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# Quasi $\mathcal{N}$-Open sets and related compactness concepts in bitopological spaces

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**Abstract**

*Three types of $N$-open sets are defined and investigated in bitopological spaces, and via them several compactness are introduced. Several relationships, examples and counter-examples regarding the new concepts are given.*

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## 1. Introduction

Let $(X, \tau)$ be a topological space and let $A$ be a subset of $X$. A point $x \in X$ is called an infinite point (resp. condensation point) of $A$ if for each $U \in \tau$ with $x \in U$, the set $U \cap A$ is infinite (resp. uncountable). The set $A$ is called $\mathcal{N}$-closed [3, 4] (resp. $\omega$-closed [6]) if it contains all its infinite points (resp. condensation points), the complement of an N-closed (resp. $\omega$-closed) set is called N-open [3, 4] (resp. $\omega$-open [7]). For a given topological space $(X, \tau)$, we will denote the class of $\omega$-open sets (resp. N-open sets) by $\tau_\omega$ (resp. $\tau_{\mathcal{N}}$). It is known that $\tau_{\mathcal{N}}$ lies between $\tau$ and $\tau_\omega$. Also, it is known that $A$ is N-open (resp. $\omega$-open) if and only if for every $x \in A$, there are $O \in \tau$ and a finite (resp. countable) set $G$ such that $x \in O - G \subseteq A$. Using $\mathcal{N}$-open sets, Al-Omari and Noorani in [3, 4] gave several characterizations of compact and strongly compact topological spaces.

As a wider structure than classical topological spaces, Kelly in [9] introduced bitopological spaces as an ordered triple $(X, \tau, \sigma)$ of a set $X$ and two topologies $\tau$ and $\sigma$. Datta in [5] introduced the notion of quasi open sets in bitopological spaces. In [1–2, 11-16, 18-20], several modifications of the concept of quasi open sets are inroduced and used to define new bitopological concepts. In this research, we define and investigate quasi $\mathcal{N}$-open sets as a new class of sets in bitopological spaces and use them to define reasonable new compactness concepts. We give some characterizations regarding compact bitopological spaces.

## 2. Three Types of $\mathcal{N}$-open Sets in Bitopological Spaces

**Definition 2.1.** [5] Let $(X, \tau, \sigma)$ be a bitopological space.

(a) The smallest topology on $X$ containing $\tau \cup \sigma$ is called the least upper bound topology on $X$.

(b) A set $A \subseteq (X, \tau, \sigma)$ is said to be semi-open (briefly, $s$-open) if it is open in the least upper bound topology on $X$.

If $\tau$ and $\sigma$ are two topologies on a set $X$, then the least upper bound topology on $X$ will be denoted by $\langle \tau, \sigma \rangle$.

**Proposition 2.2.** [1] Let $\tau$ and $\sigma$ be two topologies on a set $X$. Then $A \subseteq (X, \tau, \sigma)$ is $s$-open if and only if for each $x \in A$ there exist $U \in \tau$, and $V \in \sigma$ such that $x \in U \cap V \subseteq A$.

**Definition 2.3.** [10] A set $A \subseteq (X, \tau, \sigma)$ is said to be $u$-open if $A \in \tau \cup \sigma$.

The family of all $u$-open sets in $(X,\tau,\sigma)$ will be denoted by $u(\tau,\sigma)$.

**Definition 2.4.** [5] A set $A \subseteq (X,\tau,\sigma)$ is said to be quasi-open (briefly, $q$-open) if for every $x \in A$ there exists $U_x \in \tau$ such that $x \in U_x \subseteq A$ or $V_x \in \sigma$ such that $x \in V_x \subseteq A$. Equivalently: A set $A \subseteq (X,\tau,\sigma)$ is $q$-open if and only if $A = B \cup C$, where $B \in \tau$ and $C \in \sigma$. A set $A \subseteq (X,\tau,\sigma)$ is said to be $q$-closed if $X - A$ is $q$-open.

The family of all $q$-open sets in $(X,\tau,\sigma)$ will be denoted by $q(\tau,\sigma)$.

**Proposition 2.5.** [5] For a bitopological space $(X,\tau,\sigma)$, we have the following:

(a) $u(\tau,\sigma) \subseteq q(\tau,\sigma) \subseteq \langle\tau,\sigma\rangle$, $\tau \cup \sigma \neq q(\tau,\sigma)$ in general and $q(\tau,\sigma) \neq \langle\tau,\sigma\rangle$ in general.

