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GRAPHS r-POLAR SPHERICAL REALIZATION*

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Abstract

The graph to considered will be in general simple and finite, graphs with a nonempty set of edges. For a graph G, V(G) denote the set of vertices and E(G) denote the set of edges. Now, let $P_r = (0, 0, 0, r) \in \mathbf{R}^4, r \in \mathbf{R}^+$ *. The* $r-$*polar sphere, denoted by* S_{P_r}*, is defined by* $\left\{x \in \mathbf{R}^4 / \|x\| = 1 \wedge x \neq P_r\right\}$*: The primary target of this work is to present the concept of* $r-$*Polar Spherical Realization of a graph. That idea is the following one: If* G *is a graph and* $h : V(G) \to S_{P_r}$ *is a injective function, them the* $r-$*Polar Spherical Realization of* G*, denoted by* G^**, it is a pair* $(V(G^*), E(G^*))$ *so that* $V(G^*) = \{h(v)/v \in V(G)\}$ *and* $E(G^*) = \{arc(h(u)h(v))/uv \in E(G)\}$*, in where* $arc(h(u)h(v))$ *it is the arc of curve contained in the intersection of the plane defined by the points* $h(u), h(v), P_r$ *and the* $r-$*polar sphere.*

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1. Introduction.

The graph to considered will be in general simple and finite, graphs with a nonempty set of edges. For a graph $G, V(G)$ denote the set of vertices and $E(G)$ denote the set of edges. The cardinality of $V(G)$ is called *order* of G and habitually it is written down by $|G|$, and the cardinality of $E(G)$ is called *size* of G and habitually it is written down by $|E(G)|$. A (p,q) graph has order p and size q. Two vertices u and v are called *neighbors* if $\{u,v\}$ is an edge of G. For any vertex v of G, denote by N_v the set neighbors of v and by deg v the degree of v. To simplify the notation, and edge $\{x,y\}$ is written as xy (or yx). Other concepts used in this work and not defined explicitly can be found in the references [1], [2], [3],[4],[5], [7], [9],[10], [11],[12],[13].

1.1.- r**- sphere**:The r- sphere, $\{x \in \mathbf{R}^4 / \|x\| = r\}$, $r \in \mathbf{R}^+$, where $\| \quad \|$ is the usual norm of $\mathbf{R}^4$, it will be denoted by $S(r)$. That is, $S(r) = \{x \in \mathbf{R}^4 / \|x\| = r\}$.

1.2..- r**- Polar sphere:**The $r-$polar sphere, denoted by S_{P_r}, $r \in \mathbf{R}^+$, where $P_r = (0,0,0,r) \in \mathbf{R}^4$, is defined by $S(r) - \{P_r\}$.That is $S_{P_r} = S(r) - \{P_r\} = \{x \in S(r) \ / \ x \neq P_r\}$.

2. Preliminaries.

Theorem 1: The r**-** polar sphere admits a real tridimensional vectorial space structure, isomorphic with $\mathbf{R}^3$ as a vectorial space.

Proof: Let $v = (x,y,z) \in \mathbf{R}^3$. We define a function f_r of $\mathbf{R}^3$ in S_{P_r} by $f_r(x,y,z) = \left(\frac{2rx}{\|v\|^2+1}, \frac{2ry}{\|v\|^2+1}, \frac{2rz}{\|v\|^2+1}, r\frac{\|v\|^2-1}{\|v\|^2+1}\right)$.

Now, we will try to prove that indeed f_r is function.

Since $\left(\frac{2rx}{\|v\|^2+1}\right)^2 + \left(\frac{2ry}{\|v\|^2+1}\right)^2 + \left(\frac{2rz}{\|v\|^2+1}\right)^2 + r^2\left(\frac{\|v\|^2-1}{\|v\|^2+1}\right)^2 =$

$$= r^2\frac{4x^2+4y^2+4z^2+\|v\|^4-2\|v\|^2+1}{\left(\|v\|^2+1\right)^2} = r^2\frac{\|v\|^4+2\|v\|^2+1}{\left(\|v\|^2+1\right)^2} = r^2\frac{\left(\|v\|^2+1\right)^2}{\left(\|v\|^2+1\right)^2} = r^2,$$

then $f_r(x,y,z) \in S_{P_r}$.

Therefore, each $v \in \mathbf{R}^3$ has an image in S_{P_r}, we say $f_r(v)$. On the other hand, if $u = (a,b,c) = v = (x,y,z)$ in $\mathbf{R}^3$,then $a = x, b = y, c = z$, $\|u\| = \|v\|$and then $f_r(u) = f_r(v)$.

