Proyecciones Journal of Mathematics
Vol. 37, N^o 3, pp. 565-581, September 2018.
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An integral functional equation on groups under two measures

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Received : November 2017. Accepted : May 2018

Abstract

Let G be a locally compact Hausdorff group, let σ be a continuous involutive automorphism on G, and let μ, ν be regular, compactly supported, complex-valued Borel measures on G. We find the continuous solutions $f : G \to \mathbf{C}$ of the functional equation

$$\int_G f(\sigma(y)xt)d\mu(t) + \int_G f(xyt)d\nu(t) = f(x)f(y), \quad x, y \in G,$$

in terms of continuous characters of G. This equation provides a common generalization of many functional equations (d'Alembert's, Cauchy's, Gajda's, Kannappan's, Stetkær's, Van Vleck's equations...). So, a large class of functional equations will be solved.

Subjclass [2010] : *Primary 39B32, 39B52.*

Keywords : *Functional equation, Van Vleck, Kannappan, involutive automorphism, group character.*

1. Introduction

Let G be a group, $z_0 \in G$ be a fixed element, and $\sigma : G \to G$ be an involutive automorphism. In [2], the authors determined the general solutions $f, g : G \to \mathbf{C}$ of the functional equation

$$f(\sigma(y)xz_0) + g(xyz_0) = 2f(x)f(y), \quad x, y \in G. \tag{1.1}$$

This equation is a generalization of the functional equation

$$f(\sigma(y)xz_0) - f(xyz_0) = 2f(x)f(y), \quad x, y \in G, \tag{1.2}$$

which contains the solutions of a functional equation due to Van Vleck [10]. In 1910, he studied the continuous solutions $f : \mathbf{R} \to \mathbf{R}$, $f \neq 0$, of the functional equation

$$f(x - y + z_0) - f(x + y + z_0) = 2f(x)f(y), \quad x, y \in \mathbf{R},$$

where $z_0 > 0$ is fixed, with a view to characterize the sine function on the real line (see [7], p. 156). The functional equation (1.1) is also a generalization of the functional equation

$$f(\sigma(y)xz_0) + f(xyz_0) = 2f(x)f(y), \quad x, y \in G, \tag{1.3}$$

which generalizes a functional equation studied by Kannappan in [4]. He, in 1968, proved that a function $f : \mathbf{R} \to \mathbf{C}$ satisfies the functional equation

$$f(x - y + 2z_0) + f(x + y + 2z_0) = 2f(x)f(y), \quad x, y \in \mathbf{R}, \tag{1.4}$$

for a fixed non-zero $z_0 \in \mathbf{R}$ if and only if $f(x) = g(x-2z_0)$, where $g : \mathbf{R} \to \mathbf{C}$ is a periodic solution of the cosine functional equation $g(x+y)+g(x-y) = 2g(x)g(y)$ for all $x, y \in \mathbf{R}$ with period $4z_0$. The only non-zero continuous real-valued solutions of (1.4) (see [5], Corollary 3.14a, p. 118) are given by $f(x) = \cos(\frac{2n\pi x}{z_0})$ or $\cos(\frac{(2n+1)\pi x}{z_0})$ or $-\cos(\frac{(2n+1)\pi x}{2z_0})$ for all $x \in \mathbf{R}$, $n \in \mathbf{Z}$ (the set of integers).

Van Vleck's and Kannappan's equations have been generalized in another direction by Perkins and Sahoo in [6]. They determined the abelian solutions $f : G \to \mathbf{C}$ of each of the two functional equations

$$f(x\tau(y)z_0) \pm f(xyz_0) = 2f(x)f(y), \quad x, y \in G, \tag{1.5}$$

where $z_0 \in Z(G)$ (the center of G: the set of elements $c \in G$ that commute with every other element in G) and $\tau : G \to G$ is an involution (that is, an anti-homomorphism such that $\tau(\tau(x)) = x$ for all $x \in G$). As very recent results, Stetkær extended the results of Van Vleck and Kannappan by solving the functional equations (1.5) on semigroups (see [8, 9]).

We shall in this paper study extensions of equations (1.2) and (1.3).

To formulate our results we introduce the following notations and assumptions that will be used throughout the paper: The map $\sigma : G \to G$ denotes an involutive automorphism. That it is involutive means that $\sigma(\sigma(x)) = x$ for all $x \in G$. If $(G, +)$ is an abelian group, then the inversion $\sigma(x) := -x$ is an example of an involutive automorphism.