(b) $q(\tau,\sigma)$ is closed under arbitrary union but $q(\tau,\sigma)$ is not a topology on $X$, in general.

(c) Arbitrary intersection of $q$-closed sets is $q$-closed.

**Definition 2.6.** [1] Let $(X,\tau,\sigma)$ be a bitopological space and let $A \subseteq X$. Then

(a) $A$ is said to be $u-\omega$-open in $(X,\tau,\sigma)$ if $A \in \tau_\omega \cup \sigma_\omega$. Equivalently: $A \subseteq (X,\tau,\sigma)$ is $u-\omega$-open if and only if $A \in u(\tau_\omega,\sigma_\omega)$.

(b) $A$ is said to be $u-\omega$-closed in $(X,\tau,\sigma)$ if $X - A$ is $u-\omega$-open in $(X,\tau,\sigma)$.

(c) $A$ is said to be $s-\omega$-open in $(X,\tau,\sigma)$ if it is an open set in the least upper bound topology on $X$, of $\tau_\omega$ and $\sigma_\omega$.

**Definition 2.7.** Let $(X,\tau,\sigma)$ be a bitopological space and let $A \subseteq X$. Then

(a) $A$ is said to be $u-\mathcal{N}$-open in $(X,\tau,\sigma)$ if $A \in \tau_\mathcal{N} \cup \sigma_\mathcal{N}$. Equivalently: $A \subseteq (X,\tau,\sigma)$ is $u-\mathcal{N}$-open if and only if $A \in u(\tau_\mathcal{N},\sigma_\mathcal{N})$.

(b) $A$ is said to be $u-\mathcal{N}$-closed in $(X,\tau,\sigma)$ if $X - A$ is $u-\mathcal{N}$-open in $(X,\tau,\sigma)$.

(c) $A$ is said to be $s-\mathcal{N}$-open in $(X,\tau,\sigma)$ if it is an open set in the least upper bound topology on $X$, of $\tau_\mathcal{N}$ and $\sigma_\mathcal{N}$.

For a bitopological space $(X,\tau,\sigma)$, we will denote the family of all $u$-$\mathcal{N}$-open sets in $(X,\tau,\sigma)$ by $u$-$\mathcal{N}(\tau,\sigma)$, and the family of all $\mathcal{N}$-open sets in the topological space $(X,\langle\tau,\sigma\rangle)$ is denoted by $\langle\tau,\sigma\rangle_\mathcal{N}$.

**Theorem 2.8.** (a) Every $u$-open set in bitopological space is $u$-$\mathcal{N}$-open.

(b) Every $u$-$\mathcal{N}$-open set in bitopological space is $u$-$\omega$-open.

**Proof.** (a) Let $(X, \tau, \sigma)$ be a bitopological space and let $A$ be a $u$-open set in $(X, \tau, \sigma)$. Then $A \in u(\tau, \sigma)$. Since $\tau \subseteq \tau_{\mathcal{N}}$ and $\sigma \subseteq \sigma_{\mathcal{N}}$, then $u(\tau, \sigma) \subseteq u(\tau_{\mathcal{N}}, \sigma_{\mathcal{N}})$. It follows that $A \in u(\tau_{\mathcal{N}}, \sigma_{\mathcal{N}})$ and $A$ is $u$-$\mathcal{N}$-open.

(b) Let $(X, \tau, \sigma)$ be a bitopological space and let $A$ be a $u$-N-open set in $(X, \tau, \sigma)$. Then $A \in u(\tau_{\mathcal{N}}, \sigma_{\mathcal{N}})$. Since $\tau_{\mathcal{N}} \subseteq \tau\omega$ and $\sigma_{\mathcal{N}} \subseteq \sigma\omega$, then u$(\tau_{\mathcal{N}}, \sigma_{\mathcal{N}}) \subseteq u(\tau\omega, \sigma\omega)$. It follows that A$\in u(\tau\omega, \sigma\omega)$ and A *is* u$-\omega$-open.

The following example shows that the converse of each of the two implications in Theorem 2.8 is not true in general:

**Example 2.9.** Consider $(\mathbf{R}, \tau, \sigma)$ where $\tau$ and $\sigma$ are the left ray and the indiscrete topologies, respectively. It is clear that $\mathbf{R} - \mathbf{Q}$ is $u - \omega$-open but not $u - \mathcal{N}$-open and $\mathbf{R} - \{1\}$ is $u - \mathcal{N}$-open but not $u$-open.