If $w = (a,b,c,t) \in S_{P_r}$, then $t \neq r$ (Note that $a^2+b^2+c^2+t^2 = r^2$ and $t = r$, then $a^2+b^2+c^2 = 0$) and we could define the real numbers $\frac{a}{r-t}, \frac{b}{r-t}, \frac{c}{r-t}$. Therefore, the relation $g_r(a,b,c,t) = \left(\frac{a}{r-t}, \frac{b}{r-t}, \frac{c}{r-t}\right)$ is a function from S_{P_r} in $\mathbf{R}^3$.

It's true that, for every $v=(x,y,z)\in\mathbf{R}^3$
$(g_r\circ f_r)(v)=(g_r\circ f_r)(x,y,z)=g_r(f_r(x,y,z))=$
$=g_r\left(\frac{2rx}{\|v\|^2+1},\frac{2ry}{\|v\|^2+1},\frac{2rz}{\|v\|^2+1},r\frac{\|v\|^2-1}{\|v\|^2+1}\right)=$
$=\left(\frac{\frac{2rx}{\|v\|^2+1}}{r-r\frac{\|v\|^2-1}{\|v\|^2+1}},\frac{\frac{2ry}{\|v\|^2+1}}{r-r\frac{\|v\|^2-1}{\|v\|^2+1}},\frac{\frac{2rz}{\|v\|^2+1}}{r-r\frac{\|v\|^2-1}{\|v\|^2+1}}\right)=$
$=\left(\frac{\frac{2x}{\|v\|^2+1}}{1-\frac{\|v\|^2-1}{\|v\|^2+1}},\frac{\frac{2y}{\|v\|^2+1}}{1-\frac{\|v\|^2-1}{\|v\|^2+1}},\frac{\frac{2z}{\|v\|^2+1}}{1-\frac{\|v\|^2-1}{\|v\|^2+1}}\right)=(x,y,z)=v.$
and for every $w=(a,b,c,t)\in S_{P_r}$,we have that
$(f_r\circ g_r)(w)=(f_r\circ g_r)(a,b,c,t)=f_r(g_r(a,b,c,t))=f_r\left(\frac{a}{r-t},\frac{b}{r-t},\frac{c}{r-t}\right)=$

$=(\frac{\frac{2ra}{r-t}}{\left(\frac{a}{r-t}\right)^2+\left(\frac{b}{r-t}\right)^2+\left(\frac{c}{r-t}\right)^2+1},\frac{\frac{2rb}{r-t}}{\left(\frac{a}{r-t}\right)^2+\left(\frac{b}{r-t}\right)^2+\left(\frac{c}{r-t}\right)^2+1},\frac{\frac{2rc}{r-t}}{\left(\frac{a}{r-t}\right)^2+\left(\frac{b}{r-t}\right)^2+\left(\frac{c}{r-t}\right)^2+1},$
$r\frac{\left(\frac{a}{r-t}\right)^2+\left(\frac{b}{r-t}\right)^2+\left(\frac{c}{r-t}\right)^2-1}{\left(\frac{a}{r-t}\right)^2+\left(\frac{b}{r-t}\right)^2+\left(\frac{c}{r-t}\right)^2+1})=(\frac{2ra}{r-t}\cdot\frac{(r-t)^2}{a^2+b^2+c^2+r^2-2rt+t^2},\frac{2rb}{r-t}\cdot\frac{(r-t)^2}{a^2+b^2+c^2+r^2-2rt+t^2},$
$\frac{2rc}{r-t}\cdot\frac{(r-t)^2}{a^2+b^2+c^2+r^2-2rt+t^2},r\frac{a^2+b^2+c^2-r^2+2rt-t^2}{a^2+b^2+c^2+r^2-2rt+t^2})=$

$=\left(a,b,c,r\frac{r^2-t^2-r^2+2rt-t^2}{r^2-t^2+r^2-2rt+t^2}\right)=\left(a,b,c,r\frac{2rt-2t^2}{2r^2-2rt}\right)=\left(a,b,c,t\frac{2r-2t}{2r-2t}\right)=w.$

Therefore, f_r is invertible and bijective.

