y containing $f(x)$. Then V is also a σ_i -open subset of Y containing $f(x)$. Since f is (i, j) - θ -continuous, there exists a τ_i -open subset U of X containing x such that $f(\tau_j$ -cl(U)) $\subseteq \sigma_j$ -cl(V). Take $W = \tau_i$ -int(τ_j -cl(U)), then W is a (τ_i, τ_j) -regular open subset of X containing x . So $f(W) = f(\tau_i$ -int(τ_j -cl(U))) $\subseteq f(\tau_j$ -cl(U)) $\subseteq \sigma_j$ -cl(V). This implies that f is (i, j) -almost δ -continuous. $\square$

Corollary 4.10. *If $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is pairwise θ -continuous, then f is pairwise almost δ -continuous.*

The converse of Lemma 4.2 is not true in general as the following example shows.

Example 4.2. *Let $X = \{a, b, c\}$ with topologies*

$$\tau_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, X\}, \tau_2 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\}$$

and

$$\sigma_1 = \{\emptyset, \{a\}, \{c\}, \{a, c\}, X\}, \sigma_2 = \{\emptyset, \{b, c\}, X\}.$$

Then the function $f : (X, \tau_1, \tau_2) \rightarrow (X, \sigma_1, \sigma_2)$ defined by $f(a) = a$, $f(b) = c$ and $f(c) = b$ is 1-continuous so $(1, 2)$ -almost continuous. The

function f is also $(1, 2)$ -almost δ -continuous but it is not $(1, 2)$ - θ -continuous since $\{a\}$ is σ_1 -open set in (X, σ_1, σ_2) containing $f(a) = a$ but there is no τ_1 -open set U in (X, τ_1, τ_2) containing a such that $f(\tau_2\text{-cl}(U)) \subseteq \sigma_2\text{-cl}(\{a\}) = \{a\}$.

Corollary 4.11. *Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be an (i, j) - θ -continuous and surjective function. If X is (τ_i, τ_j) -nearly Lindelöf, then Y is (σ_i, σ_j) -almost Lindelöf.*

Proof. This is a direct consequence of Lemma 4.2 and Proposition 4.2 above. $\square$

Corollary 4.12. *Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be a pairwise θ -continuous and surjective function. If X is pairwise nearly Lindelöf, then Y is pairwise almost Lindelöf.*

Proposition 4.3. *Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be a surjective, (j, i) -almost continuous and (i, j) -precontinuous function. If X is (τ_i, τ_j) -nearly Lindelöf, then Y is (σ_i, σ_j) -almost Lindelöf.*

Proof. Let $\{V_\alpha : \alpha \in \Delta\}$ be a σ_i -open cover of Y . For each $x \in X$, take $\alpha_x \in \Delta$ such that $f(x) \in V_{\alpha_x}$. Since f is (i, j) -precontinuous, $f^{-1}(V_{\alpha_x})$ is (τ_i, τ_j) -preopen set in X , i.e., $f^{-1}(V_{\alpha_x}) \subseteq \tau_i\text{-int}(\tau_j\text{-cl}(f^{-1}(V_{\alpha_x})))$. Now

$$\left\{ \tau_i\text{-int}(\tau_j\text{-cl}(f^{-1}(V_{\alpha_x}))) : x \in X \right\}$$

forms a (τ_i, τ_j) -regular open cover of X . Since X is (τ_i, τ_j) -nearly Lindelöf, there exists a countable subset of points $x_1, \dots, x_n, \dots$ of X such that $X = \bigcup_{n \in \mathbf{N}} \tau_i\text{-int}(\tau_j\text{-cl}(f^{-1}(V_{\alpha_{x_n}})))$. Since f is (j, i) -almost

continuous and $\sigma_j\text{-cl}(V_{\alpha_{x_n}})$ is a (σ_j, σ_i) -regular closed set in Y , $\tau_j\text{-cl}(f^{-1}(V_{\alpha_{x_n}})) \subseteq f^{-1}(\sigma_j\text{-cl}(V_{\alpha_{x_n}}))$ because $f^{-1}(\sigma_j\text{-cl}(V_{\alpha_{x_n}}))$ is a τ_j -closed set in X containing $f^{-1}(V_{\alpha_{x_n}})$. So