**Theorem 2.10.** Let $(X, \tau, \sigma)$ be a bitopological space. Then $\langle \tau, \sigma \rangle_{\mathcal{N}} = \langle \tau_{\mathcal{N}}, \sigma_{\mathcal{N}} \rangle$.

**Proof.** Let $A \in \langle \tau, \sigma \rangle_{\mathcal{N}}$ and let $x \in A$. Then there exist $H, F$ such that $H \in \langle \tau, \sigma \rangle$ and a finite set $F \subseteq X$ such that $x \in H - F \subseteq A$. Since $x \in H \in \langle \tau, \sigma \rangle$, there exist $U, V$ such that $U \in \tau, V \in \sigma$ and $x \in U \cap V \subseteq H$. Note that $U - F \in \tau_{\mathcal{N}}$, $V - F \in \sigma_{\mathcal{N}}$, and $x \in (U - F) \cap (V - F) \subseteq (U \cap V) - F \subseteq H - F \subseteq A$. It follows that $A \in \langle \tau_{\mathcal{N}}, \sigma_{\mathcal{N}} \rangle$. Conversely, let $A \in \langle \tau_{\mathcal{N}}, \sigma_{\mathcal{N}} \rangle$ and let $x \in A$. Then there exist $W \in \tau_{\mathcal{N}}$ and $G \in \sigma_{\mathcal{N}}$ such that $x \in W \cap G \subseteq A$. Since $x \in W \cap G$, there exist $U \in \tau, V \in \sigma$ and finite sets $F, M \subseteq X$ such that $x \in U - F \subseteq W$ and $x \in V - M \subseteq G$. Note that $U \cap V \in \langle \tau, \sigma \rangle$ and $F \cap M$ is a finite set. Also, $x \in (U \cap V) - (F \cup M) \subseteq W \cap G \subseteq A$. Thus $A \in \langle \tau, \sigma \rangle_{\mathcal{N}}$.

**Theorem 2.11.** Let $(X, \tau, \sigma)$ be a bitopological space. Then $u$-$\mathcal{N}(\tau, \sigma) \subseteq \langle \tau, \sigma \rangle_{\mathcal{N}}$.

**Proof.** $u$-$\mathcal{N}(\tau, \sigma) = \tau_{\mathcal{N}} \cup \sigma_{\mathcal{N}} \subseteq \langle \tau_{\mathcal{N}}, \sigma_{\mathcal{N}} \rangle = \langle \tau, \sigma \rangle_{\mathcal{N}}$. Since by Theorem 2.10, $\langle \tau_{\mathcal{N}}, \sigma_{\mathcal{N}} \rangle = \langle \tau, \sigma \rangle_{\mathcal{N}}$, it follows that $u$-$\mathcal{N}(\tau, \sigma) \subseteq \langle \tau, \sigma \rangle_{\mathcal{N}}$.

The following example will show respectively, that the inclusion in Theorem 2.11 cannot be replaced by equality, in general.

**Example 2.12.** Consider the bitopological space $(\mathbf{R}, \tau_{lr}, \tau_{rr})$ and let $A = (6, 9)$. Then $A \in \langle \tau, \sigma \rangle \subseteq \langle \tau, \sigma \rangle_{\mathcal{N}}$, while $A \notin \tau_{\mathcal{N}} \cup \sigma_{\mathcal{N}} = u$-$\mathcal{N}(\tau, \sigma)$.

As defined in [1], a set $A \subseteq (X, \tau, \sigma)$ is said to be $q$-$\omega$ -open if for every $x \in A$ there exists $U_x \in \tau\omega such that$x$\in U_x \subseteq A$ or $V_x \in \sigma\omega such that$x$\in V_x \subseteq A$. Equivalently: $A \subseteq (X, \tau, \sigma)$ is $q$-$\omega$-open if and only if $A \in q(\tau\omega$ $,\sigma\omega$ $).Aset$ A$\subseteq (X, \tau, \sigma)$ is said to be $q$-$\omega$ -closed if $X - A$ is $q$-$\omega$ -open. The family of all $q$-$\omega$-open sets in $(X, \tau, \sigma)$ is denoted by $q$-$\omega(\tau, \sigma)$.

