Proyecciones Journal of Mathematics
Vol. 37, N^o 3, pp. 519-533 September 2018.
Universidad Católica del Norte
Antofagasta - Chile

On the graded classical prime spectrum of a graded module

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Received : October 2017. Accepted : March 2018

Abstract

Let G be a group with identity e. Let R be a G-graded commutative ring and M a graded R-module. In this paper, we introduce and study a new topology on $Cl.Spec_g(M)$, the collection of all graded classical prime submodules of M, called the Zariski-like topology. Then we investigate the relationship between algebraic properties of M and topological properties of $Cl.Spec_g(M)$. Moreover, we study $Cl.Spec_g(M)$ from point of view of spectral space.

Subjclass [2010] : *13A02, 16W50.*

Keywords : *Graded classical prime submodule, graded classical prime spectrum, Zariski topology.*

1. Introduction and Preliminaries

Before we state some results, let us introduce some notations and terminologies. Let G be a group with identity e and R be a commutative ring with identity 1_R. Then R is a *G-graded ring* if there exist additive subgroups R_g of R such that $R = \bigoplus_{g\in G} R_g$ and $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$. We denote this by (R, G) (see [8].) The elements of R_g are called *homogeneous* of degree g where the R_g's are additive subgroups of R indexed by the elements $g \in G$. If $x \in R$, then x can be written uniquely as $\sum_{g\in G} x_g$, where x_g is the component of x in R_g. Moreover, $h(R) = \bigcup_{g\in G} R_g$. Let I be an ideal of R. Then I is called a *graded ideal* of (R, G) if $I = \bigoplus_{g\in G}(I \bigcap R_g)$. Thus, if $x \in I$, then $x = \sum_{g\in G} x_g$ with $x_g \in I$. An ideal of a G-graded ring need not be G-graded (see [8].)

Let R be a G-graded ring and M an R-module. We say that M is a *G-graded R-module* (or *graded R-module*) if there exists a family of subgroups $\{M_g\}_{g\in G}$ of M such that $M = g \in G\bigoplus M_g$ (as abelian groups) and $R_g M_h \subseteq M_{gh}$ for all $g, h \in G$. Here, $R_g M_h$ denotes the additive subgroup of M consisting of all finite sums of elements $r_g s_h$ with $r_g \in R_g$ and $s_h \in M_h$. Also, we write $h(M) = g \in G\bigcup M_g$ and the elements of $h(M)$ are called homogeneous elements of M. Let $M = g \in G\bigoplus M_g$ be a graded R-module and N a submodule of M. Then N is called a *graded submodule* of M if $N = g \in G\bigoplus N_g$ where $N_g = N \cap M_g$ for $g \in G$. In this case, N_g is called the *g-component* of N (see [8].)

Let R be a G-graded ring and M a graded R-module. A proper graded ideal I of R is said to be a *graded prime ideal* if whenever $rs \in I$, we have $r \in I$ or $s \in I$, where $r, s \in h(R)$. The *graded radical* of I, denoted by $Gr(I)$, is the set of all $x \in R$ such that for each $g \in G$ there exists $n_g > 0$ with $x^{n_g} \in I$. Note that, if r is a homogeneous element, then $r \in Gr(I)$ if and only if $r^n \in I$ for some $n \in \mathbf{N}$. Let $Spec_g(R)$ denote the set of all graded prime ideals of R (see [11].)

A proper graded submodule N of M is said to be a *graded prime submodule* if whenever $r \in h(R)$ and $m \in h(M)$ with $rm \in N$, then either $r \in (N :_R M) = \{r \in R : rM \subseteq N\}$ or $m \in N$ (see [2].) It is shown in [2, Proposition 2.7] that if N is a graded prime submodule of M, then $P := (N :_R M)$ is a graded prime ideal of R, and N is called *graded P-prime submodule*. Let $Spec_g(M)$ denote the set of all graded prime submodules of M. Note that some graded R-modules M have no graded prime submodules. We call such graded modules g-primeless. The *graded radical* of a graded submodule N of M, denoted by $Gr_M(N)$, is defined to be the intersection of all graded prime submodules of M containing N. If N is not contained in any graded prime submodule of M, then $Gr_M(N) = M$ (see [2, 9].)

A proper graded submodule N of M is called a *graded classical prime submodule* if whenever $r, s \in h(R)$ and $m \in h(M)$ with $rsm \in N$, then either $rm \in N$ or $sm \in N$ (see [1, 4].) Of course, every graded prime submodule is a graded classical prime submodule, but the converse is not true in general (see [1], Example 2.3.) Let $Cl.Spec_g(M)$ denote the set of all graded classical prime submodules of M. Obviously, some graded R-modules M have no graded classical prime submodules; such modules are called g-Cl.primeless. The *graded classical radical* of a graded submodule N of a graded R-module M, denoted by $Gr^{cl}_M(N)$, is defined to be the intersection of all graded classical prime submodules of M containing N. If N is not contained in any graded classical prime submodule of M, then $Gr^{cl}_M(N) = M$ (see [4].) We know that $Spec_g(M) \subseteq Cl.Spec_g(M)$. As it is mentioned in ([1], Example 2.3), it happens sometimes that this containment is strict. We call M a *graded compatible* R-module if its graded classical prime submodules and graded prime submodules coincide, that is if $Spec_g(M) = Cl.Spec_g(M)$. If R is a G-graded ring, then every graded classical prime ideal of R is a graded prime ideal. So, if we consider R as a graded R-module, it is graded compatible.

