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On a new class of generalized difference sequence spaces of fractional order defined by modulus function

T. Yaying* orcid.org/0000-0003-3435-8417

*Dera Natung Govt. College, Dept. of Mathematics, Itanagar, AR, India.
tajayaying20@gmail.com

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Abstract:

Recently Baliarsingh and Dutta [11, 12] introduced the fractional difference operator Δ^α, defined by $\Delta^\alpha(x_k) = \sum_{i=0}^{\infty}(-1)^i \frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha-i+1)} x_{k+i}$ and defined new classes of generalized difference sequence spaces of fractional order $X(\Gamma, \Delta^\alpha, u)$ where $X = \{\ell_\infty, c, c_0\}$. More recently, Kadak [21] studied strongly Cesàro and statistical difference sequence space of fractional order involving lacunary sequences using the fractional difference operator Δ_v^α defined by $\Delta_v^\alpha(x_k) = \sum_{i=0}^{\infty}(-1)^i \frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha-i+1)} v_{k+i}x_{k+i}$, where $v = (v_k)$ is any fixed sequence of positive real or complex numbers.

Following Baliarsingh and Dutta [11, 12] and Kadak [21], we introduce paranormed difference sequence spaces $N_\theta\left(\Delta_v^\alpha, f, p\right)$ and $S_\theta\left(\Delta_v^\alpha, f, p\right)$ of fractional order involving lacunary sequence, θ and modulus function, f. We investigate topological structures of these spaces and examine various inclusion relations.

Keywords: Difference operator Δ_v^α; Paranormed sequence; Lacunary sequence;

MSC (2000): 46A45, 40A35, 46A80.

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1. Introduction

Throughout ω will denote the space of all real valued sequences and any subspace of ω is called sequence space. ℓ_∞, c and c_0 will denote the spaces of bounded, convergent and null sequences, respectively. These spaces are Banach spaces normed by $||x||_\infty = \sup_k |x_k|$.

The notion of difference sequence spaces was first determined by Kizmaz [1]. Later on, the notion was generalized by Et and Colak [2]. Also Et and Esi [3] generalized the sequence spaces to the sequence spaces as given below:

Let m be a non negative integer, then

$$\Delta_v^m(X) = \{x = (x_k) : \Delta_v^m(x) \in X\} \text{ for } X \in \{\ell_\infty, c, c_0\},$$

where $\Delta_v^m(x) = (\Delta_v^{m-1}(x_k) - \Delta_v^{m-1}(x_{k+1}))$, $\Delta_v^0(x) = (v_k x_k)$ and $\Delta_v^m(x_k) = \sum_{i=0}^m (-1)^i \binom{m}{i} v_{k+i} x_{k+i}$.

These spaces are Banach spaces with norm defined by

$$||x||_\Delta = \sum_{i=1}^m |v_i x_i| + \sup_k |\Delta_v^m(x_k)|.$$

Furthermore, generalized difference sequence space was studied by Et and Basarir [4], Malkowsky and Parashar [5], Et and Tripathy [22], Colak [6], and many others.

The notion of statistical convergence was independently introduced by Fast [14] and Schoenberg [15]. The concept lies on the asymptotic density of the subset E of natural number $\mathbf{N}$. A subset E of $\mathbf{N}$ is said to have asymptotic density $\delta(E)$, if $\delta(E) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^n \chi_E(k)$ exists, where χ_E is the characteristic function of E.

A sequence (x_n) is said to be statistically convergent to L if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} |\{k \in \mathbf{N} : |x_k - L| \geq \varepsilon\}| = 0,$$

where $|E|$ denotes the cardinality of the set E. In this case, we write $S - \lim x_k = L$ or $x_k \to L(S)$.