Now, we define S_{P_r} as a real vectorial space structure.
To get this, we define a binary operation in S_{P_r}, defined by $\oplus$, as
$\forall u,v\in\mathbf{R}^3, u=(a,b,c), v=(x,y,z): f_r(u)\oplus f_r(v)=f_r(a,b,c)\oplus f_r(x,y,z)=\left(\frac{2r(a+x)}{\|u+v\|^2+1},\frac{2r(b+y)}{\|u+v\|^2+1},\frac{2r(c+z)}{\|u+v\|^2+1},r\frac{\|u+v\|^2-1}{\|u+v\|^2+1}\right).$

It is evident that $f_r(u)\oplus f_r(v)=f_r(u+v)$.
Operation $\oplus$ satisfies the following properties:
(i) Closure: $\forall\widetilde{u},\widetilde{v}\in S_{P_r},\exists\widetilde{w}\in S_{P_r}:\widetilde{u},\oplus\widetilde{v}=\widetilde{w}$
Proof: Let $u=(a,b,c), v=(x,y,z)\in\mathbf{R}^3$ be so that $\widetilde{u}=f_r(u)$ and $\widetilde{v}=f_r(v)$. Define $\widetilde{w}=\left(\frac{2r(a+x)}{\|u+v\|^2+1},\frac{2r(b+y)}{\|u+v\|^2+1},\frac{2r(c+z)}{\|u+v\|^2+1},r\frac{\|u+v\|^2-1}{\|u+v\|^2+1}\right)$, then

$\widetilde{u}\oplus\widetilde{v}=f(u)\oplus f(v)=f(u+v)=\widetilde{w}$.
(ii) Associative property: $\forall\widetilde{u},\widetilde{v},\widetilde{w}\in S_{P_r}:(\widetilde{u}\oplus\widetilde{v})\oplus\widetilde{w}=\widetilde{u}\oplus(\widetilde{v}\oplus\widetilde{w})$

Proof: Let $u, v, w \in \mathbf{R}^3$ be so that $\widetilde{u} = f(u), \widetilde{v} = f(v)$ and $\widetilde{w} = f(w)$. Then $(\widetilde{u} \oplus \widetilde{v}) \oplus \widetilde{w} = f_r((u+v)+w)) = f_r(u+(v+w)) = \widetilde{u} \oplus (\widetilde{v} \oplus \widetilde{w})$.

(iii) Existence of the neutral element:
$\exists \widetilde{0} \in S_{P_r}, \forall \widetilde{u} \in S_{P_r}, : \widetilde{u} \oplus \widetilde{0} = \widetilde{0} \oplus \widetilde{u} = \widetilde{u}$
Proof: Define $\widetilde{0} = f_r((0,0,0)) = (0,0,0,-r)$ and $u = (x,y,z) \in \mathbf{R}^3$ so

that $\widetilde{u} = f_r(x,y,z) \in S_{P_r}$. Then $\widetilde{u} \oplus \widetilde{0} = \left(\frac{2r(x+0)}{\|u\|^2+1}, \frac{2r(y+0)}{\|u\|^2+1}, \frac{2r(z+0)}{\|u\|^2+1}, r\frac{\|u\|^2-1}{\|u\|^2+1} \right) = f_r(x,y,z) = \widetilde{u}$.Similarly $\widetilde{0} \oplus \widetilde{u} = \widetilde{u}$.

(iv) Existence of the inverse element:
$\forall \widetilde{u} \in S_{P_r}, \exists \widetilde{v} \in S_{P_r} : \widetilde{u} \oplus \widetilde{v} = \widetilde{v} \oplus \widetilde{u} = \widetilde{0}$
Proof: Let be so that $\widetilde{u} = f_r(x,y,z) \in S_{P_r}$and $\mathbf{0} = (0,0,0) \in \mathbf{R}^3$.
Define $v = (-x,-y,-z) \in \mathbf{R}^3$. Then $\widetilde{u} \oplus \widetilde{v} = f_r(x,y,z) \oplus f_r(-x,-y,-z) =$
$= \left(\frac{2r(x-x)}{\|0\|^2+1}, \frac{2r(y-y)}{\|0\|^2+1}, \frac{2(z-z)}{\|0\|^2+1}, r\frac{\|0\|^2-1}{\|0\|^2+1} \right) = \widetilde{0}$. Similarly $\widetilde{v} \oplus \widetilde{u} = \widetilde{0}$.

(v) Commutative property: $\forall \widetilde{u}, \widetilde{v} \in S_{P_r} : \widetilde{u} \oplus \widetilde{v} = \widetilde{v} \oplus \widetilde{u}$.
Proof: Let $u, v, w \in \mathbf{R}^3$ be so that $\widetilde{u} = f_r(u)$ and $\widetilde{v} = f_r(v)$.Then $\widetilde{u} \oplus \widetilde{v} = f_r(u+v) = f_r(v+u) = \widetilde{v} \oplus \widetilde{u}$.