**Definition 2.13.** A set $A \subseteq (X, \tau, \sigma)$ is said to be $q$-$\mathcal{N}$-open if for every $x \in A$ there exists $U_x \in \tau_{\mathcal{N}}$ such that $x \in U_x \subseteq A$ or $V_x \in \sigma_{\mathcal{N}}$ such that $x \in V_x \subseteq A$. Equivalently: $A \subseteq (X, \tau, \sigma)$ is $q$-$\mathcal{N}$-open if and only if $A \in q(\tau_{\mathcal{N}}, \sigma_{\mathcal{N}})$. A set $A \subseteq (X, \tau, \sigma)$ is said to be $q$-$\mathcal{N}$-closed if $X - A$ is $q$-$\mathcal{N}$-open. The family of all $q$-$\mathcal{N}$-open sets in $(X, \tau, \sigma)$ is denoted by $q$-$\mathcal{N}(\tau, \sigma)$.

**Theorem 2.14.** Let $(X, \tau, \sigma)$ be a bitopological space and $A \subseteq X$. Then the following are equivalent:

(a) $A$ is $q$-$\mathcal{N}$-open.

(b) For each $x \in A$ there exists $U \in u(\tau, \sigma)$ and a finite set $F \subseteq X$ such that $x \in U - F \subseteq A$.

**Proof.** (a) $\Longrightarrow$ (b): Let $A$ be a $q$-$\mathcal{N}$-open set and let $x \in A$. Since $A$ is $q$-$\mathcal{N}$-open, there exist $B, C$ such that $B \in \tau_{\mathcal{N}}$, $C \in \sigma_{\mathcal{N}}$ and $A = B \cup C$. Without loss of generality we may assume that $x \in B$. Choose $U \in \tau \subseteq u(\tau, \sigma)$ and a finite set $F \subseteq X$ such that $x \in U - F \subseteq B \subseteq A$.

(b) $\Longrightarrow$ (a): By (b), for each $x \in A$ there exists $U_x \in u(\tau, \sigma)$ and a finite set $F_x \subseteq X$ such that $x \in U_x - F_x \subseteq A$. Let $B = \bigcup\{U_x - F_x : U_x \in \tau\}$ and $C = \bigcup\{U_x - F_x : U_x \in \sigma\}$. Then $B \in \tau_{\mathcal{N}}, C \in \sigma_{\mathcal{N}}$ and $A = B \cup C$. It follows that $A$ is $q$-$\mathcal{N}$-open.

**Theorem 2.15.** Let $(X, \tau, \sigma)$ be a bitopological space. Then

(a) $u$-$\mathcal{N}(\tau, \sigma) \subseteq q$-$\mathcal{N}(\tau, \sigma)$.

(b) $q(\tau, \sigma) \subseteq q$-$\mathcal{N}(\tau, \sigma)$.

(c) $q$-$\mathcal{N}(\tau, \sigma) \subseteq \langle\tau, \sigma\rangle_{\mathcal{N}}$.

(d) $\{\emptyset, X\} \subseteq q$-$\mathcal{N}(\tau, \sigma)$.

(e) The family $q$-$\mathcal{N}(\tau, \sigma)$ is closed under arbitrary union.

(f) The family of all $q$-$\mathcal{N}$-closed sets in $(X, \tau, \sigma)$closed under arbitrary intersection.

(g) $q$-N$(\tau, \sigma) \subseteq q - \omega$ $(\tau, \sigma)$.

**Proof.** (a) Since $u(\tau, \sigma) \subseteq q(\tau, \sigma)$ and $u$-$\mathcal{N}(\tau, \sigma) = u(\tau_{\mathcal{N}}, \sigma_{\mathcal{N}})$ we have , $u(\tau_{\mathcal{N}}, \sigma_{\mathcal{N}}) \subseteq q(\tau_{\mathcal{N}}, \sigma_{\mathcal{N}}) = q$-$\mathcal{N}(\tau, \sigma)$, then $u$-$\mathcal{N}(\tau, \sigma) \subseteq q$-$\mathcal{N}(\tau, \sigma)$.

(b) Let $A \in q(\tau, \sigma)$, then there exist $B$ and $C$ such that $A = B \cup C$, where $B \in \tau$ and $C \in \sigma$. Since $\tau \subseteq \tau_{\mathcal{N}}$ and $\sigma \subseteq \sigma_{\mathcal{N}}$, then $B \in \tau_{\mathcal{N}}$ and $C \in \sigma_{\mathcal{N}}$. Thus, $A \in q(\tau_{\mathcal{N}}, \sigma_{\mathcal{N}}) = q\text{-}\mathcal{N}(\tau, \sigma)$.

(c) Since $q(\tau, \sigma) \subseteq \langle \tau, \sigma \rangle$, it follows that $q\text{-}\mathcal{N}(\tau, \sigma) = q(\tau_{\mathcal{N}}, \sigma_{\mathcal{N}}) \subseteq \langle \tau_{\mathcal{N}}, \sigma_{\mathcal{N}} \rangle$. Thus, by Theorem 2.10 (a), it follows that $q\text{-}\mathcal{N}(\tau, \sigma) \subseteq \langle \tau, \sigma \rangle_{\mathcal{N}}$.

