On stability functional differential equation with delay variable by using fixed point-theory

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Abstract:

I will explain how to use the Banach fixed point theory in the asymptotic stability of nonlinear differential equations; I will obtain appropriate generalizations and strong forms of some of the results in [2, 3, 5, 6, 10, 11, 12,13].

Specifically, in the above-mentioned paper, asymptotic stability is achieved, while I will discuss how to achieve a asymptotic stability as well as stability by making a simple observation, also circulate the previous asymptotic stability results to the Functional Differential Equations systems, not only on the scalar Functional Differential Equations as is the case in the mentioned paper.

This raises the question of how much this particular method can afford us, and what are the limitations of this technique. I will refer to the important limitation of the fixed point theory on the uniqueness of solutions only within the complete metric space area where they are not specified. If the metric space onto which the contraction mapping principle is applied is very small, i do not get a satisfactory result. I will discuss this in detail below.

Keywords: Nonlinear; Asymptotical stability; Banach fixed point theorem; Delayed functional differential equation.

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1. Introduction

Achieving the results of stability using the fixed point theory can sometimes provide better conditions for zero solution of Lyapunov methods.

The advantages of this specific method have been achieved by contraction mapping methods that require intermediate conditions of vector syndrome using the chosen variation of the parameters of the type parameters to reflect the basic equation to the integrated form.

As is known in the theory of differential equations, the common way of proving solutions is through fixed point methods. However, fairly recently, the principle of contraction mapping has been used to obtain additional properties of the solution, an attractive solutions for balance, not just the existence of curves of this solution, as is usual in classical differential equation theory.

2. Some Basic concept

2.1. Banach Contraction Mapping Principle [5, 6, 7, 8, 9]

Let (X, d) be a complete metric space. A map $P : X \rightarrow X$ is a contraction if there exists a non negative number $\rho \leq 1$ such that $d(P(x), P(y)) \leq \rho d(x, y)$. P is a strict contraction if $\rho < 1$.

A point $x \in X$ is called a fixed point of P if $P(x) = x$.

Theorem.(Contraction Mapping Principle). If $\phi : (S, d) \rightarrow (S, d)$ is a contraction mapping and if (S, d) is a complete metric space, then ϕ has a unique fixed point; that is, there is a unique $s^* \in S$ such that $\phi(s^*) = s^*$.

Definition. Let X be a non-empty set. A metric is a function $d : X \times X \rightarrow [0, +\infty)$, which satisfies the following properties for all x, y, z in X ,

- (i) $d(x, y) \geq 0$, for all $x, y \in X$. (positivity)
- (ii) $d(x, y) = 0$ iff $x = y$ for all $x, y \in X$. (non-degenerated)
- (iii) $d(x, y) = d(y, x)$ for all $x, y \in X$. (symmetry)
- (iv) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$. (triangle inequality)

A set X with a metric d defined on it is said to be a metric space, $d(x, y)$ to every pair $x, y \in X$ satisfying the above four properties, where d is distance function.

Example. Let $d : R \times R \rightarrow R$ be a function defined by $d(x, y) = |x - y|$, $x, y \in R$, we have to show that (R, d) is a **metric space**.

Solution. Since $|x - y| \geq 0$, and $|x - y| = 0$, iff $x = y$, and $|x - y| = |y - x|$, and $d(x, y) = |x - y| = |(x - z) + (z - y)| \leq |x - z| + |z - y| = d(x, z) + d(z, y)$. All properties in definition hold, thus d is a metric on R and consequently (R, d) is a metric space.

Definition. A metric space (X, d) is complete if every Cauchy sequence in X is converges to an element in X .

Definition. A complete normed linear (vector) space is called a **Banach space**.

In mathematics, a fixed point of a function is an element of the function's domain that is mapped to itself by the function. A set of fixed points is sometimes called a fixed set. That is to say, c is a fixed point of the function $f(x)$ if and only if $f(c) = c$. For example, if f is defined on the real numbers by

$$f(x) = x^2 - 3x + 4,$$

then 2 is a fixed point of f , because $f(2) = 2$.