For any complex-valued function F on G we use the notations

$$\begin{aligned} \check{F}(x) &= F(x^{-1}),\ x \in G, \\ F_e &= \frac{F + F\circ\sigma}{2} \quad \text{and} \quad F_o = \frac{F - F\circ\sigma}{2}. \end{aligned}$$

We say that F is even if $F = F_e$, and odd if $F = F_o$.

A function $A : G \to \mathbf{C}$ is called additive, if it satisfies $A(xy) = A(x) + A(y)$ for all $x, y \in G$.

A character of G is a homomorphism from G into the multiplicative group of non-zero complex numbers. It is well known that the set of characters on G is a linearly independent subset of the vector space of all complex-valued functions on G (see [7, Corollary 3.20]).

By $\mathcal{N}(G, \sigma)$ we mean the vector space of all solutions $\theta : G \to \mathbf{C}$ of the homogeneous equation

$$\theta(xy) - \theta(\sigma(y)x) = 0, \quad x, y \in G.$$

If G is a topological space, then we let $C(G)$ denotes the algebra of all continuous functions from G into $\mathbf{C}$.

Let G be a locally compact Hausdorff group, and let $M_C(G)$ denotes the space of all regular, compactly supported, complex-valued Borel measures on G. For $\mu \in M_C(G)$, we use the notations

$$\mu(f) = \int_G f(t)d\mu(t) \quad \text{and} \quad \mu^-(f) = \int_G f(t^{-1})d\mu(t) = \mu(\check{f})$$

for all $f \in C(G)$.

Let σ be a continuous involutive automorphism on G and $\mu, \nu \in M_C(G)$. The purpose of the present paper is to introduce and solve the following functional equation

$$\int_G f(\sigma(y)xt)d\mu(t) + \int_G f(xyt)d\nu(t) = f(x)f(y), \quad x, y \in G, \tag{1.6}$$

where $f \in C(G)$ is the unknown function to determine. Note that Eq. (1.2) (resp. (1.3)) results from (1.6) by taking $\mu = \frac{1}{2}\delta_{z_0}$ and $\nu = -\frac{1}{2}\delta_{z_0}$ (resp. $\mu = \nu = \frac{1}{2}\delta_{z_0}$) where δ_{z_0} is the Dirac measure concentrated at z_0.

Eq. (1.6) with $\mu = 0$ becomes

$$\int_G f(xyt)d\nu(t) = f(x)f(y), \quad x, y \in G, \tag{1.7}$$

which contains the solutions of Cauchy's functional equation on groups. When G is abelian, two important examples of (1.6) are

$$\int_G \{f(x + \sigma(y) + t) + f(x + y - t)\}d\mu(t) = f(x)f(y), \quad x, y \in G, \tag{1.8}$$

and

$$\int_G f(x + y - t)d\nu(t) = f(x)f(y), \quad x, y \in G. \tag{1.9}$$

Eq. (1.8) with $\sigma = -id$ was introduced by Gajda in [3]. He, in 1990, proved that if μ is a complex-valued regular Borel measure on G with bounded variation, then the essentially bounded non-zero solutions of Eq. (1.8), with $\sigma = -id$, are completely defined as

$$f(x) = \int_G \chi(x - t)d\mu(t) + \int_G \chi(t - x)d\mu(t) = \mu(\check{\chi})\chi(x) + \mu(\chi)\check{\chi}(x),$$

for all $x \in G$, where χ is a continuous character of G. Eq. (1.9) has been studied by Badora in [1].

In the last section, as other important consequences, we solve the following functional equations:

$$f(\sigma(y)xz_0) = f(x)f(y), \quad x, y \in G, \tag{1.10}$$

$$f(xyz_0) = f(x)f(y), \quad x, y \in G, \tag{1.11}$$

$$f(\sigma(y)xz_0) \pm f(xyz_1) = 2f(x)f(y), \quad x, y \in G, \tag{1.12}$$

$$\sum_{i=0}^{m} \alpha_i f(\sigma(y)xa_i) + \sum_{j=0}^{n} \beta_j f(xyb_j) = f(x)f(y), \quad x, y \in G, \tag{1.13}$$

where G is a group, $m, n \in \mathbf{N}$, $\alpha_i, \beta_j \in \mathbf{C}$, and $z_0, z_1, a_i, b_j \in G$ are arbitrarily fixed elements, for all $i = 0, ..., m$ and $j = 0, ..., n$. Note that each of Eqs. (1.10)-(1.13) results from (1.6) by replacing μ and ν by suitable discrete measures and that the most of them are, according to our knowledge, not in the literature even for abelian groups.