Let R be a G-graded ring and M a graded R-module. For each graded ideal I of R, the graded variety of I is the set $V^g_R(I) = \{P \in Spec_g(R) | I \subseteq P\}$. Then the set $\{V^g_R(I) | I \text{ is a graded ideal of } R\}$ satisfies the axioms for the closed sets of a topology on $Spec_g(R)$, called the Zariski topology on $Spec_g(R)$ (see [7, 10].)

In [3], $Spec_g(M)$ has endowed with quasi-Zariski topology. For each graded submodule N of M, let $V^g_*(N) = \{P \in Spec_g(M) | N \subseteq P\}$. In this case, the set $\zeta^g_*(M) = \{V^g_*(N) | \ N \text{ is a graded submodule of } M\}$ contains the empty set and $Spec_g(M)$, and it is closed under arbitrary intersections, but it is not necessarily closed under finite unions. The graded R-module M is said to be a g-Top module if $\zeta^g_*(M)$ is closed under finite unions. In this case $\zeta^g_*(M)$ satisfies the axioms for the closed sets of a unique topology τ^g_* on $Spec_g(M)$. The topology $\tau^g_*(M)$ on $Spec_g(M)$ is called the quasi-Zariski topology.

In [4], $Cl.Spec_g(M)$ has endowed with quasi-Zariski topology. For each graded submodule N of M, let $\mathbf{V}^g_*(N) = \{C \in Cl.Spec_g(M) \ | N \subseteq C\}$. In this case, the set $\eta^g_*(M) = \{\mathbf{V}^g_*(N) | N \text{ is a graded submodule of } M\}$ contains the empty set and $Cl.Spec_g(M)$, and it is closed under arbitrary intersections, but it is not necessarily closed under finite unions. The graded R-module M is said to be a g-Cl.Top module module if $\eta^g_*(M)$ is closed under finite unions. In this case $\eta^g_*(M)$ satisfies the axioms for the closed sets of a unique topology ϱ^g_* on $Cl.Spec_g(M)$. In this case, the topology $\varrho^g_*(M)$ on $Cl.Spec_g(M)$ is called the quasi-Zariski topology.

In this article, we introduce and study a new topology on $Cl.Spec_g(M)$, called the Zariski-like topology, which generalizes the Zariski topology of graded rings to graded modules. Let R be a G-graded ring and M a graded R-module. For each graded submodule N of M, we define $\mathbf{U}^g_*(N) = Cl.Spec_g(M) - \mathbf{V}^g_*(N)$ and put $\mathbf{B}^{cl}(M) = \{\mathbf{U}^g_*(N) : N$ is a graded submodule of $M\}$. Then we define $\tau^{cl}_g(M)$ to be the topology on $Cl.Spec_g(M)$ by the sub-basis $\mathbf{B}^{cl}(M)$. In fact $\tau^{cl}_g(M)$ to be the collection U of all unions of finite intersections of elements of $\mathbf{B}^{cl}(M)$. We call this topology the Zariski-like topology of M.

If N is a graded submodule (respectively proper submodule) of a graded module M we write $N \leq_g M$ (respectively $N_g M$).

2. Topology on $Cl.Spec_g(M)$

Let R be a G-graded ring and M a graded R-module. A graded submodule C of M will be called *a graded maximal classical prime* if C is a graded classical prime submodule of M and there is no graded classical prime submodule P of M such that $C \subset P$. Let $Cl.Spec_g(M)$ be endowed with the Zariski-like topology. For each subset Y of $Cl.Spec_g(M)$, We will denote the closure of Y in $Cl.Spec_g(M)$ by $cl(Y)$.

Lemma 2.1. *Let R be a G-graded ring and M a graded R-module.*

i) If Y is a nonempty subset of $Cl.Spec_g(M)$, then $cl(Y) = \bigcup_{C\in Y} \mathbf{V}^g_*(C)$.

ii) If Y is a closed subset of $Cl.Spec_g(M)$, then $Y = \bigcup_{C\in Y} \mathbf{V}^g_*(C)$.