Many different kinds of problems can be solved by means of fixed point theory. Generally, to solve a problem with fixed point theory is to find:

- (a) a set S consisting of points which would be acceptable solutions;
- (b) a mapping $P : S \rightarrow S$ with the property that a fixed point solves the problem;
- (c) a fixed point theorem stating that this mapping on this set will have a fixed point.

generally, the basic idea is to give a differential equation delay model

$$(2.1) \quad x' = f(t, x), \quad x(t_0) = \emptyset,$$

We are trying to build a reflective schema (1). In other words, we are doing some integration operations like this

$$(2.2) \quad x(t) = a(t) + \int_{t_0}^t f(t, \emptyset, s, x(\cdot)) ds.$$

The solution will be given to the functional differential equation through the fixed point theory mapping.

$$(Py)(t) = a(t) + \int_{t_0}^t f(t, \emptyset, s, y(\cdot)) ds.$$

In this way we decide that P -mapping is actually a viable representation by proving a solution of (1). Finally, the integrated solution format will provide us with stability assistance of (1). In addition, the complete metric space M will provide us with the benefits of asymptotic stability. And the M plays an important role in this, as we will see in the future.

Definition. The solution $x_\phi(t)$ of above equation is called stable, if for any $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$, such that from the inequality $|\phi(t) - \psi(t)| < \delta(\varepsilon)$ on the initial set, there follows $|x_\phi(t) - x_\psi(t)| < \varepsilon$ for all $t \geq t_0$, where $\psi(t)$ is any continuous initial function.

Definition. The solution of above equation $x_\psi(t)$ is called asymptotically stable in the large, if it is stable and $\lim_{t \rightarrow \infty} |x_\phi(t) - x_\psi(t)| = 0$, for all continuous initial functions $\psi(t)$.

3. A General Result for a Nonlinear Functional Differential Equation

The scalar delayed differential equation is study [1, 3, 4, 6, 8]

$$(3.1) \quad x' = -A(t)x(t) + f(t, x),$$

Where $A : \mathbf{R}_+ \rightarrow \mathbf{R}$ and $f : \mathbf{R}_+ \times BC \rightarrow \mathbf{R}$ are continuous where $BC = \{\emptyset \in C[\mathbf{R}_-, \mathbf{R}] : \emptyset \text{ bounded}\}$. We mean by $\mathbf{R}_+ = [0, \infty)$ and $\mathbf{R}_- = (-\infty, 0]$ respectively, the continuous function $x : \mathbf{R} \rightarrow \mathbf{R}$ denoted by $x(t)$, $x(t(\theta)) = x(t + \theta)$, for $\theta \in \mathbf{R}_-$.

We note that the author B. Zhang in [12] theorem has a necessary and adequate condition for asymptotic stability. We will focus only on the prevailing conditions to stabilize this work. In the above result, it indicates that zero is asymptotically stable if and only if $\int_0^t A(s) ds \rightarrow \infty$ as $t \rightarrow \infty$. Through analysis equation (3), realize that we can provide an integrated formula equivalent to this problem by doing something similar to what we do when we solve ODE linear order of first order, if solution of (3) exists, after we multiply it by the integrating factor:

$$\mu(t) = e^{\int_{t_0}^t A(s) ds}$$

To find

$$\frac{d}{dt}(x(t)\mu(t)) = \mu(t)f(t, x(t))$$

$$x(t) \mu(t) - x(t_0) = \int_{t_0}^t e^{\int_s^t A(u) du} f(s, x(s)) ds,$$

Within the initial condition of the differential differential equation

$$x(t) = \emptyset(0) e^{-\int_{t_0}^t A(s) ds} + \int_{t_0}^t e^{-\int_s^t A(u) du} f(s, x(s)) ds,$$

Thus we have an integral equivalent expression of the solution $x(t)$. This indicates that we are define the following mapping P defined on S :

$$(Px)(t) = \begin{cases} \emptyset(t - t_0) & \text{if } t \leq t_0 \\ \emptyset(0) e^{-\int_{t_0}^t A(s) ds} + \int_{t_0}^t e^{-\int_s^t A(u) du} f(s, x(s)) ds, & \text{if } t \geq t_0 \end{cases}$$

In order to apply fixed point theory, we must prove that P maps S to itself, $Px: \mathbf{R} \rightarrow \mathbf{R}$ is continuous, $Px(t_0) = \emptyset$, $\emptyset \in C(\delta_0)$, and $\|x_s\| \leq L$.

In B. Zhang in [12] Theorem we have the following conditions

1. $\liminf_{t \rightarrow \infty} \int_0^t A(s) ds > -\infty$.
2. $\int_0^t e^{-\int_{t_0}^t A(u) du} b(s) ds \leq \alpha < 1$ for all $t \geq 0$.
3. $|f(t, \emptyset) - f(t, \psi)| \leq b(t) \|\emptyset - \psi\|$, for all $\emptyset, \psi \in C(L)$, and $f(t, 0) = 0$.
4. $\int_0^t A(s) ds \rightarrow \infty$ as $t \rightarrow \infty$, then the zero solution of (3) is asymptotic stable.
5. $\forall \varepsilon > 0$ and $t_1 \geq 0$, then $\exists t_2 > t_1$ s.t. $t \geq t_2$, and $x(t) \in C(L)$, this implies that $|f(t, x(t))| \leq b(t)(\varepsilon + \|x\|[t_1, t])$.