(d) Since $\{\emptyset, X\} \subseteq u\text{-}\mathcal{N}(\tau, \sigma)$, then by part (a) we have $\{\emptyset, X\} \subseteq q\text{-}\mathcal{N}(\tau, \sigma)$.

(e) Since $q\text{-}\mathcal{N}(\tau, \sigma) = q(\tau_{\mathcal{N}}, \sigma_{\mathcal{N}})$ and $q(\tau, \sigma)$ is closed under arbitrary union, we get that $q(\tau_{\mathcal{N}}, \sigma_{\mathcal{N}}) = q\text{-}\mathcal{N}(\tau, \sigma)$ is closed under arbitrary union.

(f) Let $\{A_\alpha : \alpha \in \Delta\}$ be a collection of $q$-$\mathcal{N}$-closed sets of $(X, \tau, \sigma)$. Then for all $\alpha \in \Delta, X - A_\alpha$ is $q$-$\mathcal{N}$-open. Since $\alpha \in \Delta \bigcap A_\alpha = X - \alpha \in \Delta \bigcup (X - A_\alpha)$ and by part (e), $\alpha \in \Delta \bigcup (X - A_\alpha) \in q\text{-}\mathcal{N}(\tau, \sigma)$, then $\alpha \in \Delta \bigcap A_\alpha$ is $q$-$\mathcal{N}$-closed.

(g) Follows from the definitions and Theorem 2.8.

The inclusion in part (a) of Theorem 2.15 cannot be replaced by equality, in general as the following example shows:

**Example 2.16.** Consider the bitopological space $(\mathbf{R}, \tau_{lr}, \tau_{rr})$ and let $A = (-\infty, 0) \cup (1, \infty)$. Then $A \in q\text{-}\mathcal{N}(\tau, \sigma)$, but $A \notin u\text{-}\mathcal{N}(\tau, \sigma)$.

The following example shows that the inclusion in Theorem 2.15 (b) cannot be replaced by equality, in general.

**Example 2.17.** Consider the bitopological space $(\mathbf{R}, \tau_{lr}, \tau_{rr})$ and let $A = (-\infty, 1) - \{0\}$. Then $A \in q\text{-}\mathcal{N}(\tau, \sigma) - q(\tau, \sigma)$.

The following example shows that the inclusion in Theorem 2.15 (c) cannot be replaced by equality, in general.

**Example 2.18.** Consider the bitopological space $(\mathbf{R}, \tau_{lr}, \tau_{rr})$ and let $A = (0, 2)$. Then $A \in \langle \tau_{lr}, \tau_{rr} \rangle \subseteq \langle \tau_{lr}, \tau_{rr} \rangle_{\mathcal{N}}$, but $A \notin q - \mathcal{N}(\tau, \sigma)$.

The next example shows that the intersection of two $q$-$\mathcal{N}$-open sets is not $q$-$\mathcal{N}$-open in general. Therefore, the family of all $q$-$\mathcal{N}$-open sets of a bitopological space $(X, \tau, \sigma)$ does not form a topological space, in general.

**Example 2.19.** Consider the bitopological space $(\mathbf{R}, \tau_{lr}, \tau_{rr})$. Let $A = (-\infty, 2)$ and $B = (0, \infty)$. Then $A$ and $B$ are $q$-$\mathcal{N}$-open sets in $(\mathbf{R}, \tau_{lr}, \tau_{rr})$, but $A \cap B = (0, 2) \notin q\text{-}\mathcal{N}(\tau_{lr}, \tau_{rr})$.

**Theorem 2.20.** [1] Let $(X, \tau, \sigma)$ be a bitopological space. Then $q(\tau, \sigma)$ forms a topology on $X$ if and only if $q(\tau, \sigma) = \langle \tau, \sigma \rangle$.

**Theorem 2.21.** Let $(X, \tau, \sigma)$ be a bitopological space. Then $q$-$\mathcal{N}(\tau, \sigma)$ is a topology on $X$ if and only if $q$-$\mathcal{N}(\tau, \sigma) = \langle \tau, \sigma \rangle_{\mathcal{N}}$.