$$X = \bigcup_{n \in \mathbf{N}} \tau_i\text{-int}(\tau_j\text{-cl}(f^{-1}(V_{\alpha_{x_n}}))) \subseteq \bigcup_{n \in \mathbf{N}} \tau_j\text{-cl}(f^{-1}(V_{\alpha_{x_n}})) \subseteq$$

$$\bigcup_{n \in \mathbf{N}} f^{-1}(\sigma_j\text{-cl}(V_{\alpha_{x_n}})).$$

Since f is surjective,

$$Y = f(X) \subseteq \bigcup_{n \in \mathbf{N}} f(f^{-1}(\sigma_j\text{-cl}(V_{\alpha_{x_n}}))) \subseteq \bigcup_{n \in \mathbf{N}} \sigma_j\text{-cl}(V_{\alpha_{x_n}}).$$

This shows that Y is (σ_i, σ_j) -almost Lindelöf. $\square$

Corollary 4.13. *Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be a surjective, pairwise almost continuous and pairwise precontinuous function. If X is pairwise nearly Lindelöf, then Y is pairwise almost Lindelöf.*

Proposition 4.4. *Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be a surjective, (i, j) -precontinuous and (i, j) -subcontra-continuous function. If X is (τ_i, τ_j) -almost Lindelöf, then Y is σ_i -Lindelöf.*

Proof. Let $\{U_\alpha : \alpha \in \Delta\}$ be a σ_i -open cover of Y . Let $\mathcal{B}_i$ be a σ_i -open base for the topology σ_i on Y . For each $x \in X$, take $\alpha_x \in \Delta$ such that $f(x) \in U_{\alpha_x}$. Then there exists $V_{\alpha_x} \in \mathcal{B}_i$ such that $f(x) \in V_{\alpha_x} \subseteq U_{\alpha_x}$. Since f is (i, j) -subcontra-continuous, $f^{-1}(V_{\alpha_x})$ is τ_j -closed in X . Since f is (i, j) -precontinuous, $f^{-1}(V_{\alpha_x})$ is a (τ_i, τ_j) -preopen set in X , i.e., $f^{-1}(V_{\alpha_x}) \subseteq \tau_i\text{-int}(\tau_j\text{-cl}(f^{-1}(V_{\alpha_x}))) = \tau_i\text{-int}(f^{-1}(V_{\alpha_x}))$. So $f^{-1}(V_{\alpha_x}) = \tau_i\text{-int}(f^{-1}(V_{\alpha_x}))$. It follows that $f^{-1}(V_{\alpha_x})$ is (τ_j, τ_i) -clopen and hence $\{f^{-1}(V_{\alpha_x}) : x \in X\}$ forms a (τ_j, τ_i) -clopen cover of X . Since X is (τ_i, τ_j) -almost Lindelöf, there exists a countable subset of points $x_1, \dots, x_n, \dots$ of X such that $X = \bigcup_{n \in \mathbf{N}} \tau_j\text{-cl}(f^{-1}(V_{\alpha_{x_n}})) = \bigcup_{n \in \mathbf{N}} f^{-1}(V_{\alpha_{x_n}}) \subseteq \bigcup_{n \in \mathbf{N}} f^{-1}(U_{\alpha_{x_n}})$. Since f is surjective, $Y = f(X) \subseteq \bigcup_{n \in \mathbf{N}} f(f^{-1}(U_{\alpha_{x_n}})) \subseteq \bigcup_{n \in \mathbf{N}} U_{\alpha_{x_n}}$. This shows that Y is σ_i -Lindelöf. $\square$

Corollary 4.14. *Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be a surjective, pairwise precontinuous and pairwise subcontra-continuous function. If X is pairwise almost Lindelöf, then Y is Lindelöf.*

Since every (i, j) -contra-continuous function is (i, j) -subcontra-continuous, we have the following corollary.