Let $\theta = (k_r)$ be the sequence of positive integers such that $k_r = 0$, $0 < k_r < k_{r+1}$ and $h_r = k_r - k_{r-1} \to 0$ as $r \to \infty$. Then θ is called lacunary sequence. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ will be denoted by q_r. Freedman et. Al [34] introduced the sequence space N_θ given by

$$N_\theta = \left\{ (x_k) \in \omega : h_r^{-1} \sum_{k \in I_r} |x_k - L| \to 0, \text{ for some } L \right\};$$

and showed that the space N_θ is a BK space with the norm defined by

$$\|x\|_\theta = \sup_r \left(h_r^{-1} \sum_{I_r} |x_k| \right).$$

A modulus is a function $f : [0, \infty) \to [0, \infty)$ such that

(i) $f(x) = 0$ if and only if $x = 0$;

(ii) $f(x + y) \leq f(x) + f(y)$;

(iii) f is increasing;

(iv) f is continuous from right at 0.

The notion of modulus function was introduced by Nakano [17] followed by Ruckle [19], Maddox [18], Tripathy and Chandra [25] and many others, to construct various sequence spaces. The following inequality (see [20]) will be used throughout in this article:

$$|a_k + b_k|^{p_k} \leq C \left(|a_k|^{p_k} + |b_k|^{p_k} \right);$$

where $a_k, b_k \in \mathbf{C}$, $0 < p_k \leq \sup p_k = H$, $C = \max\left(1, 2^{H-1}\right)$.

Proposition 1.1. *[32] Let f be a modulus function and let $0 < \delta < 1$. Then for each $x \geq \delta$ we have $f(x) \leq 2f(1)\delta^{-1}x$.*

2. Fractional Difference Operator and Generalized Difference Sequence Space of Fractional Order

Let $\Gamma(m)$ be the Gamma function of a real number m and $m \notin \{0, -1, -2, \ldots\}$. By definition, it can be expressed as an improper integral

$$\Gamma(m) = \int_0^\infty e^{-x} x^{m-1} dx.$$

Recently, Baliarsingh and Dutta [11, 12] have introduced the generalized difference operator Δ^α, for a positive fraction α as follows:

$$\Delta^\alpha(x_k) = \sum_{i=0}^{\infty}(-1)^i \frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha-i+1)} x_{k+i}.$$

In particular, we have,

(i) $\Delta^{\frac{1}{2}}(x_k) = x_k - \frac{1}{2}x_{k+1} - \frac{1}{8}x_{k+2} - \frac{1}{16}x_{k+3} - \frac{5}{128}x_{k+4} - \dots$

(ii) $\Delta^{\frac{-1}{2}}(x_k) = x_k + \frac{1}{2}x_{k+1} + \frac{3}{8}x_{k+2} + \frac{5}{16}x_{k+3} + \frac{35}{128}x_{k+4} + \dots$

(iii) $\Delta^{\frac{2}{3}}(x_k) = x_k - \frac{2}{3}x_{k+1} - \frac{1}{9}x_{k+2} - \frac{4}{81}x_{k+3} - \frac{7}{243}x_{k+4} - \dots$

Baliarsingh [10] defined the spaces $X(\Gamma, \Delta^\alpha, u)$ for $X \in \{\ell_\infty, c, c_0\}$ using the fractional difference operator Δ^α and studied their topological properties and obtained their α, β, and γ duals.

The studies on generalized difference sequence spaces of fractional order was extended by Baliarsingh and Dutta [12, 29], Kadak and Baliarsingh [13], Serkan and Osman [30], Hasan Furkan [33], Kadak [21] and others.

Kadak in [21] determined a new classes of fractional difference sequence spaces $\Delta_v^\alpha(X)$ as follows:

$$\Delta_v^\alpha(X) = \{x = (x_k) \in \omega : \Delta_v^\alpha(X) \in X\},$$

where $\Delta_v^\alpha(x_k) = \sum_{i=0}^{\infty}(-1)^i \frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha-i+1)} v_{k+i}x_{k+i}$ and $v = (v_k)$ is a sequence of positive real numbers. Using the fractional difference operator Δ_v^α, he defined strongly Cesàro and statistical difference sequence spaces of fractional order involving lacunary sequence, θ and arbitrary sequence $p = (p_r)$ of positive real numbers.