Proof.

i) Clearly, $cl(Y) \subseteq \bigcup_{C\in Y} \mathbf{V}^g_*(C)$. Let S be a closed subset of $Cl.Spec_g(M)$ containing Y. Thus, $S = \bigcap_{i\in I}(\bigcup_{j=1}^{n_i} \mathbf{V}^g_*(N_{ij}))$, for some $N_{ij} \leq_g M$, $i \in I$ and $n_i \in \mathbf{N}$. Let $P \in \bigcup_{C\in Y} \mathbf{V}^g_*(C)$. Then, there exists $C_0 \in Y$

such that $P \in \mathbf{V}^g_*(C_0)$ and so $C_0 \subseteq P$. Since $C_0 \in S$, then for each $i \in I$ there exists j, $1 \leq j \leq n_i$, such that $N_{ij} \subseteq C_0$, and hence $N_{ij} \subseteq C_0 \subseteq P$. It follows that $P \in S$. Therefore, $\bigcup\limits_{C \in Y} \mathbf{V}^g_*(C) \subseteq S$.

ii) Clearly $Y \subseteq \bigcup\limits_{C \in Y} \mathbf{V}^g_*(C)$. For each $C \in Y$ we have $\mathbf{V}^g_*(C) = cl(\{C\}) \subseteq cl(Y) = Y$ by part(i). Hence $\bigcup\limits_{C \in Y} \mathbf{V}^g_*(C) \subseteq Y$. Therefore, $Y = \bigcup\limits_{C \in Y} \mathbf{V}^g_*(C)$.

□

Now the above lemma immediately yields the following result.

Corollary 2.2. *Let R be a G-graded ring and M a graded R-module. Then.*

1. *$cl(\{C\}) = \mathbf{V}^g_*(C)$, for all $C \in Cl.Spec_g(M)$.*
2. *$Q \in cl(\{C\})$ if and only if $C \subseteq Q$ if and only if $\mathbf{V}^g_*(Q) \subseteq \mathbf{V}^g_*(C)$.*
3. *The set $\{C\}$ is a closed in $Cl.Spec_g(M)$ if and only if C is a graded maximal classical prime submodule of M.*

The following theorem shows that for any graded R-module M, $Cl.Spec_g(M)$ is always a T_0-space.

Theorem 2.3. *Let R be a G-graded ring and M a graded R-module. Then, $Cl.Spec_g(M)$ is a T_0-space.*

Proof. Let $C_1, C_2 \in Cl.Spec_g(M)$. By Corollary 2.2, $cl(\{C_1\}) = cl(\{C_2\})$ if and only if $\mathbf{V}^g_*(C_1) = \mathbf{V}^g_*(C_2)$ if and only if $C_1 = C_2$.

Now, by the fact that a topological space is a T_0-space if and only if the closures of distinct points are distinct, we conclude that for any graded R-module M, $Cl.Spec_g(M)$ is a T_0-space. □

Let R be a G-graded ring and M a graded R-module. Let every graded classical prime submodule of M is contained in a graded maximal classical prime submodule. We define, by transfinite induction, sets X_α of graded classical prime submodule of M. To start, let X_{-1} be the empty set. Next, consider an ordinal $\alpha \geq 0$; if X_β has been defined for all ordinals $\beta < \alpha$, then let X_α be the set of those graded classical prime submodules C in M such that all graded classical prime submodules proper containing C belong to $\cup_{\beta<\alpha}X_{\beta}$. In particular, X_0 is the set of graded maximal classical prime submodules of M. If some X_{γ} contains all graded classical prime submodules of M, then we say that $dim_g^{cl}(M)$ exists, and we set $dim_g^{cl}(M)$ -the graded classical prime dimension of M to be to the smallest such γ. We write $dim_g^{cl}(M)=\gamma$ as an abbreviation for the statement that $dim_g^{cl}(M)$ exists and equal γ. In fact, if $dim_g^{cl}(M)=\gamma<\infty$, then $dim_g^{cl}(M)=sup\{ht(C)|C$ is graded classical prime submodule of M$\}$. Where $ht(C)$ is the greatest non-negative integer n such that there exists a chain of graded classical prime submodules of M, $C_0\subset C_1\subset\ldots\ldots\ldots\subset C_n=C$, and $ht(C)=\infty$ if no such n exists.

Let X be a topological space and let x_1 and x_2 be two points in X. We say that x_1 and x_2 can be separated if each lies in an open set which does not contain the other point. X is a T_1-space if any two distinct points in X can be separated. A topological space X is a T_1-space if and only if all points of X are closed in X, (see [6].)

Theorem 2.4. *Let R be a G-graded ring and M a graded R-module. Then $Cl.Spec_g(M)$ is T_1-space if and only if $dim_g^{cl}(M)\leq 0$.*

Proof. First assume that $Cl.Spec_g(M)$ is a T_1-space. If $Cl.Spac_g(M)=\phi$, then $dim_g^{cl}(M)=-1$. Also, if $Cl.Spac_g(M)$ has one element, clearly $dim_g^{cl}(M)=0$. So we can assume that $Cl.Spec_g(M)$ has more than two elements. We show that every graded classical prime submodules of M is a graded maximal classical prime submodule. To show this, let $C_1\subseteq C_2$, where $C_1,C_2\in Cl.Spec_g(M)$. Since $\{C_1\}$ is a closed set, $\{C_1\}=\bigcap\limits_{i\in I}(\bigcup\limits_{j=1}^{n_i}\mathbf{V}_*^g(N_{ij}))$, Where $N_{ij}\leq_g M$ and I is an index set. So for each $i\in I$, $C_1\in\bigcup\limits_{j=1}^{n_i}\mathbf{V}_*^g(N_{ij})$ so that there exists $1\leq t_i\leq n_i$ such that $C_1\in\mathbf{V}_*^g(N_{it_i})$. Since $C_1\subseteq C_2$, $C_2\in\mathbf{V}_*^g(N_{it_i})$ for all $i\in I$. This implies that $C_2\in\bigcup\limits_{j=1}^{n_i}\mathbf{V}_*^g(N_{ij})$, for all $i\in I$. Therefore, $C_2\in\bigcap\limits_{i\in I}(\bigcup\limits_{j=1}^{n_i}\mathbf{V}_*^g(N_{ij}))=\{C_1\}$ as desired.