**Proof.** Necessity. Suppose that $q$-$\mathcal{N}(\tau, \sigma)$ is a topology on $X$. By Theorem 2.15 (c), we need only to show that $\langle \tau, \sigma \rangle_{\mathcal{N}} \subseteq q$-$\mathcal{N}(\tau, \sigma)$. Let $A \in \langle \tau, \sigma \rangle_{\mathcal{N}}$ and let $x \in A$. Then there exists $U_x \in \langle \tau, \sigma \rangle$ and a finite set $F_x \subseteq X$ such that $x \in U_x - F_x \subseteq A$. Since $x \in U_x \in \langle \tau, \sigma \rangle$, by Proposition 2.2 there exist $H_x \in \tau$, and $G_x \in \sigma$ such that $x \in H_x \cap G_x \subseteq A$. Since $q$-$\mathcal{N}(\tau, \sigma)$ is a topology on $X$, $H_x \in \tau \subseteq q$-$\mathcal{N}(\tau, \sigma)$ and $G_x \in \sigma \subseteq q$-$\mathcal{N}(\tau, \sigma)$, then $H_x \cap G_x \in q$-$\mathcal{N}(\tau, \sigma)$ and so $(H_x \cap G_x) - F_x \in q$-$\mathcal{N}(\tau, \sigma)$. By Theorem 2.15 (e), it follows that

A=$\cup \{(H_x \cap G_x) - F_x : x \in A\}$

is $q$-$\mathcal{N}$-open.

Sufficieny. Follows because $\langle \tau, \sigma \rangle_{\mathcal{N}}$ is a topology on $X$.

## 3. Compactness

**Definition 3.1.** A cover $\mathcal{U}$ of the bitopological space $(X, \tau, \sigma)$ is called:

(a) [8] $\tau\sigma$-open if $\mathcal{U} \subseteq u(\tau, \sigma)$.

(b) [17] $p$-open if it is $\tau\sigma$-open, and $\mathcal{U}$ contains at least one nonempty member of $\tau$ and at least one nonempty member of $\sigma$.

**Definition 3.2.** [5] A bitopological space $(X, \tau, \sigma)$ is called:

(a) $s$-compact if every $\tau\sigma$-open cover of $(X, \tau, \sigma)$ has a finite subcover.

(b) $p$-compact if every $p$-open cover of $(X, \tau, \sigma)$ has a finite subcover.

**Theorem3.3**.*Let*(X,$\tau, \sigma$) be a bitopological space and let $\mathcal{A} = \{W - F : W \in u(\tau, \sigma)$ and $F \subseteq X$ is a finite set$\}$. Then $(X, \tau_{\mathcal{N}}, \sigma_{\mathcal{N}})$ is $s$-compact if and only if every cover of $X$ consists of elements of $\mathcal{A}$ has a finite subcover.

**Proof.** Necessity. Suppose $(X, \tau_{\mathcal{N}}, \sigma_{\mathcal{N}})$ is $s$-compact and let $\mathcal{H}$ be a cover of $X$ with $\mathcal{H} \subseteq \mathcal{A}$. Since $\mathcal{H} \subseteq \mathcal{A} \subseteq \tau_{\mathcal{N}} \cup \sigma_{\mathcal{N}}$, then $\mathcal{H}$ is a $\tau_{\mathcal{N}}\sigma_{\mathcal{N}}$-open cover of $(X, \tau_{\mathcal{N}}, \sigma_{\mathcal{N}})$. Since $(X, \tau_{\mathcal{N}}, \sigma_{\mathcal{N}})$ is $s$-compact, there exists a finite family of elements of $\mathcal{H}$ covers $X$.

Sufficiency. Let $\mathcal{H} = \{H_\alpha : \alpha \in \Delta\}$ be a $\tau_{\mathcal{N}}\sigma_{\mathcal{N}}$-open cover of $(X, \tau_{\mathcal{N}}, \sigma_{\mathcal{N}})$. Then, for each $\alpha \in \Delta$, $H_\alpha \in u(\tau_{\mathcal{N}}, \sigma_{\mathcal{N}})$ and $H_\alpha \in \tau_{\mathcal{N}} \cup \sigma_{\mathcal{N}}$. Hence $H_\alpha \in \tau_{\mathcal{N}}$ or $H_\alpha \in \sigma_{\mathcal{N}}$. Therefore, there exist $C_\alpha \in u(\tau, \sigma)$ and a finite set $F_\alpha$ such that $H_\alpha = C_\alpha - F_\alpha$. Hence $H_\alpha \in \mathcal{A}$ and $\mathcal{H} \subseteq \mathcal{A}$. By assumption, $\mathcal{H}$ has a finite subcover. Therefore, $(X, \tau_{\mathcal{N}}, \sigma_{\mathcal{N}})$ is $s$-compact.