Theorem 2.1. *[10]*

(a) For proper fraction, $\Delta^\alpha : \omega \to \omega$ *is a linear operator.*

(b) For $\alpha, \beta > 0$, $\Delta^\alpha(\Delta^\beta(x_k)) = \Delta^{\alpha+\beta}(x_k)$ *and* $\Delta^\alpha(\Delta^{-\alpha}(x_k)) = x_k$.

The main objective of this article is to introduce generalized paranormed difference sequence spaces $N_\theta(\Delta_v^\alpha, f, p)$ and $S_\theta(\Delta_v^\alpha, f, p)$ of fractional order involving lacunary sequence, θ and modulus function, f and to investigate topological structures of these spaces and examine various inclusion relations.

3. Main Results

By using the fractional difference operator Δ_v^α, we introduce some new generalized difference sequence spaces $N_\theta(\Delta_v^\alpha, f, p)$, $N_\theta^0(\Delta_v^\alpha, f, p)$ and $N_\theta^\infty(\Delta_v^\alpha, f, p)$ involving lacunary sequence, θ and modulus function, f as follows:

$$N_\theta(\Delta_v^\alpha, f, p) = \left\{ x = (x_k) \in \omega : \lim_{r\to\infty} h_r^{-1} \sum_{k\in I_r} f\left(|\Delta_v^\alpha x_k - L|\right)^{p_k} = 0, \text{ for some } L \right\};$$

$$N_\theta^0(\Delta_v^\alpha, f, p) = \left\{ x = (x_k) \in \omega : \lim_{r\to\infty} h_r^{-1} \sum_{k\in I_r} f\left(|\Delta_v^\alpha x_k|\right)^{p_k} = 0 \right\};$$

$$N_\theta^\infty(\Delta_v^\alpha, f, p) = \left\{ x = (x_k) \in \omega : \lim_{r\to\infty} h_r^{-1} \sum_{k\in I_r} f\left(|\Delta_v^\alpha x_k|\right)^{p_k} < \infty \right\};$$

where $\Delta_v^\alpha(x_k) = \sum_{i=0}^{\infty}(-1)^i \frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha-i+1)} v_{k+i} x_{k+i}$ and $v = (v_k)$ is a fixed sequence of positive real numbers.

Theorem 3.1. *The sequence spaces $N_\theta(\Delta_v^\alpha, f, p)$, $N_\theta^0(\Delta_v^\alpha, f, p)$ and $N_\theta^\infty(\Delta_v^\alpha, f, p)$ are linear spaces.*

Proof. We shall prove for $N_\theta^0(\Delta_v^\alpha, f, p)$. Let $x, y \in N_\theta^0(\Delta_v^\alpha, f, p)$ and α' and β' be scalars. Then there exist positive numbers $M_{\alpha'}$ and $N_{\beta'}$ such that $|\alpha'| \leq M_{\alpha'}$ and $|\beta'| \leq N_{\beta'}$. Since f is subadditive and Δ_v^α is linear,
$h_r^{-1} \sum_{k\in I_r} f\left(|\Delta_v^\alpha(\alpha' x_k + \beta' y_k|\right)^{p_k} \leq h_r^{-1} \sum_{k\in I_r} \left[f\left(|\alpha'|\,|\Delta_v^\alpha x_k|\right) + f\left(|\beta'|\,|\Delta_v^\alpha y_k|\right)\right]^{p_k}$
$\leq C\left(M_{\alpha'}\right)^H h_r^{-1} \sum_{k\in I_r} f\left(|\Delta_v^\alpha x_k|\right)^{p_k} + C\left(N_{\beta'}\right)^H h_r^{-1} \sum_{k\in I_r} f\left(|\Delta_v^\alpha y_k|\right)^{p_k}$
$\to 0$ as $r \to \infty$. This proves the linearity of $N_\theta^0(\Delta_v^\alpha, f, p)$. □