2. Solution of equation (1.6)

In this section, we solve the integral-functional equation (1.6), i.e.,

$$\int_G f(\sigma(y)xt)d\mu(t) + \int_G f(xyt)d\nu(t) = f(x)f(y), \quad x, y \in G,$$

by expressing its continuous solutions in terms of continuous characters. The following theorem is proved in [2]. For the notation $\mathcal{N}(G, \sigma)$ see the section Introduction.

Theorem 2.1. *Let G be a group, let σ be an involutive automorphism on G, and let $F_1, F_2, f : G \to \mathbf{C}$ be solutions of the functional equation*

$$F_1(xy) + F_2(\sigma(y)x) = f(x)f(y) \quad x, y \in G. \tag{2.1}$$

Then we have the following possibilities:

a) *There exists $\theta \in \mathcal{N}(G, \sigma)$ such that $F_1 = \theta$, $F_2 = -\theta$ and $f = 0$.*

b) *There exist a character χ of G with $\chi \circ \sigma \neq \chi$, constants $\alpha, \beta \in \mathbf{C}$, and a function $\theta \in \mathcal{N}(G, \sigma)$ such that*

$$\begin{aligned} F_1 &= \alpha^2\chi + \beta^2\chi \circ \sigma + \theta, \\ F_2 &= \alpha\beta(\chi + \chi \circ \sigma) - \theta, \\ f &= \alpha\chi + \beta\chi \circ \sigma. \end{aligned}$$

In this case $f \neq 0$.

c) *There exist a character χ of G with $\chi \circ \sigma = \chi$, a constant $\alpha \in \mathbf{C}$, an additive function $A : G \to \mathbf{C}$ with $A \circ \sigma = -A$, and a function $\theta \in \mathcal{N}(G, \sigma)$ such that*

$$\begin{aligned} F_1 &= \frac{1}{2}(\alpha^2 + 2\alpha A + \frac{1}{2}A^2)\chi + \theta, \\ F_2 &= \frac{1}{2}(\alpha^2 - \frac{1}{2}A^2)\chi - \theta, \\ f &= (\alpha + A)\chi. \end{aligned}$$

In this case $f \neq 0$.

Conversely, the functions given with these properties satisfy the functional equation (2.1).

Moreover, if G *is a topological group,* $f \neq 0$, *and* $F_1, F_2, f \in C(G)$, *then* $\chi, \chi \circ \sigma, A, \theta \in C(G)$.

The following lemma will be used in the proof of Theorem 2.3 in which the integral-functional equation (1.6) will be solved.

Lemma 2.2 (Lemma 4.1 of [2]). *Let* G *be a group and let* σ *be an involutive automorphism on* G. *Let* χ *be a character of* G *with* $\chi \neq \chi \circ \sigma$, $A : G \to \mathbf{C}$ *be an odd additive function,* θ *be a function in* $\mathcal{N}(G, \sigma)$, *and* α, β *be complex numbers.*

a) *If* $\alpha\chi + \beta\chi \circ \sigma + \theta = 0$, *then* $\alpha = \beta = 0$ *and* $\theta = 0$.

b) *If* $A^2 + \alpha A + \theta = 0$, *then* $A = \theta = 0$.

It is clear that $f \equiv 0$ is a solution of (1.6), so in the following theorem we are only concerned with the non-zero solutions.

Theorem 2.3. *Let* G *be a locally compact Hausdorff group, let* σ *be a continuous involutive automorphism on* G, *and let* $\mu, \nu \in M_C(G)$. *Let* $f \in C(G)\backslash\{0\}$ *be a solution of the functional equation (1.6). Then we have the following possibilities:*

a) *There exists a continuous character* χ *of* G *with* $\mu(\chi) = 0$ *and* $\nu(\chi) \neq 0$ *such that*
$$f = \nu(\chi)\chi.$$

b) *There exists a continuous character* χ *of* G *with* $\mu(\chi) \neq 0$, $\nu(\chi) \neq 0$, $\mu(\chi \circ \sigma) = \nu(\chi)$ *and* $\nu(\chi \circ \sigma) = \mu(\chi)$ *such that*
$$f = \nu(\chi)\chi + \mu(\chi)\chi \circ \sigma.$$

c) *There exists an even continuous character* χ *of* G *with* $\mu(\chi) \notin \{0, \nu(\chi), -\nu(\chi)\}$ *such that*
$$f = [\mu(\chi) + \nu(\chi)]\chi.$$