Conversely, suppose that $dim_g^{cl}(M)\leq 0$. If $dim_g^{cl}(M)=-1$, then $Cl.Spac_g(M)=\phi$, and hence it is a T_1-space. Now let $dim_g^{cl}(M)=0$. Then $Cl.Spac_g(M)\neq\phi$ and for every graded classical prime submodule of

M is a graded maximal classical prime submodule. Hence for each graded classical prime submodule C of M, $\mathbf{V}_*^g(C) = \{C\}$, and so $\{C\}$ is a closed set in $Cl.Spac_g(M)$. Hence $Cl.Spac_g(M)$ is a T_1-space. $\square$

The cofinite topology is a topology which can be defined on every set X. It has precisely the empty set and all cofinite subsets of X as open sets. As a consequence, in the cofinite topology, the only closed subset are finite sets, or the whole of X (see [6].)

Now we give a characterization for a graded module M for which $Cl.Spec_g(M)$ is the cofinite topology.

Theorem 2.5. *Let R be a G-graded ring and M a graded R-module. Then the following statements are equivalent :*

i) *$Cl.Spec_g(M)$ is the confinite topology.*

ii) *$dim_g^{cl}(M) \leq 0$ and for every graded submodule N of M either $\mathbf{V}_*^g(N) = Cl.Spec_g(M)$ or $\mathbf{V}_*^g(N)$ is finite.*

Proof. $(i) \Rightarrow (ii)$. Assume that $Cl.Spec_g(M)$ is the cofinite topology. Since every cofinite topology satisfies the T_1 axiom, by Theorem 2.4, $dim_g^{cl}(M) \leq 0$. Now assume that there exists a graded submodule N of M such that $|\mathbf{V}_*^g(N)| = \infty$ and $\mathbf{V}_*^g(N) \neq Cl.Spec_g(M)$. Then $\mathbf{U}_*^g(N) = Cl.Spec_g(M) - \mathbf{V}_*^g(N)$ is an open set in $Cl.Spec_g(M)$ with infinite complement, a contradiction. $(ii) \Rightarrow (i)$. Suppose that $dim_g^{cl}(M) \leq 0$ and for every graded submodule N of M, $\mathbf{V}_*^g(N) = Cl.Spec_g(M)$ or $\mathbf{V}_*^g(N)$ is finite. Thus every finite union $\bigcup_{j=1}^{n} \mathbf{V}_*^g(N_j)$ of graded submodules $N_j \leq_g M$ is also finite or $Cl.Spec_g(M)$. Hence any intersection of finite union $\bigcap_{i\in I}(\bigcup_{j=1}^{n} \mathbf{V}_*^g(N_{ij}))$ of graded submodules $N_{ij} \leq_g M$ is finite or $Cl.Spec_g(M)$. Hence every closed set in $Cl.Spec_g(M)$ is either finite or $Cl.Spec_g(M)$. Therefore $Cl.Spec_g(M)$ is the cofinite topology. $\square$

Suppose that X is a topological space. Let x_1 and x_2 be points in X. We say that x_1 and x_2 can be separated by neighborhoods if there exists a neighborhood U of x_1 and neighborhood V of x_2 such that $U \bigcap V = \phi$. X is a T_2-space if any two distinct points of X can be separated by neighborhoods (see [6].) It is well-known that if X is a finite space, then X is T_1-space if and only if X is the discrete space (see [6].) Thus we have the following corollary.

Corollary 2.6. *Let R be a G-graded ring and M a graded R-module such that $Cl.Spec_g(M)$ is finite. Then the following statements are equivalent:*

i) $Cl.Spec_g(M)$ *is* T_2*-space.*

ii) $Cl.Spec_g(M)$ *is* T_1*-space.*

iii) $Cl.Spec_g(M)$ *is the cofinite space.*

iv) $Cl.Spec_g(M)$ *is discrete.*

v) $dim_g^{cl}(M) \leq 0$.