Theorem 3.2. *$N_\theta^0(\Delta_v^\alpha, f, p)$ is a paranormed sequence space paranormed by*

$$g(x) = \sup_r \left(h_r^{-1} \sum_{k\in I_r} f\left(|\Delta_v^\alpha x_k|\right)^{p_k} \right)^{\frac{1}{M}}.$$

Proof. Clearly $g(\theta) = 0$ and $g(x) = g(-x)$ for all $x \in N_\theta^0(\Delta_v^\alpha, f, p)$. Using the linearity of Δ_v^α, definition of f and Minkowski's inequality, it is not difficult to show that $g(x + y) \leq g(x) + g(y)$, for any two sequences $x, y \in N_\theta^0(\Delta_v^\alpha, f, p)$.

It remains to show the continuity of the scalar multiplication. Let λ be any scalar. By definition of modulus f, we have

$$g(\lambda x) = \sup_r \left(h_r^{-1} \sum_{k \in I_r} f\left(|\Delta_v^\alpha \lambda x_k|\right)^{p_k} \right)^{\frac{1}{M}} \leq N_\lambda^{\frac{H}{M}} g(x),$$

where N_λ is a positive number such that $|\lambda| \leq N_\lambda$ and $H = \sup p_k$. Now, let $\lambda \to 0$ and $x = (x_k)$ be fixed with $g(x) \neq 0$, then for $|\lambda| < 1$,

$$h_r^{-1} \sum_{k \in I_r} f\left(|\Delta_v^\alpha \lambda x_k|\right)^{p_k} < \varepsilon, \text{ for } i > i_0.$$

Also, for $1 < i < i_0$, taking λ small enough, continuity of f implies that

$$h_r^{-1} \sum_{k \in I_r} f\left(|\Delta_v^\alpha \lambda x_k|\right)^{p_k} < \varepsilon.$$

Thus, $g(\lambda x) \to 0$ as $\lambda \to 0$. This completes the proof.

Theorem 3.3. *Let f be a modulus function, then*

$$N_\theta^0(\Delta_v^\alpha, f, p) \subset N_\theta(\Delta_v^\alpha, f, p) \subset N_\theta^\infty(\Delta_v^\alpha, f, p).$$

Proof. The first inclusion is obvious. We provide the proof of the second inclusion.
Let $x \in N_\theta(\Delta_v^\alpha, f, p)$. By definition of f, we have, $\mathrm{h}_r^{-1} \sum_{k \in I_r} f\left(|\Delta_v^\alpha x_k|\right)^{p_k} = h_r^{-1} \sum_{k \in I_r} f\left(|\Delta_v^\alpha x_k - L + L|\right)^{p_k}$
$\leq C h_r^{-1} \sum_{k \in I_r} f\left(|\Delta_v^\alpha x_k - L|\right)^{p_k} + C h_r^{-1} \sum_{k \in I_r} f\left(|L|\right)^{p_k}$.

Now, there exist a positive integer K_L such that $|L| \leq K_L$. Hence, we have,

$$h_r^{-1} \sum_{k \in I_r} f\left(|\Delta_v^\alpha x_k|\right)^{p_k} \leq C h_r^{-1} \sum_{k \in I_r} f\left(|\Delta_v^\alpha x_k - L|\right)^{p_k} + C\left(K_L f(1)\right)^H.$$

This proves the result.

Theorem 3.4. *If f, f_1, f_2 be modulus functions and $X \in \{N_\theta, N_\theta^0, N_\theta^\infty\}$, then*

(i) $X(\Delta_v^\alpha, f, p) \subset X(\Delta_v^\alpha, f \cdot f_1, p)$,

(ii) $X(\Delta_v^\alpha, f_1, p) \cap X(\Delta_v^\alpha, f_2, p) \subset X(\Delta_v^\alpha, f_1 + f_2, p)$.

Proof.