Conversely, any function f *of the forms described above solves (1.6).*

Proof. Checking that the stated functions satisfy (1.6) is done by elementary calculations, that we leave out. So it is left to show that any solution $f \in C(G)\backslash\{0\}$ of (1.6) falls into one of the indicated forms. Define $F_1, F_2 : G \to \mathbf{C}$ by

$$F_1(x) = \int_G f(xt)d\nu(t) \quad \text{and} \quad F_2(x) = \int_G f(xt)d\mu(t) \tag{2.2}$$

for all $x \in G$. Since $\mu, \nu \in M_C(G)$ and $f \in C(G)$, we have $F_1, F_2 \in C(G)$. Using these new functions defined in (2.2), the equation (1.6) becomes

$$F_1(xy) + F_2(\sigma(y)x) = f(x)f(y), \quad x, y \in G.$$

Since $f \neq 0$, we know from Theorem 2.1 that there are only the following two cases:

Case 1: There exist a continuous character χ of G with $\chi \circ \sigma \neq \chi$, constants $\alpha, \beta \in \mathbf{C}$, and a continuous function $\theta \in \mathcal{N}(G, \sigma)$ such that

$$\begin{aligned} F_1 &= \alpha^2\chi + \beta^2\chi \circ \sigma + \theta, \\ F_2 &= \alpha\beta(\chi + \chi \circ \sigma) - \theta, \\ f &= \alpha\chi + \beta\chi \circ \sigma. \end{aligned}$$

Since $F_1(x) = \int_G f(xt)d\nu(t)$ and $F_2(x) = \int_G f(xt)d\mu(t)$ for all $x \in G$, we have:

$$\alpha^2\chi(x) + \beta^2\chi \circ \sigma(x) + \theta(x) = \alpha\chi(x)\nu(\chi) + \beta\chi \circ \sigma(x)\nu(\chi \circ \sigma),$$

and

$$\alpha\beta[\chi(x) + \chi \circ \sigma(x)] - \theta(x) = \alpha\chi(x)\mu(\chi) + \beta\chi \circ \sigma(x)\mu(\chi \circ \sigma)$$

for all $x \in G$. We reformulate the last two equations as follows

$$\begin{aligned} \alpha[\alpha - \nu(\chi)]\chi(x) + \beta[\beta - \nu(\chi \circ \sigma)]\chi \circ \sigma(x) + \theta(x) &= 0, \\ \alpha[\beta - \mu(\chi)]\chi(x) + \beta[\alpha - \mu(\chi \circ \sigma)]\chi \circ \sigma(x) + (-\theta)(x) &= 0 \end{aligned}$$

for all $x \in G$. According to Lemma 2.2(a), we obtain

$$\begin{cases} \alpha[\alpha - \nu(\chi)] & = & 0 \\ \beta[\beta - \nu(\chi\circ\sigma)] & = & 0 \\ \alpha[\beta - \mu(\chi)] & = & 0. \\ \beta[\alpha - \mu(\chi\circ\sigma)] & = & 0 \\ \theta & = & 0 \end{cases} \tag{2.3}$$

Since $f = \alpha\chi + \beta\chi\circ\sigma$ and $f \neq 0$, then at least one of α and β is non-zero.

Subcase 1.1: Suppose that $\beta = 0$. Hence $\alpha \neq 0$. From (2.3) we see that $\alpha = \nu(\chi)$ and $\mu(\chi) = 0$. This solution is included in case (a) in our statement.

Subcase 1.2: Suppose that $\alpha = 0$. Hence $\beta \neq 0$. From (2.3) we see that $\beta = \nu(\chi\circ\sigma)$ and $\mu(\chi\circ\sigma) = 0$. So we are in case (a) with the continuous character $\chi\circ\sigma$ replacing χ.