**Theorem 3.4.** For a bitopological space $(X, \tau, \sigma)$, the following are equivalent:

(a) $(X, \tau, \sigma)$ is $s$-compact.
(b) $(X, \tau_{\mathcal{N}}, \sigma_{\mathcal{N}})$ is $s$-compact.
(c) Each cover of $X$ of elements of $q$-$\mathcal{N}(\tau, \sigma)$, has a finite subcover.
(d) Each cover of $X$ of elements of $q(\tau, \sigma)$, has a finite subcover.

**Proof.** (a) $\Longrightarrow$ (b): Suppose $(X, \tau, \sigma)$ is $s$-compact. We will apply Theorem 3.3. Let $\mathcal{A} = \{W - F : W \in u(\tau, \sigma)$ and $F \subseteq X$ is a finite set$\}$ and let $\mathcal{H} \subseteq \mathcal{A}$ be a cover of $X$, say $\mathcal{H} = \{W_\alpha - F_\alpha : \alpha \in \Delta$, where $W_\alpha \in u(\tau, \sigma)$ and $F_\alpha \subseteq X$ is a finite set$\}$. Note that $\alpha \in \Delta \bigcup W_\alpha = X$. Then by (a), there exists a finite set $\Delta' \subseteq \Delta$ such that $\{W_\alpha : \alpha \in \Delta'\}$ covers $X$. Put $G = \alpha \in \Delta' \bigcup F_\alpha$ and for each $x \in G$, choose $\alpha_x \in \Delta$ such that $x \in (W_{\alpha_x} - F_{\alpha_x})$. Thus, $\{W_\alpha - F_\alpha : \alpha \in \Delta'\} \cup \{W_{\alpha_x} - F_{\alpha_x} : x \in G\}$ is a finite subcover of $\mathcal{H}$.

(b) $\Longrightarrow$ (c): Suppose $(X, \tau_{\mathcal{N}}, \sigma_{\mathcal{N}})$ is $s$-compact. Let $\mathcal{H} = \{H_\alpha : \alpha \in \Delta\}$ be a cover of $X$ consists of elements of $q$-$\mathcal{N}(\tau, \sigma)$. For each $\alpha \in \Delta$, there exist $A_\alpha \in \tau_{\mathcal{N}}$ and $B_\alpha \in \sigma_{\mathcal{N}}$ such that $H_\alpha = A_\alpha \cup B_\alpha$. Since $\{A_\alpha \cup B_\alpha : \alpha \in \Delta\}$ covers $X$ and $\{A_\alpha, B_\alpha : \alpha \in \Delta\} \subseteq u(\tau_{\mathcal{N}}, \sigma_{\mathcal{N}})$, then by (b), there exists a finite set $\Delta' \subseteq \Delta$ such that $\{A_\alpha, B_\alpha : \alpha \in \Delta'\}$ covers $X$. It follows that $\{H_\alpha : \alpha \in \Delta'\}$ is a finite subcover of $\mathcal{H}$.

(c) $\Longrightarrow$ (d): Let $\mathcal{H}$ be a cover of $X$ with $\mathcal{H} \subseteq q(\tau, \sigma)$. Since $q(\tau, \sigma) \subseteq q$-$\mathcal{N}(\tau, \sigma)$, then $\mathcal{H} \subseteq q$-$\mathcal{N}(\tau, \sigma)$. Therefore, by (c) $\mathcal{H}$ has a finite subcover.

(d) $\Longrightarrow$ (a): Since $u(\tau, \sigma) \subseteq q(\tau, \sigma)$, then by (d) every cover cover of $X$ with elements of $u(\tau, \sigma)$ has a finite subcover. It follows that $(X, \tau, \sigma)$ is $s$-compact .

**Theorem 3.5.** For a bitopological space $(X, \tau, \sigma)$, the following are equivalent:

(a) $(X, \tau, \sigma)$ is $p$-compact.
(b) $(X, \tau_{\mathcal{N}}, \sigma_{\mathcal{N}})$ is $p$-compact.