(i) We shall prove for $N_\theta^0(\Delta_v^\alpha, f, p)$. Let $\varepsilon > 0$ and choose $0 < \delta < 1$ such that $f(t) < \varepsilon$ for $0 \leq t \leq \delta$. We write $y_k = f_1(|\Delta_v^\alpha x_k|)$ and consider

$$\sum_{k \in I_r} f(y_k)^{p_k} = \sum_1 f(y_k)^{p_k} + \sum_2 f(y_k)^{p_k}$$

where the first summation runs over $y_k \leq \delta$ and the second summation runs over $y_k > \delta$. Since f is continuous, we have

$$\sum_1 f(y_k)^{p_k} < h_r \varepsilon^H \tag{3.1}$$

Also,

$$y_k < \frac{y_k}{\delta} \leq 1 + \frac{y_k}{\delta}.$$

Hence, by using Proposition 1.1, we can write,

$$h_r^{-1} \sum_2 f(y_k)^{p_k} \leq \max(1, (2f(1)\delta^{-1})^H) h_r^{-1} \sum_{k \in I_r} y_k \tag{3.2}$$

Using equations 3.1 and 3.2, we get $N_\theta^0(\Delta_v^\alpha, f, p) \subset N_\theta^0(\Delta_v^\alpha, f \cdot f_1, p)$

(ii) The proof of (ii) follows from the inequality

$$(f_1 + f_2)(|\Delta_v^\alpha x_k|)^{p_k} \leq C f_1(|\Delta_v^\alpha x_k|)^{p_k} + C f_2(|\Delta_v^\alpha x_k|)^{p_k}$$

The following result is an immediate consequence of Theorem 3.4 (i).

Corollary 3.5. *Let f be a modulus function. Then $X(\Delta_v^\alpha, p) \subset X(\Delta_v^\alpha, f, p)$ where $X \in \{N_\theta, N_\theta^0, N_\theta^\infty\}$.*

Theorem 3.6. *Let $0 < p_k < q_k$ and $\left(\frac{q_k}{p_k}\right)$ be bounded then $X(\Delta_v^\alpha, f, q) \subset X(\Delta_v^\alpha, f, p)$.*

Proof. The proof of the theorem is easy and hence omitted.

4. Lacunary statistical convergence of fractional order defined by modulus function

In this section we introduce generalized lacunary statistical convergence of fractional order defined by a modulus function as follows:

$$S_\theta(\Delta_v^\alpha, f, p) = \Big\{ x = (x_k) \in \omega : \lim_{r\to\infty} h_r^{-1} \left| \{ k \in I_r : f\left(|\Delta_v^\alpha x_k - L| \right)^{p_k} \geq \varepsilon \} \right| = 0,$$

$$\text{for some } L \Big\}.$$

When $p = (p_k) = 1$, we shall denote $S_\theta(\Delta_v^\alpha, f, p)$ by $S_\theta(\Delta_v^\alpha, f)$.

Note that when $f(x) = x$, $p = (p_k) = 1$, then $S_\theta(\Delta_v^\alpha, f, p)$ reduces to $S_\theta(\Delta_v^\alpha)$ as studied by Kadak [21]. When $f(x) = x$, $p = (p_k) = 1$ and $\alpha = m \in \mathbf{N}$, then $S_\theta(\Delta_v^\alpha, f, p)$ reduces to $S_\theta(\Delta_v^m)$ as studied by Et [28]. The class of lacunary convergence has been studied from different aspects by Fridy and Orhan [31], Tripathy and Baruah [24], Tripathy and Mahanta [23], Tripathy and Dutta [26], Tripathy et al. [27] and many others.

Theorem 4.1. *Let θ be a lacunary sequence. Then $S(\Delta_v^\alpha, f) \subset S_\theta(\Delta_v^\alpha, f)$, if* $\liminf q_r > 1$.

Proof. Let $\liminf q_r > 1$, then there exist a $\delta > 0$ such that $1 + \delta \leq q_r$, for sufficiently large r. Since $h_r = k_r - k_{r-1}$, which implies that $\frac{h_r}{k_r} \geq \frac{\delta}{1+\delta}$.