Theorem 2.7. *Let R be a G-graded ring and M a graded R-module such that M has ACC on intersection of graded classical prime submodules. Then, $Cl.Spec_g(M)$ is a quasi-compact space*

Proof. Suppose M is a graded R-module such that M has ACC on intersection of graded classical prime submodules. Let be a family of open sets covering $Cl.Spec_g(M)$, and suppose that no finite subfamily of covers $Cl.Spec_g(M)$. Since $\mathbf{V}_*^g(0) = Cl.Spec_g(M)$, then we may use the ACC on the intersection of graded classical prime submodules to choose a graded submodule N maximal with respect to the property that no finite subfamily of covers $\mathbf{V}_*^g(N)$. We claim that N is a graded classical prime submodule of M, for if not, then there exist $m_\lambda \in h(M)$ and $r_g, s_h \in h(R)$, such that $r_g s_h m_\lambda \in N$, $r_g m_\lambda \notin N$ and $s_h m_\lambda \notin N$. Thus $NN + Rr_g m_\lambda$ and $NN + Rs_h m_\lambda$. Hence, without loss of generality, there must exist a finite subfamily $'$ of that covers both $\mathbf{V}_*^g(N + Rr_g m_\lambda)$ and $\mathbf{V}_*^g(N + Rs_h m_\lambda)$. Let $C \in \mathbf{V}_*^g(N)$. Since $r_g s_h m_\lambda \in N$, $r_g s_h m_\lambda \in C$ and since C is graded classical prime, $r_g m_\lambda \in C$ or $s_h m_\lambda \in C$. Thus either $C \in \mathbf{V}_*^g(N + Rr_g m_\lambda)$ *or* $C \in \mathbf{V}_*^g(N + Rs_h m_\lambda)$, and hence $\mathbf{V}_*^g(N) \subseteq \mathbf{V}_*^g(N + Rr_g m_\lambda) \bigcup \mathbf{V}_*^g(N + Rs_h m_\lambda)$. Thus, $\mathbf{V}_*^g(N)$ is covered with the finite subfamily $'$, a contradiction. Therefore, N is a graded classical prime submodule of M.

Now, choose $W \in$ such that $N \in W$. Hence N must have a neighborhood $\bigcap_{i=1}^n \mathbf{U}_*^g(P_i)$, for some graded submodule P_i of M and $n \in \mathbf{N}$, such that $\bigcap_{i=1}^n \mathbf{U}_*^g(P_i) \subseteq W$. We claim that for each i $(1 \leq i \leq n)$, $N \in \mathbf{U}_*^g(P_i + N) \subseteq \mathbf{U}_*^g(P_i)$. To see this, assume that $C \in \mathbf{U}_*^g(P_i + N)$, i.e., $P_i + NC$. So P_iC, i.e., $C \in \mathbf{U}_*^g(P_i)$. On the other hand, $N \in \mathbf{U}_*^g(P_i)$, i.e., P_iN. Therefore, $P_i + NC$, i.e., $C \in \mathbf{U}_*^g(P_i + N)$. Consequently, $N \in \bigcap\limits_{i=1}^{n} \mathbf{U}_*^g(P_i + N) \subseteq \bigcap\limits_{i=1}^{n} \mathbf{U}_*^g(P_i) \subseteq W$.

Hence $\bigcap_{i=1}^{n}\mathbf{U}^g_*(P_i^{'})$, where $P_i^{'} := P_i + N$, is a neighborhood of N such that $\bigcap_{i=1}^{n}\mathbf{U}^g_*(P_i^{'}) \subseteq W$. Since for each i $(1 \leq i \leq n)$, then $NP_i^{'}$, $\mathbf{V}^g_*(P_i^{'})$ can be covered by some finite subfamily $_i^{'}$ of . But, $\mathbf{V}^g_*(N)\backslash[\bigcup\limits_{i=1}^{n}\mathbf{V}^g_*(P_i^{'})] = \mathbf{V}^g_*(N)\backslash[\bigcap\limits_{i=1}^{n}\mathbf{U}^g_*(P_i^{'})]^c = [\bigcap\limits_{i=1}^{n}\mathbf{U}^g_*(P_i^{'})] \cap \mathbf{V}^g_*(N) \subseteq W$, and so $\mathbf{V}^g_*(N)$ can be covered by $_1^{'}\bigcup_2^{'}\bigcup ... \bigcup\ _n^{'}\bigcup\{W\}$, contrary to our choice of N. Thus, there must exist a finite subfamily of which covers $Cl.Spec_g(M)$. Therefore, $Cl.Spec_g(M)$ is a quasi-compact space. $\square$

3. Graded modules whose Zariski-like topologies are spectral spaces

A topological space X is called *irreducible* if $X \neq \phi$ and every finite intersection of non-empty open sets of X is non-empty. A (non-empty) subset Y of a topology space X is called *an irreducible set* if the subspace Y of X is irreducible, equivalently if Y_1 and Y_2 are closed subset of X and satisfy $Y \subseteq Y_1 \cup Y_2$, then $Y \subseteq Y_1$ or $Y \subseteq Y_2$ (see [6].)

Let Y be a closed subset of a topological space. An element $y \in Y$ is called *a generic point* of Y if $Y = cl(\{y\})$. Note that a generic point of the irreducible closed subset Y of a topological space is unique if the topological space is a T_0-space (see [5].)

A *spectral space* is a topological space homomorphic to the prime spectrum of a commutative ring equipped with the Zariski topology. Spectral spaces have been characterized by Hochster [5] as the topological space W which satisfy the following conditions:

i) W is a T_0-space.

ii) W is quasi-compact.

iii) the quasi-compact open subsets of W are closed under finite intersections and form an open basis.

iv) each irreducible closed subset of W has a generic point.

