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On asymptotic commutativity degree of finite groups

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Abstract:

The aim of this paper is to give a detailed account of a problema posed by P. Lescot regarding asymptotic commutativity degree of finite groups.

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1. Introduction

The commutativity degree $d(G)$ of a finite group G is the probability that two randomly chosen elements of G commute (see [6]).

Therefore,

$$d(G) = \frac{|\{(x,y) \in G \times G : xy = yx\}|}{|G|^2}.$$

In the year 2001, Lescot [7] has computed $d(D_{2n})$ and $d(Q_{2^{n+1}})$ where D_{2n} is a dihedral group

presented by $\langle a, b : a^n = b^2 = 1, bab^{-1} = a^{-1}\rangle$ and $Q_{2^{n+1}}$ is a quaternion group presented by $\langle a, b : a^{2^n} = 1, b^2 = a^{2^{n-1}}, bab^{-1} = a^{-1}\rangle$. It was shown that

$$d(D_{2n}) \to \frac{1}{4} \quad \text{and} \quad d(Q_{2^{n+1}}) \to \frac{1}{4}$$

as $|D_{2n}| \to \infty$ and $|Q_{2^{n+1}}| \to \infty$.

Then Lescot asked, "*whether there are other natural families of groups with the same property*".

Let G_n be a family of finite non-abelian groups such that $|G_n| \to \infty$ as $n \to \infty$. Then the limit of $d(G_n)$ as $n \to \infty$ is called the asymptotic commutativity degree of G_n. In this regard, the problem posed by Lescot can be restate in the following way:

Question 1: Is there any family of finite groups other than D_{2n} and $Q_{2^{n+1}}$ whose asymptotic commutativity degree is $\frac{1}{4}$?

In the year 2008, Doostie and Maghasedi [2] have computed the commutativity degree of the following classes of finite groups:

$$G_1(m,n) = \langle a, b, c : a^2 = b^n = c^{2m} = 1, c^{-1}aca = 1, c^{-1}bcb = 1\rangle \quad \text{and}$$

$$G_2(m,n) = \langle a, b, c : a^{2^n} = b^{2^n} = c^2 = 1, c^{-1}ac = b, c^{-1}bc = a\rangle.$$

They have shown that

$$d(G_1(m,n)) = \begin{cases} \dfrac{n+3}{4n}, & \text{if } n \text{ is odd} \\ \dfrac{n+6}{4n}, & \text{if } n \text{ is even} \end{cases} \quad \text{and } d(G_2(m,n)) = \frac{2^n+3}{2^{n+2}}.$$

Therefore, as n tends to infinity we have

$$d(G_1(m,n)) \to \frac{1}{4} \text{ and } d(G_2(m,n)) \to \frac{1}{4}.$$

Thus the families $G_1(m,n)$ and $G_2(m,n)$ give affirmative answer to Question 1.

In the year 2010, Castelaz (see [1, Chapter 4]) computed the asymptotic commutativity degree for several different classes of finite groups including the dicyclic groups Q_{4m} presented by $\langle a,b : a^{2m} = 1, b^2 = a^m, bab^{-1} = a^{-1}\rangle$ and the semidihedral groups SD_n presented by $\langle a,b : a^{2^{n-1}} = b^2 = 1, bab^{-1} = a^{-1+2^{n-2}}\rangle$. Castelaz showed that

$$d(Q_{4m}) \to \frac{1}{4} \text{ and } d(SD_n) \to \frac{1}{4},$$

as the orders of Q_{4m} and SD_n tend to infinity.

In 2013, the author has computed $d(C_{n\theta}C_{2m})$, where $\theta : C_{2m} \longrightarrow Aut(C_n)$ is the homomorphism such that $\theta(b)$, for a generator b of C_{2m}, is the inversing automorphism of C_n (see [9]). The author also have shown that

$$d\left(C_n \times_\theta C_{2m}\right) \to \frac{1}{4} \text{ as } n \to \infty.$$

Recently, Dutta [3] have shown that the asymptotic commutativity degree of the group $M_{2m,n}$ for $n > 2$ presented by $\langle a,b : a^n = b^{2m} = 1, bab^{-1} = a^{-1}\rangle$ is $\frac{1}{4}$.

It is worth mentioning that Erovenko and Sury [4] have computed $d(A \wr B)$ where A, B are two finite abelian group and $\wr$ stands for wreath product. They showed that $d(A \wr B) \to \frac{1}{n^2}$ as $|A| \to \infty$ if B is fixed of order $n > 1$.

Doostie and Maghasedi [2] have also computed the commutativity degree of the groups namely $G_3(m,n)$ presented by

$$\langle a,b,c : a^{2^{n-1}} = c^{2m} = 1, b^2 = a^{2^{n-2}}, b^{-1}aba = c^{-1}aca = c^{-1}bcb = 1\rangle.$$

They have shown that

$$d(G_3(m,n)) = \frac{2^{n-3}+3}{2^n} \text{ and so } d(G_3(m,n)) \to \frac{1}{8}$$

as n tends to infinity. Motivated by these facts one may ask the following question.

Question 2: Let $k > 1$ be any positive integer. Is there any family of finite groups whose asymptotic commutativity degree is $\frac{1}{k}$?

In this paper, we answer Question 2 affirmatively. Further we shall show that the reciprocal of every positive integer can be realized as $d(G)$ for some finite group G. It is worth mentioning that the central problem in the study of commutativity degree of finite groups is to find the rational numbers in the interval $(0,1]$ that can be realized as $d(G)$ for some finite group G.

2. Main Results

We begin with the following three useful results.