Corollary 4.15. *Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be a surjective, (i, j) -precontinuous and (i, j) -contra-continuous function. If X is (τ_i, τ_j) -almost Lindelöf, then Y is σ_i -Lindelöf.*

Corollary 4.16. *Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be a surjective, pairwise precontinuous and pairwise contra-continuous function. If X is pairwise almost Lindelöf, then Y is Lindelöf.*

Many types of open functions between topological spaces are studied such as open, contra open, weakly open etc. We extend these types of open functions into bitopological spaces as follows.

Definition 4.3. A function $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is said to be

- (a) i -open if the functions $f : (X, \tau_i) \rightarrow (Y, \sigma_i)$ is open, f is called open if it is i -open for each $i = 1, 2$;
- (b) i -contra open if $f(U)$ is σ_i -closed for every τ_i -open set U in X , f is called contra open if it is i -contra open for each $i = 1, 2$.
- (c) (i, j) -contra open if $f(U)$ is σ_j -closed for every τ_i -open set U in X , f is called pairwise contra open if it is both $(1, 2)$ -contra open and $(2, 1)$ -contra open.

Definition 4.4. A bitopological space (X, τ_1, τ_2) is said to be

- (i) i - P -space if countable intersection of i -open sets in X is i -open. X is called P -space if it is i - P -space for each $i = 1, 2$;
- (ii) (i, j) - P -space if countable intersection of i -open sets in X is j -open. X is called B - P -space if it is $(1, 2)$ - P -space and $(2, 1)$ - P -space.

Definition 4.5. [11, 15] A subset S of a bitopological space X is said to be (i, j) -almost Lindelöf relative to X if for every cover $\{U_\alpha : \alpha \in \Delta\}$ of S by i -open subsets of X such that $S \subseteq \bigcup_{\alpha \in \Delta} U_\alpha$, there exists a countable subset $\{\alpha_n : n \in \mathbf{N}\}$ of Δ such that $S \subseteq \bigcup_{n \in \mathbf{N}} j\text{-cl}(U_{\alpha_n})$. S is called pairwise almost Lindelöf relative to X if it is both $(1, 2)$ -almost Lindelöf relative to X and $(2, 1)$ -almost Lindelöf relative to X .

Theorem 4.3. [11, 15] Let X be a bitopological space and $S \subseteq X$. The following are equivalent:

- (i) S is (i, j) -almost Lindelöf relative to X ;
- (ii) for every family $\{C_\alpha : \alpha \in \Delta\}$ of i -closed subsets of X such that $\left(\bigcap_{\alpha \in \Delta} C_\alpha\right) \cap S = \emptyset$, there exists a countable subfamily $\{C_{\alpha_n} : n \in \mathbf{N}\}$ such that

$$\left(\bigcap_{n \in \mathbf{N}} j\text{-int}(C_{\alpha_n})\right) \cap S = \emptyset;$$

- (iii) for every family $\{C_\alpha : \alpha \in \Delta\}$ of i -closed subsets of X for which every countable subfamily $\{C_{\alpha_n} : n \in \mathbf{N}\}$ satisfies $\left(\bigcap_{n \in \mathbf{N}} j\text{-int}(C_{\alpha_n})\right) \cap S = \emptyset$, the intersection $\left(\bigcap_{\alpha \in \Delta} C_\alpha\right) \cap S = \emptyset$.

Now we have one question. Let $f : X \rightarrow Y$ be a function from X into (i, j) -almost Lindelöf space Y . What conditions the function f should satisfy in order for X to be also (i, j) -almost Lindelöf? The following theorem will answer this question. Recall that from elementary general topology, if a function $f : X \rightarrow Y$ is an open function, then $f(\text{int}(A)) \subseteq \text{int}(f(A))$ for each $A \subseteq X$ (see [5]).