Let M be a G-graded R-Module and Y a subset of $Cl.Spec_g(M)$. We will denote $\bigcap\limits_{C\in Y} C$ by $\Im(Y)$ (note that if $Y = \phi$, then $\Im(Y) = M$).

Lemma 3.1. *Let R be a G-graded ring and M a graded R-module. Then for each $C \in Cl.Spec_g(M)$, $\mathbf{V}^g_*(C)$ is irreducible.*

Proof. Suppose that $\mathbf{V}_*^g(C) \subseteq Y_1 \cup Y_2$, where Y_1 and Y_2 are closed sets. Since $C \in \mathbf{V}_*^g(C)$, either $C \in Y_1$ or $C \in Y_2$. Without loss of generality we can assume that $C \in Y_1$. We have $Y_1 = \bigcap_{i \in I}(\bigcup_{j=1}^{n_i} \mathbf{V}_*^g(N_{ij}))$, for some I, $n_i(i \in I)$, and $N_{ij} \leq_g M$. Thus $C \in \bigcup_{j=1}^{n_i} \mathbf{V}_*^g(N_{ij})$, for all $i \in I$. It follows that $\mathbf{V}_*^g(C) \subseteq \bigcup_{j=1}^{n_i} \mathbf{V}_*^g(N_{ij})$, for all $i \in I$. Thus $\mathbf{V}_*^g(C) \subseteq Y_1$. Therefore $\mathbf{V}_*^g(C)$ is irreducible. □

Theorem 3.2. *Let R be a G-graded ring, M a graded R-module and $Y \subseteq Cl.Spec_g(M)$.*

i) If Y is irreducible, then $\Im(Y)$ is a graded classical prime submodule.

ii) If $\Im(Y)$ is a graded classical prime submodule and $\Im(Y) \in cl(Y)$, then Y is irreducible.

Proof. (i) Assume that Y is an irreducible subset of $Cl.Spec_g(M)$. Clearly, $\Im(Y) = \bigcap\limits_{C \in Y} C_g M$ and $Y \subseteq \mathbf{V}_*^g(\Im(Y))$. Let I, J be graded ideals of R and N be a graded submodule of M such that $IJN \subseteq \Im(Y)$. It is easy to see that $Y \subseteq \mathbf{V}_*^g(IJN) \subseteq \mathbf{V}_*^g(IN) \bigcup \mathbf{V}_*^g(JN)$. Since Y is irreducible, either $Y \subseteq \mathbf{V}_*^g(IN)$ or $Y \subseteq \mathbf{V}_*^g(JN)$. If $Y \subseteq \mathbf{V}_*^g(IN)$, then $IN \subseteq C$, for all $C \in Y$. Thus $IN \subseteq \Im(Y)$. If $Y \subseteq \mathbf{V}_*^g(JN)$, then $JN \subseteq C$, for all $C \in Y$. Hence $JN \subseteq \Im(Y)$. Thus by [1, Theorem 2.1.], $\Im(Y)$ is a graded classical prime submodule of M. (ii) Assume that $C := \Im(Y)$ is a graded classical prime submodule of M and $C \in cl(Y)$. It is easy to see that $cl(Y) = \mathbf{V}_*^g(C)$. Now let $Y \subseteq Y_1 \bigcup Y_2$, where Y_1, Y_2 are closed sets. Then we have $\mathbf{V}_*^g(C) = cl(Y) \subseteq Y_1 \bigcup Y_2$. Since $\mathbf{V}_*^g(C) \subseteq Y_1 \bigcup Y_2$ and by Lemma 3.1, $\mathbf{V}_*^g(C)$ is irreducible, $\mathbf{V}_*^g(C) \subseteq Y_1$ or $\mathbf{V}_*^g(C) \subseteq Y_2$. Hence either $Y \subseteq Y_1$ or $Y \subseteq Y_2$. Thus Y is irreducible. □

Corollary 3.3. *Let R be a G-graded ring, M a graded R-module and N a graded submodule of M. Then the subset $\mathbf{V}_*^g(N)$ of $Cl.Spec_g(M)$ is irreducible if and only if $Gr_M^{cl}(N)$ is a graded classical prime submodule. Consequently, $Cl.Spec_g(M)$ is irreducible if and only if $Gr_M^{cl}(M)$ is a graded classical prime submodule.*

Proof. ($\Rightarrow$)Let $Y := \mathbf{V}^g_*(N)$ be an irreducible subset of $Cl.Spec_g(M)$. Then we have $\Im(Y) = Gr^{cl}_M(N)$ so that $Gr^{cl}_M(N)$ is a graded classical prime submodule of M by Theorem 3.2(i).