Then by using the fact $\emptyset \in C(\delta_0)$, and $\|x(s)\| \leq L$ and condition (2,3) from B. Zhang in [12] theorem, we have that

$$\begin{aligned} |(Px)(t)| &\leq \left| \emptyset(0) e^{-\int_{t_0}^t A(s) ds} + \int_{t_0}^t e^{-\int_s^t A(u) du} f(s, x(s)) ds \right| \\ &\leq \delta_0 e^{-\int_{t_0}^t A(s) ds} + \int_{t_0}^t e^{-\int_s^t A(u) du} b(s) \|x(s)\| ds \\ &\leq \delta_0 K + L \int_{t_0}^t e^{-\int_s^t A(u) du} b(s) ds \\ &\leq \delta_0 K + L\alpha \leq L \end{aligned}$$

Now we will show that $(Px)(t) \rightarrow 0$ as $t \rightarrow \infty$, since $x(t) \rightarrow 0$ as $t \rightarrow \infty$ and $\exists t_1 > t_0$, s.t. $|x(t)| < \varepsilon \quad \forall t \geq t_1$, then by condition (5) of theorem B. Zhang in [12] $\exists t_2 \geq t_1$ s.t. $t \geq t_2$ this implies that $|f(t, x(t))| \leq b(t)(\varepsilon + \|x\|_{[t_1, t]})$ for $t \geq t_2$

$$\begin{aligned} & \left| \int_{t_0}^t e^{-\int_s^t A(u)du} f(s, x(s)) ds \right| \\ & \leq \left| \int_{t_0}^{t_2} e^{-\int_s^t A(u)du} f(s, x(s)) ds \right| + \left| \int_{t_2}^t e^{-\int_s^t A(u)du} f(s, x(s)) ds \right| \\ & \leq \int_{t_0}^{t_2} e^{-\int_s^t A(u)du} |f(s, x(s))| ds + \int_{t_2}^t e^{-\int_s^t A(u)du} |f(s, x(s))| ds \\ & \leq \int_{t_0}^{t_2} e^{-\int_s^t A(u)du} b(s) \|x(s)\| ds + \int_{t_2}^t e^{-\int_s^t A(u)du} b(s) (\varepsilon + \|x\|_{[t_1, t]}) ds \end{aligned}$$

Since $\|x\|_{[t_1, t]} \leq \varepsilon$ for $t \geq t_2$, then we get

$$\begin{aligned} & \leq \|x(t)\| \int_{t_0}^{t_2} e^{-\int_s^t A(u)du} b(s) ds + \int_{t_2}^t e^{-\int_s^t A(u)du} b(s) (2\varepsilon) ds \\ & \leq L \int_{t_0}^{t_2} e^{-\int_s^t A(u)du - \int_{t_2}^t A(u)du} b(s) ds + 2\varepsilon \alpha \\ & \leq \alpha L e^{-\int_{t_2}^t A(u)du} + 2\varepsilon \alpha \end{aligned}$$

By condition (4) of theorem B. Zhang in [12] $\exists t_3 \geq t_2$ s.t.

$$\delta_0 e^{-\int_{t_0}^t A(u)du} + L e^{-\int_{t_2}^t A(u)du} < \varepsilon$$

This two estimate implies that $t \geq t_3$, and

$$\begin{aligned} |(Px)(t)| & \leq \left| \emptyset(0) e^{-\int_{t_0}^t A(s)ds} + \int_{t_0}^t e^{-\int_s^t A(u)du} f(s, x(s)) ds \right| \\ & \leq \delta_0 e^{-\int_{t_0}^t A(s)ds} + \alpha L e^{-\int_{t_2}^t A(u)du} + 2\alpha \varepsilon < 3\varepsilon \end{aligned}$$

This prove that $(Px)(t) \rightarrow 0$ as $t \rightarrow \infty$, and $Px \in S$ for $\forall x \in S$, and $P : S \rightarrow S$ is well define, now to proving P is a contraction mapping on S , let $x, y \in S$:

$$\begin{aligned} |(Px)(t) - (Py)(t)| & \leq \int_{t_0}^t e^{-\int_s^t A(u)du} |f(s, x(s)) - f(s, y(s))| ds \\ & \leq \int_{t_0}^t e^{-\int_s^t A(u)du} b(s) \|x(s) - y(s)\| ds \\ & \leq \sup_{s \in [t_0, t]} \|x(s) - y(s)\| \int_{t_0}^t e^{-\int_s^t A(u)du} b(s) ds \\ & \leq \alpha \rho(x, y) \end{aligned}$$

By Contraction Mapping Principle theorem $\exists$ unique fixed point $x \in S$ that solve equation (3) for each $\emptyset \in C(\delta_0)$, and we have $x(t) = x(t, t_0, \emptyset)$ converge to zero as $t \rightarrow \infty$.

