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Saturations of submodules with respect to subsets of a ring

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Abstract

Let R be a commutative ring with identity and M a unital R-module. For any submodule N of M and non-empty subset T of R, let $S_T^M(N) = \{m \in M : rm \in N \text{ for some } r \in R \setminus T\}$. In this article we study conditions under which $S_T^M(N)$ is a submodule of M and investigate when it is a union of prime submodules.

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1. Introduction

Let R be a ring, M be an R-module and N be a submodule of M. The saturation of N with respect to $p \in Spec(R)$ is the contraction of N_p in M and designated by $S_p(N)$. It is known that $S_p(N) = \{x \in M : sx \in N$ for some $s \in R \setminus p\}$. It is shown that $S_p(N)$ is a p-prime submodule of M if and only if $(S_p(N) : M) = p$ [2, Theorem 2.3]. Moreover, if M is a finitely generated R-module and N a proper submodule of M such that $p = (N : M)$, then $S_p(N)$ is a prime submodule of M [2, Corollary 3.4]. In particular, if $N = pM$ and $p \supseteq Ann(M)$, then $S_p(N)$ is a prime submodule of M ([2, Corollary 3.8] or by combining [3, Theorem 2.1] and [3, Proposition 3.8]). In this paper, we replace the ideal p by an arbitrary subset T of R and investigate the properties of $S_T^M(N) = \{m \in M : rm \in N$ for some $r \in R \setminus T\}$, in particular considering when it is a submodule of M. We show that if M is a Noetherian module and N is a submodule of M with $(N : M) \subseteq T$, then $S_T(N)$ is a union of prime submodules of the form $(N :_M r)$ for some $r \in R \setminus T$.

2. Saturation of submodules

In this note all rings are commutative rings with identity and all modules are unital. For a submodule N of M, by $(N : M)$ we mean the ideal $\{r \in R : rM \subseteq N\}$ of R. Also, if there is no ambiguity we write $S_T(N)$ instead of $S_T^M(N)$.

The first few results gather together some basic properties of saturation.

Proposition 2.1. *Let R be a ring and, T_1 and T_2 be subsets of R. For an R-module M and submodules N, K of M with $N \subseteq K$, we have;*

(1) $N \subseteq S_T(N)$.

(2) $(N : M) \subseteq S_T(N : M) \subseteq (S_T(N) : M)$.

(3) If $S_T(K)$ is a submodule of M, then $S_T(\frac{K}{N}) = \frac{S_T(K)}{N}$.

(4) $S_T(N) \subseteq S_T(K)$.

(5) $S_{T_1}(S_{T_2}(N)) = S_{T_2}(S_{T_1}(N))$. *Moreover, if $T_1 \subseteq T_2$, then $S_{T_1}(N) \supseteq S_{T_2}(N)$.*

Proof. Clear. □ Let $T = \{0\}$ and $N = 0$. Then $S_T(N) = \{x \in M : rx = 0 \text{ for some } r \in R \setminus \{0\}\}$ is the set of all torsion elements of M. If R is an integral domain, $S_T(M)$ will be a submodule of M but this is not true in general. For example, let $R = M = \mathbf{Z}_6$. Then $\bar{2}, \bar{3} \in S_0(0)$ but $\bar{3} - \bar{2} \notin S_0(0)$ since $\bar{1} \notin S_0(0)$.

Lemma 2.2. *Let R be a ring and M be an R-module. Let N be a submodule of M and T be a subset of R such that $(N : M)T$. Then $S_T(N) = M$. However, the converse is not true in general.*

Proof. Let $r \in (N : M) \setminus T$. Then $rM \subseteq N$ with $r \in R \setminus T$ and hence $M \subseteq S_T(N)$. For the last part, it suffices to take $M = \mathbf{Q}$, $R = N = \mathbf{Z}$ and $T = \{0\}$. □ If $(N : M)T$, then $(R \setminus T) \cap (N : M) \neq \emptyset$ and hence $S_T(N) = M$. So this will not give us any information. Therefore we always assume that $(N : M) \subseteq T$.