Subcase 1.3: We now suppose that $\alpha \neq 0$ and $\beta \neq 0$. From (2.3) we see that $\alpha = \nu(\chi) = \mu(\chi\circ\sigma)$ and $\beta = \mu(\chi) = \nu(\chi\circ\sigma)$. This solution is included in case (b). This completes case 1.

Case 2: There exist a continuous character χ of G with $\chi\circ\sigma = \chi$, a constant $\alpha \in \mathbf{C}$, an additive function $A \in C(G)$ with $A\circ\sigma = -A$, and a continuous function $\theta \in \mathcal{N}(G,\sigma)$ such that

$$\begin{aligned} F_1 &= \frac{1}{2}(\alpha^2 + 2\alpha A + \frac{1}{2}A^2)\chi + \theta, \\ F_2 &= \frac{1}{2}(\alpha^2 - \frac{1}{2}A^2)\chi - \theta, \\ f &= (\alpha + A)\chi. \end{aligned}$$

Since $F_1(x) = \int_G f(xt)d\nu(t)$ and $F_2(x) = \int_G f(xt)d\mu(t)$ for all $x \in G$, then a small computation shows that

$$\begin{aligned} &\frac{1}{2}[\alpha^2 + 2\alpha A(x) + \frac{1}{2}A^2(x)]\chi(x) + \theta(x) \\ &\quad = \alpha\nu(\chi)\chi(x) + \nu(\chi)\chi(x)A(x) + \nu(\chi A)\chi(x) \end{aligned}$$

and

$$\begin{aligned}&\frac{1}{2}[\alpha^2-\frac{1}{2}A^2(x)]\chi(x)-\theta(x)\\&\quad=\alpha\mu(\chi)\chi(x)+\mu(\chi)\chi(x)A(x)+\mu(\chi A)\chi(x)\end{aligned}$$

for all $x\in G$. We reformulate the last two equations as follows

$$\begin{aligned}A^2+4[\alpha-\nu(\chi)]A+\theta_1=0,\\A^2+4\mu(\chi)A+\theta_2=0,\end{aligned}$$

where $\theta_1(x):=4(\frac{\theta}{\chi})(x)+2\alpha^2-4\alpha\nu(\chi)-4\nu(\chi A)$ and $\theta_2(x):=4(\frac{\theta}{\chi})(x)-2\alpha^2+4\alpha\mu(\chi)+4\mu(\chi A)$ for all $x\in G$. Since χ is even we have $\theta_1,\theta_2\in\mathcal{N}(G,\sigma)$. According to Lemma 2.2(b), we get that $A=\theta_1=\theta_2=0$ and hence $f=\alpha\chi$. Since $f\neq 0$, we have $\alpha\neq 0$. By definition of θ_1 and θ_2, we infer that $4\alpha\nu(\chi)-2\alpha^2=2\alpha^2-4\alpha\mu(\chi)$ which implies that $\alpha=\mu(\chi)+\nu(\chi)$. So we are in case ($a$), ($b$) or ($c$). This finishes the proof. □

As consequences of Theorem 2.3 one can obtain the following corollaries.

Corollary 2.4. *Let G be a locally compact Hausdorff group, let σ be a continuous involutive automorphism on G, and let $\mu\in M_C(G)$. Then a function $f\in C(G)\setminus\{0\}$ satisfies the functional equation*

$$\int_G f(\sigma(y)xt)d\mu(t)=f(x)f(y),\quad x,y\in G,$$

if and only if there exists an even continuous character χ of G with $\mu(\chi)\neq 0$ such that

$$f=\mu(\chi)\chi.$$

Proof. The proof follows on putting $\nu=0$ in Theorem 2.3. □

Corollary 2.5. *Let G be a locally compact Hausdorff group and let $\nu\in M_C(G)$. Then a function $f\in C(G)\setminus\{0\}$ satisfies the functional equation*

$$\int_G f(xyt)d\nu(t)=f(x)f(y),\quad x,y\in G,$$

if and only if there exists a continuous character χ of G with $\nu(\chi)\neq 0$ such that

$$f=\nu(\chi)\chi.$$

Proof. The proof follows on putting $\mu = 0$ in Theorem 2.3. □

Corollary 2.6. *Let G be a locally compact Hausdorff group, let σ be a continuous involutive automorphism on G, and let $\mu \in M_C(G)$. Then a function $f \in C(G) \setminus \{0\}$ satisfies the functional equation*