Theorem 4.4. *Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be an i -contra open and j -open function and for each $y \in Y$, let $\{f^{-1}(y)\}$ be (τ_i, τ_j) -almost Lindelöf relative to X . If X is (τ_j, τ_i) - P -space and Y is (σ_i, σ_j) -almost Lindelöf, then X is (τ_i, τ_j) -almost Lindelöf.*

Proof. Let $\{C_\alpha : \alpha \in \Delta\}$ be a family of τ_i -closed subsets of X for which every countable subfamily $\{C_{\alpha_n} : n \in \mathbf{N}\}$ satisfies $\bigcap_{n \in \mathbf{N}} \tau_j\text{-int}(C_{\alpha_n}) \neq \emptyset$. Let

$M = \Delta^{\mathbf{N}}$, i.e., each $\mu \in M$ is of the form $\mu = (\alpha_1, \alpha_2, \dots, \alpha_n, \dots)$, $\alpha_k \in \Delta$ for each $k \in \mathbf{N}$. Put $C_\mu = \bigcap_{n \in \mathbf{N}} \tau_j\text{-int}(C_{\alpha_n}) \neq \emptyset$, $\mu \in M$. Since X is (τ_j, τ_i) - P -space, then $\{C_\mu : \mu \in M\}$ is a family of τ_i -open subsets of X for which the countable subfamily $\{C_{\mu_m} : m \in \mathbf{N}\}$ satisfies $\bigcap_{m \in \mathbf{N}} \tau_j\text{-int}(C_{\mu_m}) \neq \emptyset$. Since

f is i -contra open function, then $\{f(C_\mu) : \mu \in M\}$ is a family of σ_i -closed subsets of Y such that $\bigcap_{m \in \mathbf{N}} f(\tau_j\text{-int}(C_{\mu_m})) \supseteq f\left(\bigcap_{m \in \mathbf{N}} \tau_j\text{-int}(C_{\mu_m})\right) \neq \emptyset$.

Hence $\bigcap_{m \in \mathbf{N}} \sigma_j\text{-int}(f(C_{\mu_m})) \neq \emptyset$ since f is a j -open function. So there exists

a countable subfamily $\{f(C_{\mu_m}) : m \in \mathbf{N}\}$ such that $\bigcap_{m \in \mathbf{N}} \sigma_j\text{-int}(f(C_{\mu_m})) \neq$

$\emptyset$. Since Y is (σ_i, σ_j) -almost Lindelöf, by Theorem 4.1(iii), $\bigcap_{\mu \in M} f(C_\mu) \neq \emptyset$

and hence there exists $y_0 \in Y$ such that $y_0 \in f(C_\mu)$ for each $\mu \in M$. It follows that $\{f^{-1}(y_0)\} \cap C_\mu \neq \emptyset$ for each $\mu \in M$, thus $\{f^{-1}(y_0)\}$ intersects

all countable subfamily $\{C_{\alpha_n} : n \in \mathbf{N}\}$ such that $\left(\bigcap_{n \in \mathbf{N}} \tau_j\text{-int}(C_{\alpha_n})\right) \cap$

$\{f^{-1}(y_0)\} \neq \emptyset$. Since $\{f^{-1}(y_0)\}$ is (τ_i, τ_j) -almost Lindelöf relative to X , by

Theorem 4.3(iii), we have $\left(\bigcap_{\alpha \in \Delta} C_\alpha\right) \cap \{f^{-1}(y_0)\} \neq \emptyset$ and thus $\bigcap_{\alpha \in \Delta} C_\alpha \neq \emptyset$.

This, by Theorem 4.1, implies that X is (τ_i, τ_j) -almost Lindelöf. $\square$

5. Mappings on (i, j) -weakly Lindelöf Spaces

Definition 5.1. [13] A bitopological space (X, τ_1, τ_2) is said to be (i, j) -weakly Lindelöf if for every i -open cover $\{U_\alpha : \alpha \in \Delta\}$ of X , there exists a countable subset $\{\alpha_n : n \in \mathbf{N}\}$ of Δ such that $X = j\text{-cl} \left(\bigcup_{n \in \mathbf{N}} U_{\alpha_n} \right)$. X is called pairwise weakly Lindelöf if it is both $(1, 2)$ -weakly Lindelöf and $(2, 1)$ -weakly Lindelöf.

Fawakhreh and Kılıçman [7] proved that the almost continuous image of a weakly Lindelöf space is weakly Lindelöf. For the case of (τ_i, τ_j) -weakly Lindelöf spaces we give the following theorem.

Theorem 5.1. Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be a surjective and pairwise almost continuous function. If X is (τ_i, τ_j) -weakly Lindelöf, then Y is (σ_i, σ_j) -weakly Lindelöf.