Let $x \in S(\Delta_v^\alpha, f)$. Then for $\varepsilon > 0$,

$$\begin{aligned} \frac{1}{k_r} \left| \{ k \leq k_r : f\left(|\Delta_v^\alpha x_k - L| \right) \geq \varepsilon \} \right| &\geq \frac{1}{k_r} \left| \{ k \in I_r : f\left(|\Delta_v^\alpha x_k - L| \right) \geq \varepsilon \} \right| \\ &\geq \frac{\delta}{1+\delta} h_r^{-1} \left| \{ k \in I_r : f\left(|\Delta_v^\alpha x_k - L| \right) \geq \varepsilon \} \right|; \end{aligned}$$

This esthablishes the result.

Theorem 4.2. *Let θ be a lacunary sequence. Then $S_\theta(\Delta_v^\alpha, f) \subset S(\Delta_v^\alpha, f)$, if* $\limsup q_r < \infty$.

Proof. Let $\limsup q_r < \infty$, then there is a $K > 0$ such that $q_r < K$, for all r. Let $x \in S_\theta(\Delta_v^\alpha, f)$ and let $\tau_r = \left| \{ k \in I_r : f\left(|\Delta_v^\alpha x_k - L| \right) \geq \varepsilon \} \right|$.

Now by definition, for $\varepsilon > 0$ there is an integer r_0 such that

$$h_r^{-1} \tau_r < \varepsilon \text{ for all } r > r_0. \tag{4.1}$$

Now let $\gamma = \max\{\tau_r : 1 \le r \le r_0\}$ and let n be any integer satisfying $k_{r-1} < n \le k_r$; then we can write,

$$\begin{aligned}
\frac{1}{n}\left|\{k \le n : f(|\Delta_v^\alpha x_k - L|) \ge \varepsilon\}\right| &\le \frac{1}{k_{r-1}}\left|\{k \le k_r : f(|\Delta_v^\alpha x_k - L|) \ge \varepsilon\}\right| \\
&= \frac{1}{k_{r-1}}\{\tau_1 + \tau_2 + \ldots + \tau_{r_0} + \tau_{r_0+1} + \ldots + \tau_r\} \\
&\le \frac{\gamma}{k_{r-1}} r_0 + \frac{1}{k_{r-1}}\left\{h_{r_0+1}\frac{\tau_{r_0+1}}{h_{r_0+1}} + \ldots + h_r\frac{\tau_r}{h_r}\right\} \\
&\le \frac{\gamma}{k_{r-1}} r_0 + \frac{1}{k_{r-1}}\left(\sup_{r<r_0}\frac{\tau_r}{h_r}\right)(h_{r_0+1} + \ldots + h_r) \\
&\le \frac{\gamma}{k_{r-1}} r_0 + \varepsilon\frac{k_r - k_{r_0}}{k_{r-1}} \quad \text{(using equation 4.1)} \\
&\le \frac{\gamma}{k_{r-1}} r_0 + \varepsilon q_r \\
&\le \frac{\gamma}{k_{r-1}} r_0 + \varepsilon K.
\end{aligned}$$

This establishes the result.

Following result is the direct consequence of theorems 4.1 and 4.2.

Corollary 4.3. *Let θ be a lacunary sequence. Then $S(\Delta_v^\alpha, f) = S_\theta(\Delta_v^\alpha, f)$, if $1 < \liminf q_r \le \limsup q_r < \infty$.*

Theorem 4.4. *Let f be a modulus function and $H = \sup_k p_k$. Then $N_\theta(\Delta_v^\alpha, f, p) \subset S_\theta(\Delta_v^\alpha)$.*