Lemma 2.3. *Let R be a ring, M be an R-module and N be a submodule of M. Then $S_T(N)$ is a submodule of M if and only if it is closed under addition.*

Proof. Clear. □

Proposition 2.4. *Let R be a valuation ring and M be an R-module. Then $S_T(N)$ is a submodule of M for any submodule N of M.*

Proof. Let N be a submodule of M and $m_1, m_2 \in S_T(N)$. Then there exist $r_1, r_2 \in R \setminus T$ such that $r_1m_1, r_2m_2 \in N$. Since the ideals of R are totally ordered by inclusion, we may assume that $(r_1) \subseteq (r_2)$. Thus $r_1 = rr_2$ for some $r \in R$ and hence $r_1(m_1 + m_2) = rr_2(m_1 + m_2) \in N$, that is $m_1 + m_2 \in S_T(N)$. By Lemma 2.3, we have the result. □ Let M be an R-module. A proper submodule P of M is called a prime or p-prime submodule of M, if for $r \in R$ and $m \in M$, $rm \in P$ implies that $r \in p = (P : M)$ or $m \in P$.

Proposition 2.5. *Let N be a proper submodule of M and $T = (N : M)$. Suppose that $S_T(N)$ is a proper submodule of M. Then*

(1) *$S_T(N)$ is a prime submodule of M.*

(2) *T is a prime ideal of R if and only if $S_T(S_T(N)) = S_T(N)$.*

Proof. (1) Let $rm \in S_T(N)$ and $m \notin S_T(N)$. Then there exists $r' \in R \setminus T$ such that $r'(rm) \in N$. Since $m \notin S_T(N)$, $(r'r)m \in N$ implies that $r'r \in T = (N : M)$. Thus $r'(rM) \subseteq N$ and hence $rM \subseteq S_T(N)$. Therefore $r \in (S_T(N) : M)$ and hence $S_T(N)$ is a prime submodule of M.

(2) ($\Rightarrow$) Let T be a prime ideal of R. By Proposition 2.1, $S_T(N) \subseteq S_T(S_T(N))$. Let $m \in S_T(S_T(N))$. Then there exist $r, r' \in R \setminus T$ such that $r'rm \in N$. Since T is assumed to be prime, $r'r \notin T$ and hence $m \in S_T(N)$. Thus we have $S_T(S_T(N)) \subseteq S_T(N)$. ($\Leftarrow$) Suppose that T is not a prime ideal of R. Since $(N : M) \subseteq (S_T(N) : M)$ and $S_T(N)$, by (1), is a prime submodule of M, we have $(N : M) \subset (S_T(N) : M)$. Thus we have $(S_T(N) : M)T$ and hence by Lemma 2.2, $S_T(S_T(N)) = M$. Therefore $S_T(S_T(N)) \neq S_T(N)$. $\square$

Proposition 2.6. *Let P be a prime submodule of M and T be a subset of R such that $S_T(P) \neq M$. Then $S_T(P) = P$.*

Proof. Let $m \in S_T(P)$. Then there exists $r \in R \setminus T$ such that $rm \in P$. Since $S_T(P) \neq M$, by Lemma 2.2, we have $(N : M) \subseteq T$ and hence $r \notin (N : M)$. But P is prime and therefore $rm \in P$ implies that $m \in P$, that is $S_T(P) \subseteq P$. So we have the equality. $\square$

Proposition 2.7. *Let R be a ring and M be an R-module.*

(1) If $N \subseteq K$ are submodules of M such that $S_T^M(N)$ is a submodule of M, then $S_T^K(N)$ is a submodule of K.

(2) If $M = i \in I \oplus M_i$ and $S_T^M(i \in I \oplus N_i)$ is a submodule of M, where N_i is a submodule of M_i, then $S_T^{M_i}(N_i)$ is a submodule of M_i and $S_T^M(i \in I \oplus N_i) = i \in I \oplus S_T^{M_i}(N_i)$.

Proof. (1) Clearly $S_T^K(N)$ is the intersection of two submodules $S_T^M(N)$ and K of M and hence we have the result. (2) Let $x,\ y \in S_T^{M_j}(N_j)$. Then $x,\ y \in S_T^M(i \in I \oplus N_i)$ and hence there exists $r \in R \setminus T$ such that $r(x+y) \in i \in I \oplus N_i$, by hypothesis. Thus $r(x+y) \in (i \in I \oplus N_i) \cap M_j = N_j$ with $r \in R \setminus T$, that is $x + y \in S_T^{M_j}(N_j)$. $\square$