($\Leftarrow$) By [4, Proposition 3.4(1)], for each graded submodule N of M, $\mathbf{V}^g_*(N) = \mathbf{V}^g_*(Gr^{cl}_M(N))$. Now let $Gr^{cl}_M(N)$ is a graded classical prime submodule of M. Then $Gr^{cl}_M(N) \in \mathbf{V}^g_*(N)$, and hence by Theorem 3.2 (ii), $\mathbf{V}^g_*(N)$ is irreducible. $\square$

Lemma 3.4. *Let R be a G-graded ring and M a graded R-module. Then*

i) Every $C \in Cl.Spec_g(M)$ is a generic point of the irreducible closed subset $\mathbf{V}^g_*(C)$.

ii) Every finite irreducible closed subset of $Cl.Spec_g(M)$ has a generic point.

Proof.

i) is clear by Corollary 2.2(i).

ii) Let Y be an irreducible closed subset of $Cl.Spec_g(M)$ and $Y = \{C_1, C_2, ..., C_n\}$, where $C_i \in Cl.Spec_g(M)$, $n \in \mathbf{N}$. By Lemma 2.1(i), $Y = cl(Y) = \mathbf{V}^g_*(C_1) \bigcup \mathbf{V}^g_*(C_2) \bigcup .... \bigcup \mathbf{V}^g_*(C_n)$. Since Y is irreducible, $Y = \mathbf{V}^g_*(C_i)$ for some $i(1 \leq i \leq n)$. Now by (i), C_i is a generic point of Y.

$\square$

Theorem 3.5. *Let R be a G-graded ring and M a graded R-module such that $Cl.Spec_g(M)$ is finite. Then $Cl.Spec_g(M)$ is a spectral space (with the Zariski-like topology). Consequently, for each finite graded R-module M, $Cl.Spec_g(M)$ is a spectral space.*

Proof. Since $Cl.Spec_g(M)$ is finite, every subset of $Cl.Spec_g(M)$ is quasi-compact. Hence the quasi-compact open sets of $Cl.Spec_g(M)$ are closed under finite intersection and form an open basis (note: this basis is $\beta = \{\mathbf{U}^g_*(N_1) \bigcap \mathbf{U}^g_*(N_2) \bigcap ... \bigcap \mathbf{U}^g_*(N_k) : N_i \leq_g M,\ 1 \leq i \leq k,$ for some $k \in \mathbf{N}\})$. Also by Theorem 2.3, $Cl.Spec_g(M)$ a T_0-space. Moreover, every irreducible closed subset of $Cl.Spec_g(M)$ has a generic point by Lemma 3.4. Therefore $Cl.Spec_g(M)$ is a spectral space by Hochster's characterization. □

Let X be a topological space. By the *patch topology* on X, we mean the topology which has as a sub-basis for its closed sets the closed sets and compact open sets of the original space. By a patch we mean a set closed in the patch topology. The patch topology associated to a spectral space is compact and T_2-space (see [5].)

Definition 3.6. *Let R be a G-graded ring and M a graded R-module, and let $P_*^g(M)$ be the family of all subsets of $Cl.Spec_g(M)$ of the form $\mathbf{V}_*^g(N) \bigcap \mathbf{U}_*^g(K)$, where N, $K \leq_g M$. Clearly $P_*^g(M)$ contains both $Cl.Spec_g(M)$ and ϕ because $Cl.Spec_g(M) = \mathbf{V}_*^g(0) \bigcap \mathbf{U}_*^g(M)$ and $\phi = \mathbf{V}_*^g(M) \bigcap \mathbf{U}_*^g(0)$. Let $T_*^g(M)$ be the collection of all unions of finite intersections of elements of $P_*^g(M)$. Then, $T_*^g(M)$ is a topology on $Cl.Spec_g(M)$ and is called the patch-like topology of M, in fact, $P_*^g(M)$ is a sub-basis for the patch-like topology of M.*

Theorem 3.7. *Let R be a G-graded ring and M a graded R-module. Then, $Cl.Spec_g(M)$ with the patch-like topology is a T_2-space.*

Proof. Suppose distinct points $C_1, C_2 \in Cl.Spec_g(M)$. Since $C_1 \neq C_2$, then either C_1C_2 or C_2C_1. Assume that C_1C_2. By Definition 3.6, $P_1 := \mathbf{U}_*^g(M) \bigcap \mathbf{V}_*^g(C_1)$ is a patch-like-neighborhood of C_1 and $P_2 := \mathbf{U}_*^g(C_1) \bigcap \mathbf{V}_*^g(C_2)$ is a patch-like-neighborhood of C_2. Clearly, $\mathbf{U}_*^g(C_1) \bigcap \mathbf{V}_*^g(C_1) = \phi$, and thus $P_1 \bigcap P_2 = \phi$. Therefore, $Cl.Spec_g(M)$ is a T_2-space. □

The proof of the next theorem is similar to the proof of Theorem 2.7.