Let $\varepsilon > 0$, $\varepsilon < L$ are given we will find $\delta < \varepsilon$ s.t. $\delta K + \alpha \varepsilon < \varepsilon$, then

$$x(t) = x(t, t_0, \emptyset) = \emptyset(0) e^{-\int_{t_0}^t A(s) ds} + \int_{t_0}^t e^{-\int_s^t A(u) du} f(s, x(s)) ds,$$

We prove that $|x(t)| < \varepsilon$, $\forall t \geq t_0$, and $\exists T > t_0$ notice that $x(t_0) < \delta < \varepsilon$ for $t_0 \leq s < T$. But $|x(T)| = \varepsilon$, then

$$\begin{aligned} |x(T)| &\leq \delta e^{-\int_{t_0}^T A(s) ds} + \int_{t_0}^T e^{-\int_s^T A(u) du} b(s) \|x(s)\| ds, \\ &\leq \delta K + \alpha \varepsilon < \varepsilon, \end{aligned}$$

Which is contradiction to the definition of T . This means that there T is not exist and $x(t) < \varepsilon$ for $\forall t \geq t_0$, then the zero solution of equation (3) is asymptotic stable.

References

- [1] L. C. Becker and T. A. Burton, "Stability, fixed points and inverses of delays", *Proceedings of the Royal Society of Edinburgh: section a mathematics*, vol. 136, no. 2, pp. 245–275, Apr. 2006, doi: 10.1017/S0308210500004546
- [2] T. A. Burton, "Fixed points and stability of a nonconvolution equation", *Proceedings of the American Mathematical Society*, vol. 132, no. 12, pp. 3679–3688, Dec. 2004, doi: 10.1090/S0002-9939-04-07497-0
- [3] T. A. Burton, "Fixed points, stability, and exact linearization", *Nonlinear analysis: theory, methods & applications*, vol. 61, no. 5, pp. 857–870, May 2005, doi: 10.1016/j.na.2005.01.079
- [4] T. A. Burton, "The case for stability by fixed point theory", *Dynamics of continuous, discrete and impulsive systems series a: mathematical analysis*, vol. 13B, suppl., pp. 253–263, 2006. [On line]. Available: <https://bit.ly/351iN5e>
- [5] T. A. Burton, *Stability by fixed point theory for functional differential equations*. Mineola : Dover Publ. Inc., 2006.
- [6] T. A. Burton and T. Furumochi, "Fixed points and problems in stability theory for ordinary and functional differential equations". *Dynamic systems and applications*, vol. 10, no. 1, pp. 89–116, 2001. [On line]. Available: <https://bit.ly/2KzJVP9>

- [7] T. A. Burton, "Stability by fixed point theory or liapunov theory: a comparison", *Fixed point theory*, vol. 4, no. 1, pp. 15-32, 2003. [On line]. Available: <https://bit.ly/2KxPFci>
- [8] T. A. Burton, "Stability by fixed point methods for highly nonlinear delay equations", *Fixed point theory*, vol. 5, no. 1, pp. 3-20, 2004. [On line]. Available: <https://bit.ly/2W1lfDn>
- [9] T. A. Burton and T. Furumochi, "Fixed points and problems in stability theory for ordinary and functional differential equations", *Dynamic systems and applications*, vol. 10, pp. 89-116, 2001. [On line]. Available: <https://bit.ly/2xc0e1N>
- [10] T. Burton and I. Purnaras, "Positive kernels, fixed points, and integral equations", *Electronic journal of qualitative theory of differential equations*, no. 44, pp. 1-21, Jun. 2018. doi: 10.14232/ejqtde.2018.1.44
- [11] E. Zeidler, *Nonlinear functional analysis and its applications*, vol. 1. New York: Springer, 1986.
- [12] B. Zhang, "Contraction mapping and stability in a delay-differential equation", *Proceedings of dynamic systems and applications*, vol. 4, pp. 183-190, 2004. [On line]. Available: <https://bit.ly/2xa93Jh>
- [13] B. Zhang, "Fixed points and stability in differential equations with variable delays", *Nonlinear analysis: theory, methods & applications*, vol. 63, no. 5-7, pp. 233-242, Nov. 2005, doi: 10.1016/j.na.2005.02.081