$$\int_G \{f(\sigma(y)xt) + f(xyt)\}d\mu(t) = f(x)f(y), \quad x, y \in G,$$

if and only if there exists a continuous character χ of G with $\mu(\chi) \neq 0$ and $\mu(\chi \circ \sigma) = \mu(\chi)$ such that

$$f = \mu(\chi)(\chi + \chi \circ \sigma).$$

Proof. The proof follows on putting $\nu = \mu$ in Theorem 2.3. □

Corollary 2.7. *Let G be a locally compact Hausdorff group, let σ be a continuous involutive automorphism on G, and let $\mu \in M_C(G)$. Then a function $f \in C(G) \setminus \{0\}$ satisfies the functional equation*

$$\int_G \{f(\sigma(y)xt) - f(xyt)\}d\mu(t) = f(x)f(y), \quad x, y \in G,$$

if and only if there exists a continuous character χ of G with $\mu(\chi) \neq 0$ and $\mu(\chi \circ \sigma) = -\mu(\chi)$ such that

$$f = -\mu(\chi)(\chi - \chi \circ \sigma).$$

Proof. The proof follows on putting $\nu = -\mu$ in Theorem 2.3. □

In view Corollary 2.5, we obtain the following.

Corollary 2.8 ([1]). *Let $(G, +)$ be a locally compact abelian Hausdorff group and let $\nu \in M_C(G)$. Then a function $f \in C(G) \setminus \{0\}$ satisfies the functional equation*

$$\int_G f(x + y - t)d\nu(t) = f(x)f(y), \quad x, y \in G,$$

if and only if there exists a continuous character χ of G with $\nu(\check{\chi}) \neq 0$ such that

$$f = \nu(\check{\chi})\chi.$$

Proof. The proof follows on replacing ν by ν^- in Corollary 2.5. $\square$

In the following corollary, we solve the integral-functional equation (1.8), i.e.,

$$\int_G \{f(x+\sigma(y)+t)+f(x+y-t\}d\mu(t)=f(x)f(y), \quad x,y\in G.$$

In view of this result we determine the continuous solutions of Gajda's equation, i.e., Eq. (1.8) with $\sigma=-id$.

Corollary 2.9. *Let $(G,+)$ be a locally compact abelian Hausdorff group, let σ be a continuous involutive automorphism on G, and let $\mu\in M_C(G)$. Let $f\in C(G)\backslash\{0\}$ be a solution of the functional equation (1.8). Then we have the following possibilities:*

a) There exists a continuous character χ of G with $\mu(\chi)=0$ and $\mu(\check{\chi})\neq 0$ such that

$$f=\mu(\check{\chi})\chi.$$

b) There exists a continuous character χ of G with $\mu(\chi)\neq 0$, $\mu(\check{\chi})\neq 0$, $\mu(\chi\circ\sigma)=\mu(\check{\chi})$ and $\mu(\check{\chi}\circ\sigma)=\mu(\chi)$ such that

$$f=\mu(\check{\chi})\chi+\mu(\chi)\chi\circ\sigma.$$

c) There exists an even continuous character χ of G with $\mu(\chi)\notin\{0,\mu(\check{\chi}),-\mu(\check{\chi})\}$ such that

$$f=[\mu(\chi)+\mu(\check{\chi})]\chi.$$

Conversely, any function f of the forms described above solves (1.8).

Proof. The proof follows on putting $\nu=\mu^-$ in Theorem 2.3. $\square$

Corollary 2.10 ([3]). *Let $(G,+)$ be a locally compact abelian Hausdorff group and let $\mu\in M_C(G)$. Then a function $f\in C(G)\setminus\{0\}$ satisfies the functional equation*

$$\int_G \{f(x-y+t)+f(x+y-t)\}d\mu(t)=f(x)f(y), \quad x,y\in G,$$

if and only if there exists a continuous character χ of G such that

$$f=\mu(\check{\chi})\chi+\mu(\chi)\check{\chi}.$$

Proof. The proof follows on putting $\sigma = -id$ in Corollary 2.9. $\square$

As another consequence of Theorem 2.3, we have the following result on the solution of the functional equation

$$(2.4) \int_G \{f(x+\sigma(y)+t) - f(x+y-t)\} d\mu(t) = f(x)f(y), \quad x, y \in G.$$