**Proof.** (a) $\Longrightarrow$ (b): Let $(X, \tau, \sigma)$ be $p$-compact. Let $\mathcal{H} = \{H_\alpha : \alpha \in \Delta\}$ be a $p$-open cover of $(X, \tau_{\mathcal{N}}, \sigma_{\mathcal{N}})$. Choose $\alpha_1, \alpha_2 \in \Delta$ such that $H_{\alpha_1} \in \tau_{\mathcal{N}} - \{\emptyset\}$ and $W_{\alpha_2} \in \sigma_{\mathcal{N}} - \{\emptyset\}$. For each $\alpha \in \Delta$, there exists an indexed set $\omega_{\ \alpha}$ such that $H_\alpha = \beta \in \omega_{\ \alpha} \bigcup (V_\beta - F_\beta)$ where $\{F_\beta : \beta \in \omega_{\ \alpha}$ is a family of finite subsets of $X$ and $\{V_\beta : \beta \in \omega_{\ \alpha} \subseteq \tau$ or $\{V_\beta : \beta \in \omega_{\ \alpha} \subseteq \sigma$. For every $i = 1, 2$, choose $\beta_i \in \omega_{\ \alpha_i}$ such that $V_{\beta_1} \in \tau - \{\emptyset\}, V_{\beta_2} \in \sigma - \{\emptyset\}$. Thus, $\{V_\beta : \beta \in$

$\alpha \in \Delta\bigcup\omega_\alpha\}$ is a $p$-open cover of $(X, \tau, \sigma)$. Since $(X, \tau, \sigma)$ is $p$-compact, then there exists a finite set $\Delta' \subseteq \Delta$ such that for every $\alpha \in \Delta'$, there exists a finite set $\Gamma_\alpha \subseteq\omega_{\ \alpha}$ such that $\{V_\beta : \beta \in \alpha \in \Delta'\bigcup\Gamma_\alpha\}$ covers $X$. Take $G = \bigcup\{F_\beta : \beta \in \alpha \in \Delta'\bigcup\Gamma_\alpha\}$. Then $G$ is finite and $\{V_\beta - F_\beta : \beta \in \alpha \in \Delta'\bigcup\Gamma_\alpha\}$ is a cover of $X - G$. For each $x \in G$, choose $\alpha_x \in \Delta$ such that $x \in H_{\alpha_x}$. Thus, $\{H_\alpha : \alpha \in \Delta'\} \cup \{H_{\alpha_x} : x \in G\}$ is a finite subcover of $\mathcal{H}$.

(b) $\Longrightarrow$ (a): Let $(X, \tau_\mathcal{N}, \sigma_\mathcal{N})$ be $p$-compact. Let $\mathcal{H}$ be a $p$-open cover of $(X, \tau, \sigma)$. Then $\mathcal{H}$ is a $p$-open cover of $(X, \tau_\mathcal{N}, \sigma_\mathcal{N})$ and hence it has a finite subcover.

Recall that a subset $A$ of a bitopological space $(X, \tau, \sigma)$ is called an $s$-compact subset of $(X, \tau, \sigma)$ if every $\tau\sigma$-open cover of $A$ has a finite subcover.

**Theorem 3.6.** Let $(X, \tau, \sigma)$ be an $s$-compact bitopological space and $A$ be a $q$-$\mathcal{N}$-closed subset in $(X, \tau, \sigma)$. Then $A$ is an $s$-compact subset of $(X, \tau, \sigma)$.

**Proof.** Let $(X, \tau, \sigma)$ be $s$-compact. Let $\mathcal{H}$ be a $\tau\sigma$-open cover of $A$ and so $\mathcal{H} \subseteq q$-$\mathcal{N}(\tau, \sigma)$. Note that $X - A \in q$-$\mathcal{N}(\tau, \sigma)$ and so there are $B \in \tau$ and $C \in \sigma$ such that $X - A = B \cup C$. Thus $\mathcal{H} \cup \{X - A\}$ is a cover of $X$ consists of elements of $q$-$\mathcal{N}(\tau, \sigma)$. Since $(X, \tau, \sigma)$ is $s$-compact, by Theorem 3.4, $\mathcal{H} \cup \{X - A\}$ has a finite subcover $\mathcal{M}$. Let $\mathcal{M}_1 = \mathcal{M} - \{X - A\}$. Then $\mathcal{M}_1$ is finite, $\mathcal{M}_1 \subseteq \mathcal{H}$, and $\mathcal{M}_1$ covers $A$. It follows that $A$ is an $s$-compact subset of $(X, \tau, \sigma)$.

**Corollary 3.7.** Let $(X, \tau, \sigma)$ be $s$-compact and $A \subseteq X$. If $A$ is $u$-$\mathcal{N}$-closed subset in $(X, \tau, \sigma)$, then $A$ is an $s$-compact subset of $(X, \tau, \sigma)$.

**Proof.** Follows from Theorem 3.6.

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