Theorem 3.8. *Let R be a G-graded ring and M a graded R-module such that M has ACC on intersection of graded classical prime submodules. Then $Cl.Spec_g(M)$ with the patch-like topology is a compact space.*

Theorem 3.9. *Let R be a G-graded ring and M a graded R-module such that M has ACC on intersection of graded classical prime submodules. Then every irreducible closed subset of $Cl.Spec_g(M)$ (with the Zariski-like topology) has a generic point.*

Proof. Let Y be an irreducible closed subset of $Cl.Spec_g(M)$. By Definition 3.6 for each $C \in Y$, $\mathbf{V}^g_*(C)$ is an open subset of $Cl.Spec_g(M)$ with the patch-like topology. On the other hand since $Y \subseteq Cl.Spec_g(M)$ is closed with the Zariski-like topology, the complement of Y is open by this topology. This yields that the complement of Y is open with the patch-like topology. So $Y \subseteq Cl.Spec_g(M)$ is closed with the patch-like topology. Since $Cl.Spec_g(M)$ is a compact space in patch-like topology by Theorem 3.8 and Y is closed in $Cl.Spec_g(M)$, we have Y is compact space in patch-like topology. Now $Y = \bigcup_{C \in Y} \mathbf{V}^g_*(C)$ by Lemma 2.1(ii) and each $\mathbf{V}^g_*(C)$ is open in patch-like topology. Hence there exists a finite set $Y_1 \subseteq Y$ such that $Y = \bigcup_{C \in Y_1} \mathbf{V}^g_*(C)$. Since Y is irreducible, $Y = \mathbf{V}^g_*(C) = cl(\{C\})$ for some $C \in Y$. Therefore, C is a generic point for Y. □

We need the following evident lemma

Lemma 3.10. *Assume τ_1 and τ_2 are two topologies on X such that $\tau_1 \subseteq \tau_2$. If X is quasi-compact in τ_2 , then X is also quasi-compact in τ_1.*

Theorem 3.11. *Let R be a G-graded ring and M a graded R-module such that M has ACC on intersection of graded classical prime submodules. Then for each $n \in \mathbf{N}$, and graded submodules $N_i (1 \leq i \leq n)$ of M, $\mathbf{U}^g_*(N_1) \bigcap \mathbf{U}^g_*(N_2) \bigcap ... \bigcap \mathbf{U}^g_*(N_n)$ is a quasi-compact subset of $Cl.Spec_g(M)$ with the Zariski-like topology.*

Proof. Clearly, for each $n \in \mathbf{N}$, and each graded submodules $N_i (1 \leq i \leq n)$ of M, $\mathbf{U}^g_*(N_1) \bigcap \mathbf{U}^g_*(N_2) \bigcap ... \bigcap \mathbf{U}^g_*(N_n)$ is a closed set in $Cl.Spec_g(M)$ with patch-like topology. By Theorem 3.8, $Cl.Spec_g(M)$ is a compact space with the patch-like topology and since every closed subset of a compact space is compact, $\mathbf{U}^g_*(N_1) \bigcap \mathbf{U}^g_*(N_2) \bigcap ... \bigcap \mathbf{U}^g_*(N_n)$ is compact in $Cl.Spec_g(M)$ with patch-like topology and so by Lemma 3.10, it is quasi-compact in $Cl.Spec_g(M)$ with the Zariski-like topology. □

Corollary 3.12. *Let R be a G-graded ring and M a graded R-module such that M has ACC on intersection of graded classical prime submodules. Then Zariski-like quasi-compact open sets of $Cl.Spec_g(M)$ are closed under finite intersections.*

Proof. It suffices to show that the intersection $Q = Q_1 \bigcap Q_2$ of two Zariski-like quasi-compact open sets Q_1 and Q_2 of $Cl.Spec_g(M)$ is Zariski-like quasi-compact set. Each $Q_i, i = 1, 2$, is a finite union of members of the open base $\beta = \{\mathbf{U}_*^{\mathbf{g}}(\mathbf{N_1}) \bigcap \mathbf{U}_*^{\mathbf{g}}(\mathbf{N_2}) \bigcap ... \bigcap \mathbf{U}_*^g(N_n) : N_i \leq_g M, 1 \leq i \leq n,$ for some $n \in \mathbf{N}\}$. Hence $Q = \bigcup_{i=1}^{m}(\bigcap_{j=1}^{n_i} \mathbf{U}_*^g(N_j))$. Let Γ be any open cover of Q. So Γ also covers each $\bigcap_{j=1}^{n_i} \mathbf{U}_*^g(N_j)$ which is Zariski-like quasi-compact by Theorem 3.11. Thus each $\bigcap_{j=1}^{n_i} \mathbf{U}_*^g(N_j)$ has a finite subcover of Γ and so dose Q. □

Theorem 3.13. *Let R be a G-graded ring and M a graded R-module such that M has ACC on intersection of graded classical prime submodules. Then $Cl.Spec_g(M)$ (with the Zariski-like topology) is a spectral space.*

Proof. By Theorem 2.3, $Cl.Spec_g(M)$ is a T_0-space. Also, by Theorem 3.11., $Cl.Spec_g(M)$ is quasi-compact and has a basis of quasi-compact open subsets. Moreover, by Corollary 3.12, the family of quasi-compact open subset of $Cl.Spec_g(M)$ is closed under finite intersections. Finally, every irreducible closed subset of $Cl.Spec_g(M)$ has generic point by Theorem 3.9. Thus $Cl.Spec_g(M)$ is spectral space by Hochster's characterization. □

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