Example 2.8. *Let $R = M = \mathbf{Z}_4 \oplus \mathbf{Z}_4$, $M_1 = \mathbf{Z}_4 \oplus 0$, $M_2 = 0 \oplus \mathbf{Z}_4$, $N_1 = 2\mathbf{Z}_4 \oplus 0$, $N_2 = 0 \oplus 2\mathbf{Z}_4$ and consider T to be $\{(\bar{0}, \bar{0}), (\bar{0}, \bar{2}), (\bar{2}, \bar{0}), (\bar{2}, \bar{2})\}$. Then $S_T^{M_1}(N_1) = M_1$ and $S_T^{M_2}(N_2) = M_2$ are submodules of M_1 and M_2 but $S_T^M(N_1 \oplus N_2) = S_T^{M_1 \oplus M_2}(N_1 \oplus N_2)$ is not a submodule of M. Since $(\bar{3}, \bar{0})$, $(\bar{0}, \bar{3}) \in S_T^M(N_1 \oplus N_2)$ but $(\bar{3}, \bar{0}) + (\bar{0}, \bar{3}) = (\bar{3}, \bar{3}) \notin S_T^M(N_1 \oplus N_2)$.*

Proposition 2.9. *Let W be a multiplicatively closed subset of R, M be an R-module and N be a submodule of R such that $S_T^M(N)$ is a submodule of M. Then $S_T^{M_W}(N_W)$ is an R-submodule of M_W.*

Proof. $(S_T^M(N))_W$ is a submodule of M_W and we will show that it is equal to $(S_T^{M_W}(N_W))$. Let $\frac{m}{w} \in (S_T^M(N))_W$. Then there exists $r \in R \setminus T$ such that $rm \in N$ and hence $\frac{rm}{w} \in N_W$, that is, $\frac{m}{w} \in S_T^{M_W}(N_W)$. Now, let $\frac{m}{w} \in S_T^{M_W}(N_W)$ for some $m \in M$ and $w \in W$. Then there exists $r \in R \setminus T$ such that $r\frac{m}{w} \in N_W$. So, $rw'm \in N$ for some $w' \in W$ and hence $r\frac{m}{w} = r\frac{w'm}{w'w} \in N_W$. This means that $\frac{m}{w} \in (S_T^M(N))_W$. Hence $(S_T^M(N))_W = (S_T^{M_W}(N_W))$. $\square$

Proposition 2.10. *Let $f : R \to S$ be a ring homomorphism and M be an S-module. Consider M as an R-module with the scalar multiplication defined by $r.m = f(r)m$. For a S-submodule N of M the following hold:*

(1) If f is injective, then $S_T^M(N) \subseteq S_{f(T)}^M(N)$.

(2) If f is surjective, then $S_{f(T)}^M(N) \subseteq S_T^M(N)$.

Proof. (1) Let f be injective and $m \in S_T^M(N)$. Then $r.m \in N$ for some $r \in R \setminus T$ and since f is injective, $f(r) \notin f(T)$. Hence $m \in S_{f(T)}^M(N)$.

(2) Let f be surjective and $m \in S_{f(T)}^M(N)$. Then $sm \in N$ for some $s \in S \setminus f(T)$ and since f is surjective, $s = f(r)$ for some $r \in R \setminus T$. Hence $r.m = f(r)m = sm \in N$. Therefore $m \in S_T^M(N)$. $\square$

Let R be a ring. An ideal I of R is said to be irreducible if it cannot be written as the intersection of two ideals, both of which properly contain it.

Theorem 2.11. *Let R be a ring.*

(1) For every cyclic R-module M and arbitrary submodule N of M, $S_T(N)$ with $T = (N : M)$, is a submodule of M if and only if $S_T(T)$ is a (prime) ideal of R.

(2) Let T be an ideal of R. Then for every R-module M and for every submodule N of M, $S_T(N)$ is a submodule of M if and only if every ideal of R contained in T is an irreducible ideal of R.

Proof. (1) ($\Rightarrow$) Consider the cyclic R-module R and let I be an arbitrary ideal of R. Then $T = (I : R) = I$ and $S_T(I) = S_T(T)$ is an ideal of R. If I is a proper ideal of R, then $S_T(I) \neq R$ and hence, by Proposition 2.5, it is a prime ideal of R . ($\Leftarrow$) Let N be a submodule of M and $x_1, x_2 \in S_T(N)$. Then there exist $r_1, r_2 \in R \setminus T$, such that $r_1x_1, r_2x_2 \in N$. Since $S_T(T)$ is a prime ideal of R, $r_1r_2 \notin T$ and hence $r_1r_2(x_1 + x_2) \in N$ implies that $x_1 + x_2 \in S_T(N)$. Now, by Lemma 2.3, we have the result.