Corollary 2.11. *Let $(G,+)$ be a locally compact abelian Hausdorff group, let σ be a continuous involutive automorphism on G, and let $\mu \in M_C(G)$. Let $f \in C(G)\backslash\{0\}$ be a solution of the functional equation (2.4). Then we have the following possibilities:*

a) *There exists a continuous character χ of G with $\mu(\chi) = 0$ and $\mu(\check{\chi}) \neq 0$ such that*

$$f = -\mu(\check{\chi})\chi.$$

b) *There exists a continuous character χ of G with $\mu(\chi) \neq 0$, $\mu(\check{\chi}) \neq 0$, $\mu(\chi \circ \sigma) = -\mu(\check{\chi})$ and $\mu(\check{\chi} \circ \sigma) = -\mu(\chi)$ such that*

$$f = -\mu(\check{\chi})\chi + \mu(\chi)\chi \circ \sigma.$$

c) *There exists an even continuous character χ of G with $\mu(\chi) \notin \{0, \mu(\check{\chi}), -\mu(\check{\chi})\}$ such that*

$$f = [\mu(\chi) - \mu(\check{\chi})]\chi.$$

Conversely, any function f of the forms described above solves (2.4).

Proof. The proof follows on putting $\nu = -\mu^-$ in Theorem 2.3. $\square$

In view of Corollary 2.11, we obtain the following.

Corollary 2.12. *Let $(G,+)$ be a locally compact abelian Hausdorff group and let $\mu \in M_C(G)$. Then a function $f \in C(G) \backslash \{0\}$ satisfies the functional equation*

$$\int_G \{f(x-y+t) - f(x+y-t)\} d\mu(t) = f(x)f(y), \quad x, y \in G,$$

if and only if there exists a continuous character χ of G with $\mu(\chi) = 0$ and $\mu(\check{\chi}) \neq 0$ such that

$$f = -\mu(\check{\chi})\chi.$$

Proof. The proof follows on putting $\sigma = -id$ in Corollary 2.11. $\square$

3. Results corresponding to measures with finite support

In this section let G be a group, σ be an involutive automorphism on G, $m, n \in \mathbf{N}$, $\alpha_i, \beta_j \in \mathbf{C}$, and $z_0, z_1, a_i, b_j \in G$ be arbitrarily fixed elements, for all $i = 0, ..., m$ and $j = 0, ..., n$. To illustrate our theory, we continue by discussing the solution of Eq. (1.6) but now when μ and ν are supported by finite sets. We get the solutions from our theory by equipping G with the discrete topology.

Corollary 3.1. *The non-zero solutions $f : G \to \mathbf{C}$ of the functional equation*

$$f(\sigma(y)xz_0) = f(x)f(y), \quad x, y \in G,$$

are the functions of the form $f = \chi(z_0)\chi$, where χ is an even character of G.

Proof. The proof follows on putting $\mu = \delta_{z_0}$ in Corollary 2.4. $\square$

Corollary 3.2. *The non-zero solutions $f : G \to \mathbf{C}$ of the functional equation*

$$f(xyz_0) = f(x)f(y), \quad x, y \in G, \tag{3.1}$$

are the functions of the form $f = \chi(z_0)\chi$, where χ is a character of G.

Proof. The proof follows on putting $\nu = \delta_{z_0}$ in Corollary 2.5. $\square$

Eq. (3.1) is solved in [9] by Stetkær. In the following two corollaries we solve special cases of Eq. (1.6) that are, according to our knowledge, not in the literature even for abelian groups.

Corollary 3.3. *The non-zero solutions $f : G \to \mathbf{C}$ of the functional equation*

$$f(\sigma(y)xz_0) + f(xyz_1) = 2f(x)f(y), \quad x, y \in G,$$

are the functions of the forms:

a) *There exists a character χ of G with $\chi \circ \sigma(z_0) = \chi(z_1)$ and $\chi \circ \sigma(z_1) = \chi(z_0)$ such that*

$$f = \frac{\chi(z_1)}{2}\chi + \frac{\chi(z_0)}{2}\chi \circ \sigma.$$

b) *There exists an even character χ of G with $\chi(z_0) \notin \{\chi(z_1), -\chi(z_1)\}$ such that*

$$f = \frac{\chi(z_0) + \chi(z_1)}{2}\chi.$$

Proof. The proof follows on putting $\mu = \frac{1}{2}\delta_{z_0}$ and $\nu = \frac{1}{2}\delta_{z_1}$ in Theorem 2.3. □