Proof. Let $\{U_\alpha : \alpha \in \Delta\}$ be a σ_i -open cover of Y . Then $\{\sigma_i\text{-int}(\sigma_j\text{-cl}(U_\alpha)) : \alpha \in \Delta\}$ is a (σ_i, σ_j) -regular open cover of Y . Since f is (i, j) -almost continuous,

$$f^{-1}(\sigma_i\text{-int}(\sigma_j\text{-cl}(U_\alpha)))$$

is τ_i -open set in X . So $\{f^{-1}(\sigma_i\text{-int}(\sigma_j\text{-cl}(U_\alpha))) : \alpha \in \Delta\}$ forms a τ_i -open cover of X . Since X is (τ_i, τ_j) -weakly Lindelöf, there exists a countable subset $\{\alpha_n : n \in \mathbf{N}\}$ of Δ such that

$$\begin{aligned} X &= \tau_j\text{-cl} \left(\bigcup_{n \in \mathbf{N}} f^{-1}(\sigma_i\text{-int}(\sigma_j\text{-cl}(U_{\alpha_n}))) \right) \\ &\subseteq \tau_j\text{-cl} \left(\bigcup_{n \in \mathbf{N}} f^{-1}(\sigma_j\text{-cl}(U_{\alpha_n})) \right) \\ &= \tau_j\text{-cl} \left(f^{-1} \left(\bigcup_{n \in \mathbf{N}} \sigma_j\text{-cl}(U_{\alpha_n}) \right) \right) \\ &\subseteq \tau_j\text{-cl} \left(f^{-1} \left(\sigma_j\text{-cl} \left(\bigcup_{n \in \mathbf{N}} U_{\alpha_n} \right) \right) \right). \end{aligned}$$

Since $\sigma_j\text{-cl}\left(\bigcup_{n\in\mathbf{N}} U_{\alpha_n}\right)$ is a (σ_j, σ_i) -regular closed set in Y and f is (j, i) -almost continuous, then $f^{-1}\left(\sigma_j\text{-cl}\left(\bigcup_{n\in\mathbf{N}} U_{\alpha_n}\right)\right)$ is τ_j -closed in X . So

$$X = \tau_j\text{-cl}\left(f^{-1}\left(\sigma_j\text{-cl}\left(\bigcup_{n\in\mathbf{N}} U_{\alpha_n}\right)\right)\right) = f^{-1}\left(\sigma_j\text{-cl}\left(\bigcup_{n\in\mathbf{N}} U_{\alpha_n}\right)\right).$$

Since f is surjective, we have

$$Y = f(X) = f\left(f^{-1}\left(\sigma_j\text{-cl}\left(\bigcup_{n\in\mathbf{N}} U_{\alpha_n}\right)\right)\right) \subseteq \sigma_j\text{-cl}\left(\bigcup_{n\in\mathbf{N}} U_{\alpha_n}\right).$$

This shows that Y is (σ_i, σ_j) -weakly Lindelöf. $\square$

Corollary 5.1. *Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be a surjective and pairwise almost continuous function. If X is pairwise weakly Lindelöf, then Y is pairwise weakly Lindelöf.*

Similar argument with (i, j) -almost Lindelöf, we cannot directly say that (i, j) -weakly Lindelöf is an i -bitopological property since i -continuity and pairwise almost continuity are independent concepts. Since every continuous function is pairwise almost continuous, Theorem 5.1 and Corollary 5.1 easily imply the following result.

Corollary 5.2. *The (i, j) -weakly Lindelöf property and the pairwise weakly Lindelöf property are bitopological properties.*

We have defined the concept of (i, j) -nearly paracompact and pairwise nearly paracompact property in [13, 14]. Next we will introduce another new notion related to the generalization of nearly paracompactness into bitopological space.

Definition 5.2. *A bitopological space X is said to be $(i, j)_1$ -nearly paracompact if every cover of X by (i, j) -regular open sets admits an i -open refinement which is j -locally finite. X is called B -nearly paracompact if it is both $(1, 2)_1$ -nearly paracompact and $(2, 1)_1$ -nearly paracompact.*

Definition 5.3. [10, 14] A bitopological space X is said to be (i, j) -semiregular if for each $x \in X$ and for each i -open set V of X containing x , there is an i -open set U such that $x \in U \subseteq i\text{-int}(j\text{-cl}(U)) \subseteq V$. X is called pairwise semiregular if it is both $(1, 2)$ -semiregular and $(2, 1)$ -semiregular.