Consequently $\langle S_{P_r}, \oplus \rangle$ is an abelian group.

Here, we would define an external operation $\odot : \mathbf{R} \times S_{P_r} \to S_{P_r}$ as

$\forall \alpha \in \mathbf{R}, \forall v = (x,y,z) \in \mathbf{R}^3 : \alpha \odot f_r(v) = \left(\frac{2\alpha r x}{\alpha^2\|v\|^2+1}, \frac{2\alpha r y}{\alpha^2\|v\|^2+1}, \frac{2\alpha r z}{\alpha^2\|v\|^2+1}, r\frac{\alpha^2\|v\|^2-1}{\alpha^2\|v\|^2+1} \right)$
It is evident that $\alpha \odot f_r(v) = f_r(\alpha v)$ and likewise $\alpha \odot f_r(v) \in S_{P_r}$

We should prove the following properties:
(i) $\forall \alpha, \beta \in \mathbf{R}, \forall v \in \mathbf{R}^3 : (\alpha+\beta) \odot f_r(v) = [\alpha \odot f_r(v)] \oplus [\beta \odot f_r(v)]$,
Proof:$(\alpha+\beta) \odot f_r(v) = f_r((\alpha+\beta)v) = f_r(\alpha v + \beta v) = f_r(\alpha v) \oplus f_r(\beta v) = [\alpha \odot f_r(v)] \oplus [\beta \odot f_r(v)]$

(ii) $\forall \alpha \in \mathbf{R}, \forall u, v \in \mathbf{R}^3 : \alpha \odot [f_r(u) \oplus f_r(v)] = [\alpha \odot f_r(u)] \oplus [\alpha \odot f_r(v)]$.
Proof: $\alpha \odot [f_r(u) \oplus f_r(v)] = \alpha \odot f_r(u+v) = f_r(\alpha(u+v)) = f_r(\alpha u + \alpha v) = f_r(\alpha u) \oplus f_r(\alpha v) = [\alpha \odot f_r(u)] \oplus [\alpha \odot f_r(v)]$.

(iii) $\forall \alpha, \beta \in \mathbf{R}, : \alpha \odot (\beta \odot f_r(v)) = (\alpha\beta) \odot f_r(v)$.
Proof: $\alpha \odot (\beta \odot f_r(v)) = \alpha \odot f_r(\beta v) = f_r(\alpha\beta v) = (\alpha\beta) \odot f_r(v)$

(iv) $\forall v \in \mathbf{R}^3 : 1 \odot f_r(v) = f_r(v)$ (Here 1 is the neutral multiplicative from $\mathbf{R}$)
Proof: $1 \odot f_r(v) = f_r(1v) = f_r(v)$
From the above points, we have that S_{P_r} is a $\mathbf{R}$-vectorial space.

Now, it will be proven that S_{P_r} has dimension three as real vectorial space and that f_r defines a linear transformation F of $\mathbf{R}^3$ in S_{P_r}.
First, we will prove that $\{w_1, w_2, w_3\}$ is a basis for S_{P_r}, where
$w_1 = f_r(e_1) = (r, 0, 0, 0), w_2 = f_r(e_2) = (0, r, 0, 0), w_3 = f_r(e_3) = (0, 0, r, 0)$

If $v = (x, y, z, t) \in S_{P_r}, t \neq r$, then there exist some $\alpha, \beta, \gamma \in \mathbf{R}$ so that $v = \alpha \odot f_r(e_1) \oplus \beta \odot f_r(e_2) \oplus \gamma \odot f_r(e_3)$.
Consequently $\alpha \odot f_r(e_1) = f_r(\alpha, 0, 0), \beta \odot f_r(e_2) = f_r(0, \beta, 0), \gamma \odot f_r(e_3) = f_r(0, 0, \gamma)$.