Lemma 1. [5] *For any two finite group H and K we have*

$$d(H \times K) = d(H)d(K).$$

Proposition 2. [10] *If G is a finite p-group, where p is a prime, and $G' \subseteq Z(G)$, then*

$$d(G) = \frac{1}{|G'|}\left(1 + \sum_{\substack{K \leq G', \\ G'/K \text{ cyclic}}} \frac{(p-1)\,|G' : K|}{p\,|G : K^*|}\right).$$

where $K^* = \{x \in G : [G, x] \subseteq K\} \trianglelefteq G$ and $\frac{G}{K^*} \cong \prod(C_{p^{n_i}} \times C_{p^{n_i}})$ with $p \leq p^{n_i} \leq p^{n_1} = p^k = |G' : K|$.

A consequence of the above results is given below.

Corollary 3. *Let G be a finite group and $|G'| = p$, a prime. If $G' \subseteq Z(G)$, then $\frac{G}{Z(G)} \cong (C_p \times C_p)^s$, for some $s \geq 1$, and*

$$d(G) = \frac{1}{p}\left(1 + \frac{p-1}{p^{2s}}\right).$$

Proof. If $G' \subseteq Z(G)$ then G is nilpotent of class 2. Hence, $G = P_1 \times P_2 \times \cdots \times P_k$ where P_i's are Sylow p_i-subgroups of G corresponding to the primes p_i dividing $|G|$. Since $G' = P_1' \times P_2' \times \cdots \times P_k'$ and $|G'| = p$ we must have $|P_1'| = p$ and $|P_2'| = \cdots = |P_k'| = 1$, assuming that P_1 is a Sylow p-subgroup. Therefore, $P_2, \ldots, P_k$ are abelian groups and hence $Z(G) = Z(P_1) \times P_2 \times \cdots \times P_k$. By [10, Proposition 2], it follows that

$$\frac{G}{Z(G)} \cong \frac{P_1}{Z(P_1)} \cong (C_p \times C_p)^s \tag{2.1}$$

for some $s \geq 1$. Again, by Lemma 1 and Proposition 2, we have

$$d(G) = \prod_{i=1}^{m} d(P_i) = d(P_1) = \frac{1}{p}\left(1 + \frac{(p-1)|P_1' : \{1\}|}{p|P_1 : \{1\}^*|}\right) \tag{2.2}$$

since $K = \{1\}$ is the only proper subgroup of P_1' such that P_1'/K is cyclic. Hence, the result follows from (2.1) and (2.2) noting that $\{1\}^* = Z(P_1)$. $\square$

We now state and prove the first main result of this section which give affirmative answer to Question 2.

Theorem 4. *There exists a family of finite groups having asymptotic commutativity degree $\frac{1}{k}$ for every integer $k > 1$.*

Proof. Let $k = p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}$ be the prime factorization of k. Consider the families $ES(n_i, p_i)$ of extra-special p_i-groups of order $p_i^{2n_i+1}$ for $i = 1, 2, \ldots, m$. By Lemma 1 and Corollary 3, we have

$$d((ES(n_i, p_i))^{k_i}) = \left(\frac{1}{p_i} + \frac{p_i - 1}{p_i^{2n_i+1}} \right)^{k_i}$$

where $(ES(n_i, p_i))^{k_i}$ is the direct product of k_i copies of $ES(n_i, p_i)$.

Hence, the result follows from Lemma 1, considering the family

$$(ES(n_1, p_1))^{k_1} \times (ES(n_2, p_2))^{k_2} \times \cdots \times (ES(n_m, p_m))^{k_m}$$

obtained by extra-special p-groups noting that

$$d((ES(n_i, p_i))^{k_i}) \to \frac{1}{p_i^{k_i}} \quad \text{as } n_i \to \infty.$$

The following theorem shows that the reciprocal of every positive integer can be realized as $d(G)$ of some finite group G.

Theorem 5. *There exists a finite group G such that $d(G) = \frac{1}{n}$ for every positive integer n.*

Proof. We shall prove the theorem by induction on n. For $n = 1$, we may take G to be any abelian group. If $n = 2$, we may take, $G = S_3$. So, assume that $n \geq 3$ and that the theorem is true for all positive integers k less than n.

Case 1. $n \equiv 0$ or $2 \pmod 4$. In this case, $n = 2^\alpha . m$, where α, m are positive integers and m is odd. Clearly $m < n$. So, by induction hypothesis there exists a finite group G such that $d(G) = \frac{1}{m}$. Hence, using the fact that $d(S_3) = \frac{1}{2}$ and Lemma 1, we have

$$d(G \times (S_3)^\alpha) = d(G).(d(S_3))^\alpha = \frac{1}{m.2^\alpha} = \frac{1}{n}.$$

Case 2. $n \equiv 1 \pmod 4$. In this case, $\frac{n+3}{4}$ is a positive integer and $\frac{n+3}{4} < n$. So, by induction hypothesis, there exists a finite group G such that $d(G) = \frac{4}{n+3}$. Hence,

$$d(D_{2n} \times G) = \frac{n+3}{4n}.\frac{4}{n+3} = \frac{1}{n}.$$

Case 3. $n \equiv 3 \pmod 4$. In this case, $\frac{n+1}{4}$ is a positive integer and $\frac{n+1}{4} < n$. So, by induction hypothesis, there exists a finite group G such that $d(G) = \frac{4}{n+1}$. Hence,

$$d(D_{6n} \times G) = \frac{3n+3}{12n}.\frac{4}{n+1} = \frac{1}{n}.$$

This completes the proof.

We conclude this paper noting that the above two theorems are also obtained by Castelaz considering different families of finite groups (see Corollary 4.3.2 and Corollary 5.3.3 of [1]).

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