(2) ($\Leftarrow$) Let M be an R-module, N be a submodule of M. Assume that every ideal contained in T is an irreducible ideal of R. Let $m_1, m_2 \in S_T(N)$. Then there exist $r_1, r_2 \in R\setminus T$ such that $r_1m_1, r_2m_2 \in N$. If $Rr_1 \cap Rr_2 \subseteq T$, then $Rr_1 \subseteq Rr_2$ or $Rr_2 \subseteq Rr_1$. Let, for example, $Rr_1 \subseteq Rr_2$. Then $r_1(m_1 + m_2) = r_1m_1 + r_1m_2 = r_1m_1 + r'r_2m_2 \in N$ and hence $m_1 + m_2 \in S_T(N)$. If $Rr_1 \cap Rr_2 T$, choose $r \in Rr_1 \cap Rr_2 \setminus T$. Then there exist $r'_1, r'_2 \in R$ such that $r(m_1+m_2) = r'_1r_1m_1+r'_2r_2m_2 \in N$ and hence $m_1+m_2 \in S_T(N)$. ($\Rightarrow$) Let I, J be incomparable ideals of R such that $I \cap J \subseteq T$. We will show that there exists an R-module M and a submodule N of M such that $S_T(N)$ is not a submodule of M. Let $a \in I \setminus J$ and $b \in J \setminus I$, $M = \frac{R}{(a)} \oplus \frac{R}{(b)}$ and $N = 0$. Then $a(\bar{1}, \bar{0}) = (\bar{0}, \bar{0}) = b(\bar{0}, \bar{1})$ and hence $(\bar{1}, \bar{0})$, $(\bar{0}, \bar{1}) \in S_T(0)$. Now, it is clear that there is no element $r \in R \setminus T$ such that $r(\bar{1}, \bar{1}) = (\bar{0}, \bar{0})$ and hence $S_T(0)$ is not a submodule of M. □

Let R be a ring and M be an R-module. For a submodule N of M, let $s_T(N) = \{m \in M : rm \in N$ for some $r \in R \setminus T$ such that $rs \in T$ for some $s \in R$ implies that $s \in T\}$. It is clear that, $s_T(N)$ is an R-submodule of M for every submodule N of M.

Theorem 2.12. *Let R be a ring and T be a subset of R such that $Z(R) \subseteq T$.*

(1) $s_T(N) = N$ for every R-module and for every submodule N of that module if and only if R is a total quotient ring.

(2) $S_T(N) = N$ for every R-module and for every submodule N of that module if and only if R is a field.

Proof. (1) ($\Rightarrow$) Let M be an R-module, N be a submodule of M and $m \in s_T(N)$. Then there exists $r \in R \setminus T$ such that $rm \in N$. Since R is a total quotient ring and $r \notin Z(R)$, r is a unit and hence $m \in N$. ($\Leftarrow$) Let $r \in R$ be a regular element of R and consider the submodule Rr of the R-module R. Since $r.1 \in Rr$, we have $1 \in s_T(Rr) = Rr$ and hence $Rr = R$, that is r is a unit.

(2) ($\Rightarrow$) Let M be an R-module, N be a submodule of M and $m \in S_T(N)$. Then there exists $r \in R \setminus T$ such that $rm \in N$. Since R is a field, r is a unit and hence $m \in N$. ($\Leftarrow$) Let $r \in R$ be a non-zero element of R and consider the submodule Rr of the R-module R. Since $r.1 \in Rr$, we have $1 \in S_T(Rr) = Rr$ and hence $Rr = R$, that is r is a unit. $\square$

Theorem 2.13. *Let R be a ring. The following conditions are equivalent:*

(1) $s_{(N:M)}(N) = S_{(N:M)}(N)$ for every R-module M and its submodule N;

(2) For every R-module M and its submodule N, either $s_{(N:M)}(N) = M$ or $s_{(N:M)}(N)$ is a prime submodule of M.

Proof. (1) $\Rightarrow$ (2) Suppose that $s_{(N:M)}(N)$ be a proper submodule of M. Hence, by (1), $S_{(N:M)}(N)$ is a proper submodule of M. Therefore, by Proposition 2.5, $S_{(N:M)}(N)$ and hence $s_{(N:M)}(N)$ is a prime submodule of M.

(2) $\Rightarrow$ (1) Let M be an R-module and N be a proper submodule of M. By (2), $s_{(N:M)}(N)$ is a prime ideal of R. Now, it is clear that $s_{(N:M)}(N) = S_{(N:M)}(N)$. $\square$

3. $S_T(N)$ as a union of prime submodules

In this section, we consider the question of when for an R-module M, the set $S_T(M)$ is a union of prime submodules.