Therefore $(x, y, z, t) = f_r(\alpha, 0, 0) \oplus f_r(0, \beta, 0) \oplus f_r(0, 0, \gamma) =$
$= f_r(\alpha, \beta, 0) \oplus f_r(0, 0, \gamma) = \left(\frac{2r\alpha}{\alpha^2+\beta^2+\gamma^2+1}, \frac{2r\beta}{\alpha^2+\beta^2+\gamma^2+1}, \frac{2r\gamma}{\alpha^2+\beta^2+\gamma^2+1}, r\frac{\alpha^2+\beta^2+\gamma^2-1}{\alpha^2+\beta^2+\gamma^2+1}\right)$.
Therefore
$x = \frac{2r\alpha}{\alpha^2+\beta^2+\gamma^2+1}, y = \frac{2r\beta}{\alpha^2+\beta^2+\gamma^2+1}, z = \frac{2r\gamma}{\alpha^2+\beta^2+\gamma^2+1}, t = r\frac{\alpha^2+\beta^2+\gamma^2-1}{\alpha^2+\beta^2+\gamma^2+1} \Longrightarrow$
$\alpha^2 + \beta^2 + \gamma^2 = \frac{r+t}{r-t} \Longrightarrow$

$$\Longrightarrow x = \frac{2r\alpha}{\frac{r+t}{r-t}+1}, y = \frac{2r\beta}{\frac{r+t}{r-t}+1}, z = \frac{2r\gamma}{\frac{r+t}{r-t}+1} \Longrightarrow \alpha = \frac{x}{r-t}, \beta = \frac{y}{r-t}, \gamma = \frac{z}{r-t}$$

As a neutral element for the $\oplus$ operation we have $-P_r = (0, 0, 0, -r)$, then $\alpha \odot f_r(e_1) \oplus \beta \odot f_r(e_2) \oplus \gamma \odot f_r(e_3) = -P_r \Rightarrow \alpha = \beta = \gamma = 0$.

Finally, the function F_r form $\mathbf{R}^3$ in S_{P_r} defined by $F_r(x, y, z) = f_r(x, y, z)$ is linear.

Corolary 1.1 : Any simple and finite graph G is realizable over S_{P_r}.
PROOF : Using a graph theorem that ensures all simple and finite graph G is realizable over $\mathbf{R}^3$, its proof could be verified in [3], then we obtain the requested in the prior Theorem 1.

With the following definition we claim to obtain an always limited distance measure between two graphs, limited by a fixed number μ. In our research, $\mu = 3\pi^2 r^3$ [6],[8].

3 Graphs $r-$Polar Spheric Realization

If G is a simple and finite graph, and $h : V(G) \to S_{P_r}$ is an injective function, then the $r-$*Polar Spheric Realization* from G, denoted by G^* is a couple $(V(G^*), E(G^*))$ such that $V(G^*) = \{h(v)/v \in V(G)\} \subset S_{P_r}$ y $E(G^*) = \{arc\,(h(u)h(v))\,/uv \in V(G)\}$, where $arc\,(h(u)h(v))$ is the arc of the curve contained in the intersection of the plane, defined by the points $h(u), h(v), P$ and the $r-$polar sphere.

In the context of the polar spheric realization we consider the following restrictions:

(1) As Ω we record the class of all finite and simple graphs class.

(2) As Ω^* we record the families of all the polar spheric realizations form Ω.

(3) If $G, H \in \Omega$ y $G \neq H$, then $V(G^*) \cap V(H^*) = \phi$ and $E(G^*) \cap E(H^*) = \phi$.

(4) If $G \in \Omega$, $G^* \in \Omega^*$. (So that all the Ω and Ω^* elements are defined as univocal.).

Some $r-$polar spheric realization examples are:

1.- G is the graph defined by $V(G) = \{1, 2, 3\}$ and $E(G) = \{12, 23, 13\}$, then the $1-$polar spheric realization from G is

$V(G^*) = \{i, j, k\}$ and $E(G^*) = \{arc\,(ij)\,, arc\,(ik)\,, arc\,(jk)\}$,

where $h\,(1) = i = (1, 0, 0, 0)\,, h\,(2) = j = (0, 1, 0, 0)\,, h\,(3) = k = (0, 0, 1, 0)$

2.- G is the graph defined by $V(G) = \{a, b\}$ and $E(G) = \{ab\}$, then the $1-$polar spheric realization from G is $V(G^*) = \left\{\left(\frac{1}{\sqrt{3}}, \frac{\sqrt{2}}{\sqrt{3}}, 0, 0\right), \left(0, \frac{\sqrt{3}}{2}, \frac{1}{2}, 0\right)\right\}$ and $E(G^*) = \{arc\,(h\,(a)\,h\,(b))\}$., where $h\,(a) = \left(\frac{1}{\sqrt{3}}, \frac{\sqrt{2}}{\sqrt{3}}, 0, 0\right), h\,(b) = \left(0, \frac{\sqrt{3}}{2}, \frac{1}{2}, 0\right)$.

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