Proof. Let $x \in N_\theta(\Delta_v^\alpha, f, p)$ and $\varepsilon > 0$ be given. Then,

$$\begin{aligned}
h_r^{-1}\sum_{k\in I_r} f(|\Delta_v^\alpha x_k - L|)^{p_k} &= h_r^{-1}\sum_{\substack{k\in I_r \\ |\Delta_v^\alpha x_k - L| \ge \varepsilon}} f(|\Delta_v^\alpha x_k - L|)^{p_k} \\
&\quad + h_r^{-1}\sum_{\substack{k\in I_r \\ |\Delta_v^\alpha x_k - L| < \varepsilon}} f(|\Delta_v^\alpha x_k - L|)^{p_k} \\
&\ge h_r^{-1}\sum_{\substack{k\in I_r \\ |\Delta_v^\alpha x_k - L| \ge \varepsilon}} f(|\Delta_v^\alpha x_k - L|)^{p_k} \\
&\ge h_r^{-1}\sum_{k\in I_r} f(\varepsilon)^{p_k} \\
&\ge h_r^{-1}\sum_{k\in I_r} \min\left(f(\varepsilon)^{\inf p_k}, f(\varepsilon)^H\right) \\
&\ge h_r^{-1}\left|\{k \in I_r : |\Delta_v^\alpha x_k - L| \ge \varepsilon\}\right| \min\left(f(\varepsilon)^{\inf p_k}, f(\varepsilon)^H\right)
\end{aligned}$$

Taking the limit as $r \to \infty$, we have

$$\lim_{r\to\infty} h_r^{-1} \left|\{k \in I_r : |\Delta_v^\alpha x_k - L| \geq \varepsilon\}\right|$$
$$\leq \frac{1}{\min\left(f(\varepsilon)^{\inf p_k}, f(\varepsilon)^H\right)} \lim_{r\to\infty} h_r^{-1} \sum_{k\in I_r} f\left(|\Delta_v^\alpha x_k - L|\right)^{p_k} = 0.$$

This establishes the result.

Theorem 4.5. *Let f be bounded and $0 < h = \inf p_k \leq p_k \leq \sup p_k = H < \infty$. Then $S_\theta(\Delta_v^\alpha) \subset N_\theta(\Delta_v^\alpha, f, p)$.*

Proof. Since f is bounded, there exists some K such that $f(x) < K$ for all $x \geq 0$. Now,

$$\begin{aligned} h_r^{-1} \sum_{k\in I_r} f\left(|\Delta_v^\alpha x_k - L|\right)^{p_k} &= h_r^{-1} \sum_{\substack{k\in I_r \\ |\Delta_v^\alpha x_k - L| \geq \varepsilon}} f\left(|\Delta_v^\alpha x_k - L|\right)^{p_k} \\ &\quad + h_r^{-1} \sum_{\substack{k\in I_r \\ |\Delta_v^\alpha x_k - L| < \varepsilon}} f\left(|\Delta_v^\alpha x_k - L|\right)^{p_k} \\ &\leq h_r^{-1} \sum_{k\in I_r} \max\left(K_h, K_H\right) + h_r^{-1} \sum_{k\in I_r} f(\varepsilon)^{p_k} \\ &\leq \max\left(K_h, K_H\right) h_r^{-1} \left|\{k \in I_r : |\Delta_v^\alpha x_k - L| \geq \varepsilon\}\right| + \max\left(f(\varepsilon)^h, f(\varepsilon)^H\right) \end{aligned}$$

Hence $x \in N_\theta(\Delta_v^\alpha, f, p)$. The following result is an immediate consequence of the Theorem 4.4 and Theorem 4.5.

Corollary 4.6. *Let f be bounded and $0 < h = \inf p_k \leq p_k \leq \sup p_k = H < \infty$. Then $S_\theta(\Delta_v^\alpha) = N_\theta(\Delta_v^\alpha, f, p)$.*

5. Conclusion

Fractional order difference sequence space has been an active field of research during the recent times. Many authors have introduced different classes of difference sequence spaces of fractional order, obtained their α, β and γ duals and matrix transformations. In this article we tend to generalize the findings of the previous authors using modulus function. We expect that the introduced notions and the results might be a reference for further studies in this field. For further studies one can investigate and generalize this results using multiplier sequences, sequence of modulus functions, etc.

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