Proposition 5.1. An (i, j) -weakly Lindelöf, (i, j) -semiregular and $(i, j)_1$ -nearly paracompact bitopological space X is (i, j) -almost Lindelöf.

Proof. Let $\{U_\alpha : \alpha \in \Delta\}$ be an (i, j) -regular open cover of X . Since X is $(i, j)_1$ -nearly paracompact, this cover admits an i -open refinement which is j -locally finite $\{V_\lambda : \lambda \in \Lambda\}$. Since X is (i, j) -weakly Lindelöf, there exists a countable subfamily $\{V_{\lambda_n} : n \in \mathbf{N}\}$ such that $X = j\text{-cl}\left(\bigcup_{n \in \mathbf{N}} V_{\lambda_n}\right)$. Since the family $\{V_{\lambda_n} : n \in \mathbf{N}\}$ is j -locally finite, then $j\text{-cl}\left(\bigcup_{n \in \mathbf{N}} V_{\lambda_n}\right) = \bigcup_{n \in \mathbf{N}} j\text{-cl}(V_{\lambda_n})$ (see [6]). Choosing, for $n \in \mathbf{N}$, $\alpha_n \in \Delta$ such that $V_{\lambda_n} \subseteq U_{\alpha_n}$, we obtain $X = \bigcup_{n \in \mathbf{N}} j\text{-cl}(V_{\lambda_n}) = \bigcup_{n \in \mathbf{N}} j\text{-cl}(U_{\alpha_n})$. By Theorem 3.1 in [15], X is (i, j) -almost Lindelöf. $\square$

Corollary 5.3. A pairwise weakly Lindelöf, pairwise semiregular and B -nearly paracompact bitopological space X is pairwise almost Lindelöf.

Proposition 5.2. An (i, j) -weakly Lindelöf, (i, j) -regular and $(i, j)_1$ -nearly paracompact bitopological space X is i -Lindelöf.

Proof. Let $\{U_\alpha : \alpha \in \Delta\}$ be an i -open cover of X . Since X is (i, j) -regular, then it is (i, j) -semiregular. So X is (i, j) -almost Lindelöf by Proposition 5.1. Since X is (i, j) -regular, then X is i -Lindelöf by Proposition 3.4 in [15]. $\square$

Corollary 5.4. A pairwise weakly Lindelöf, pairwise regular and B -nearly paracompact bitopological space X is Lindelöf.

By using Proposition 5.1 and Theorem 5.1 above, we obtain the following corollary.

Corollary 5.5. Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be a surjective and pairwise almost continuous function. If X is (τ_i, τ_j) -weakly Lindelöf and Y is (σ_i, σ_j) -semiregular and (σ_i, σ_j) -nearly paracompact space, then Y is (σ_i, σ_j) -almost Lindelöf.

Moreover by using Proposition 5.2 and Theorem 5.1 above, we conclude the following corollary.

Corollary 5.6. *Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be a surjective and pairwise almost continuous function. If X is (τ_i, τ_j) -weakly Lindelöf and Y is (σ_i, σ_j) -regular and $(\sigma_i, \sigma_j)_1$ -nearly paracompact space, then Y is σ_i -Lindelöf.*

Corollary 5.7. *Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be a surjective and pairwise almost continuous function. If X is pairwise weakly Lindelöf and Y is pairwise regular and B -nearly paracompact space, then Y is Lindelöf.*

Theorem 5.2. *Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be a (j, i) -contra open and j -open function and for each $y \in Y$, let $\{f^{-1}(y)\}$ be (τ_i, τ_j) -weakly Lindelöf relative to X . If Y is (σ_i, σ_j) -weakly Lindelöf, then X is (τ_i, τ_j) -weakly Lindelöf.*

Proof. The proof is similar to the proof of Theorem 4.4 by using certain results in [13], so we omit the detail. $\square$

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