Corollary 3.4. *The non-zero solutions $f : G \to \mathbf{C}$ of the functional equation*

$$f(\sigma(y)xz_0) - f(xyz_1) = 2f(x)f(y), \quad x, y \in G,$$

are the functions of the forms:

a) *There exists a character χ of G with $\chi \circ \sigma(z_0) = -\chi(z_1)$ and $\chi \circ \sigma(z_1) = -\chi(z_0)$ such that*

$$f = -\frac{\chi(z_1)}{2}\chi + \frac{\chi(z_0)}{2}\chi \circ \sigma.$$

b) *There exists an even character χ of G with $\chi(z_0) \notin \{\chi(z_1), -\chi(z_1)\}$ such that*

$$f = \frac{\chi(z_0) - \chi(z_1)}{2}\chi.$$

Proof. The proof follows on putting $\mu = \frac{1}{2}\delta_{z_0}$ and $\nu = -\frac{1}{2}\delta_{z_1}$ in Theorem 2.3. □

As a consequence of Corollary 3.3 (or Corollary 2.6) we have:

Corollary 3.5 (Corollary 4.5 of [2]). *The non-zero solutions $f : G \to \mathbf{C}$ of the functional equation*

$$f(\sigma(y)xz_0) + f(xyz_0) = 2f(x)f(y), \quad x, y \in G,$$

are the functions of the form $f = \frac{\chi(z_0)}{2}(\chi + \chi \circ \sigma)$, where χ is a character of G such that $\chi \circ \sigma(z_0) = \chi(z_0)$.

Proof. The proof follows on putting $z_1 = z_0$ in Corollary 3.3. □

With $z_1 = z_0$ in Corollary 3.4 we obtain:

Corollary 3.6 (Corollary 4.3 of [2]). *The non-zero solutions $f : G \to \mathbf{C}$ of the functional equation*

$$f(\sigma(y)xz_0) - f(xyz_0) = 2f(x)f(y), \quad x, y \in G,$$

are the functions of the form $f = -\frac{\chi(z_0)}{2}(\chi - \chi \circ \sigma)$, where χ is a character of G such that $\chi \circ \sigma(z_0) = -\chi(z_0)$.

We complete the paper with an important result concerning Eq. (1.6) which generalizes all previous results of this section.

Corollary 3.7. *The non-zero solutions $f : G \to \mathbf{C}$ of the functional equation*

$$\sum_{i=0}^{m} \alpha_i f(\sigma(y)xa_i) + \sum_{j=0}^{n} \beta_j f(xyb_j) = f(x)f(y), \quad x, y \in G,$$

are the functions of the forms:

a) There exists a character χ of G with $\sum_{i=0}^{m} \alpha_i \chi(a_i) = 0$ and $\sum_{i=0}^{n} \beta_i \chi(b_i) \neq 0$ such that

$$f = \sum_{i=0}^{n} \beta_i \chi(b_i)\chi.$$

b) There exists a character χ of G with $\sum_{i=0}^{m} \alpha_i \chi(a_i) \neq 0$, $\sum_{i=0}^{n} \beta_i \chi(b_i) \neq 0$, $\sum_{i=0}^{m} \alpha_i \chi \circ \sigma(a_i) = \sum_{i=0}^{n} \beta_i \chi(b_i)$ and $\sum_{i=0}^{n} \beta_i \chi \circ \sigma(b_i) = \sum_{i=0}^{m} \alpha_i \chi(a_i)$ such that

$$f = \sum_{i=0}^{n} \beta_i \chi(b_i)\chi + \sum_{i=0}^{m} \alpha_i \chi(a_i)\chi \circ \sigma.$$

c) There exists an even character χ of G with

$$\sum_{i=0}^{m} \alpha_i \chi(a_i) \notin \left\{0, \sum_{i=0}^{n} \beta_i \chi(b_i), -\sum_{i=0}^{n} \beta_i \chi(b_i)\right\}$$

such that

$$f = \left[\sum_{i=0}^{m} \alpha_i \chi(a_i) + \sum_{i=0}^{n} \beta_i \chi(b_i)\right] \chi.$$

Proof. The proof follows on putting

$$\mu = \sum_{i=0}^{m} \alpha_i \delta_{a_i} \quad \text{and} \quad \nu = \sum_{i=0}^{n} \beta_i \delta_{b_i}$$

in Theorem 2.3. $\square$

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