Let R be a ring and M be an R-module. It is known that if I is a maximal of $\{(0 : m) : 0 \neq m \in M\}$, then I is a prime ideal of R [1, p.4, Theorem 6]. Also, if M is a finitely generated nonzero module, then $Z(M)$ is the union of a finite number of prime ideals each of which is $(0 : m)$ for some nonzero element m of M [1, p.55, Theorem 80]. In this section, we will see that $S_T(N)$, for some submodule N of M with $S_T(N) \neq M$ of some finitely generated modules M over Noetherian rings has similar properties.

Theorem 3.1. *Let M be an R-module and L be a submodule of M such that $S_T(L) \neq M$ with $T = (L : M)$. Let $\Sigma = \{N : NM$ such that $N \subseteq \cup(L :_M r_\lambda)$ for some $\{r_\lambda\} \subseteq R\}$. Then a maximal element of Σ is prime.*

Proof. Let $N = \cup(L :_M r_\lambda)$ be a maximal element of Σ and let $rm \in N$ but $m \notin N$. Assume that $rr_\lambda \notin T$ for every $\lambda \in \Lambda$. So, for every $\lambda \in \Lambda$, $(L :_M r_\lambda) \subseteq (L :_M rr_\lambda)$ and hence $N_1 = \cup(L :_M rr_\lambda) \supseteq \cup(L :_M r_\lambda) = N$. Also, it is clear that N_1 is a submodule of M with $N_1 \subseteq S_T(L)$ and hence by maximality of N we have $N_1 = N$. Now $rm \in N$ implies that $rm \in (L :_M r_\lambda)$, for some $\lambda \in \Lambda$. Therefore $m \in (L :_M rr_\lambda) \subseteq N_1 = N$, a contradiction. This shows that there must exist a $\lambda \in \Lambda$ such that $rr_\lambda \in T$. But then $rM \subseteq (L :_M r_\lambda) \subseteq N$, that is $r \in (N : M)$ and hence N is prime. $\square$

Theorem 3.2. *Let M be an R-module and L be a submodule of M such that $S_T(L) \neq M$ with $T = (L : M)$. Then $S_T(L)$ is a union of prime submodules of M.*

Proof. For $m \in S_T(L)$, set $\Sigma_m = \{N : NM, m \in N \subseteq S_T(L)$ and $N = \cup(L : r_\lambda)$ for some $\{r_\lambda\} \subseteq R\}$. Suppose that $rm \in S_T(L)$ for some $r \in R \setminus T$. Then $m \in (L :_M r) \subseteq S_T(L)$ and hence $\Sigma_m \neq$. Partially order Σ_m by inclusion. By Zorn's Lemma, Σ_m has a maximal element N_m which is a prime submodule of M, by Theorem 3.1 and hence $S_T(L) = \cup_{m \in S_T(L)} N_m$, that is $S_T(L)$ is a union of prime submodules of M. $\square$

Theorem 3.3. *Let L ba submodule of M and $\Phi_L = \{(L :_M r) \subseteq M : r \in R \setminus T\}$. Then a maximal element of Φ_L is a prime submodule of M.*

Proof. Let $P = (L :_M r)$ be a maximal element of Φ_L. Let $r'm \in P$ and $m \notin P$. Then $rr'm \in L$ but $rm \notin L$. Since P is maximal and $P \subseteq (L :_M rr')$, either $P = (L :_M rr')$ or $(L :_M rr') = M$. If $P = (L :_M rr')$, then $m \in (L :_M rr') = P$, a contradiction. Thus $(L :_M rr') = M$, that is $r(r'M) = rr'M \subseteq L$. Therefore $r'M \subseteq (L :_M r) = P$. Thus $r' \in (P : M)$ and hence P is prime. $\square$

Theorem 3.4. *Let M be a Noetherian R-module . For a submodule L of M and any subset T of R with $(L : M) \subseteq T$, $S_T(L) = \cup\{P : P$ is a prime submodule of M, $P = (L :_M r)$ for some $r \in R \setminus T\}$.*

Proof. Let $m \in S_T(L)$ and so $rm \in L$ for some $r \in R \setminus T$. Then $m \in (L :_M r)M$, since $(L : M) \subseteq T$. Let $\Phi_{L,r} = \{(L :_M r') : r' \in R \setminus T, (L :_M r) \subseteq (L :_M r')\}$. Then, by hypothesis, $\Phi_{L,r}$ has a maximal element $(L :_M s)$ which is a prime submodule of M by Theorem 3.3. Therefore, $m \in (L :_M r') \subseteq (L :_M s)$ and hence $S_T(L)$ is a union of prime submodules of the form $(L :_M s)$ for some $s \in R \setminus T$. $\square$

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