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Appendix A

Results on semigroups

We recall here some results on generation and regularity properties of a strongly continuous semigroup by a second order, possibly degenerated, differential operator.

We study essentially two cases: differential operators acting on a space of "continuous functions" and differential operators acting on a L^2 space.

These results are perhaps not new but it was impossible to find a full proof of all the properties we need of such semigroups in a few basic and accessible references. Proofs essentially follow adapting arguments from [83]. The details have been spelled out in [43].

We denote by $C_b^n(\mathbb{R}^d;\mathbb{R})$ be the Banach space of bounded continuous functions with bounded continuous partial derivatives up to the order n endowed with the norm

$$\|f\|_{C_b^n} = \left\| \left(\sum_{|\alpha|\leq n} |\partial^\alpha f|^2 \right)^{1/2} \right\|_\infty$$

where $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$ is a multindex and $|\alpha| = \alpha_1 + \dots + \alpha_d$, $\partial^\alpha f = \partial_1^{\alpha_1} \dots \partial_d^{\alpha_d} f$.

The closed subspace of continuous functions vanishing at infinity with continuous partial derivatives up to the order n vanishing at infinity will be denoted by $C_0^n(\mathbb{R}^d;\mathbb{R})$.

The closed subspace of continuous functions having a limit as $|x|$ tends to infinity with continuous partial derivatives up to the order n having a limit as $|x|$ tends to infinity will be denoted by $C_\ell^n(\mathbb{R}^d;\mathbb{R})$.

Let φ_γ ($\gamma > 0$) be the functions

$$\varphi_\gamma : \mathbb{R}^d \to \mathbb{R}, \qquad \varphi_\gamma(x) = \left(1 + |x|^2\right)^{\gamma/2}.$$

We denote by $C_{-\gamma}^n(\mathbb{R}^d;\mathbb{R})$ be the Banach space of bounded continuous functions with bounded continuous partial derivatives up to the order n such that $\varphi_\gamma \partial^\alpha f$

vanishes as $|x|$ goes to infinity endowed with the norm

$$\|f\|_{C^n_{-\gamma}} = \left\| \varphi_\gamma \left(\sum_{|\alpha| \le n} |\partial^\alpha f|^2 \right)^{1/2} \right\|_\infty$$

The linear manifold of infinitely differentiable functions with compact support will be denoted $C^\infty_c(I\!R^d; I\!R)$.

A similar notation will be used for spaces of continuous complex-valued functions.

The following are the main results proved in the appendix

Theorem A.1 *Let $n \geq 2$ be a fixed integer. For $x \in I\!R^d$, let $(a_{jk}(x))_{1 \le j,k \le d}$ be a $d \times d$ symmetric non-negative matrix, $(b_j)_{1 \le j \le d}$ a d-dimensional vector and $c(x)$ a real number. Suppose that:*

1. *the functions a_{jk} belong to $C^n(I\!R^d; I\!R)$ and their second order partial derivatives belong to $C^{n-2}_b(I\!R^d; I\!R)$ for every $j, k \in \{1, \ldots, d\}$.*
2. *the functions b_j are differentiable and their first order partial derivatives belong to $C^{n-1}_b(I\!R^d; I\!R)$ for every $j \in \{1, \ldots, d\}$.*
3. *the function c belongs to $C^n_b(I\!R^d; I\!R)$.*

Then, for every $\gamma \geq 0$, the closure in $C^0_{-\gamma}(I\!R^d; I\!R)$ of the operator

$$Af = \frac{1}{2} \sum_{j,k=1}^{d} a_{jk} \partial_j \partial_k f + \sum_{j=1}^{d} b_j \partial_j f + cf \tag{A.1}$$

defined on the domain $C^\infty_c(I\!R^d; I\!R)$ is the infinitesimal generator of a strongly continuous semigroup $(T(t))_{t \ge 0}$ on $C^0_{-\gamma}(I\!R^d; I\!R)$. Moreover

1. *for every $t \geq 0$ and every non-negative $f \in C^0_{-\gamma}(I\!R^d; I\!R)$ the function $T(t)f$ is also non-negative,*
2. *for every $t \geq 0$ and every k with $2 \le k \le n$ the linear manifold $C^k_{-\gamma}(I\!R^d; I\!R)$ is invariant under $T(t)$ and the resolvent operators $R(\lambda; A)$ $(\lambda > 0)$,*
3. *if $\gamma = 0$ the operators $(T(t))_{t \ge 0}$ are contractive if and only if $c \le 0$.*

Corollary A.2 *Under the assumptions of Theorem A.1 suppose $c = 0$. Let D be the linear manifold generated by $C^\infty_c(I\!R^d; I\!R)$ and constant functions. Then the closure in $C^0_\ell(I\!R^d; I\!R)$ of the operator A defined on the domain D by (A.1) is the infinitesimal generator of a strongly continuous contraction semigroup $(T(t))_{t \ge 0}$ on $C^0_\ell(I\!R^d; I\!R)$. Moreover the semigroup $(T(t))_{t \ge 0}$ is Markov (i.e. $T(t)1 = 1$).*

Theorem A.3 *Suppose that the hypotheses of Theorem A.1 hold. Then the closure in* $L^2(\mathbb{R}^d;\mathbb{C})$ *of the operator*

$$Gf = \frac{1}{2}\sum_{j,k=1}^{d} a_{jk}\partial_j\partial_k u + \sum_{j=1}^{d} b_j\partial_j u + cu \tag{A.2}$$

defined on the domain $C_c^\infty(\mathbb{R}^d;\mathbb{C})$ *is the infinitesimal generator of a strongly continuous semigroup* $(P(t))_{t\geq 0}$ *on* $L^2(\mathbb{R}^d;\mathbb{C})$. *Moreover the semigroup* $(P(t))_{t\geq 0}$ *is contraction if and only if* G *is dissipative i.e.*

$$2\Re e\langle u, Gu\rangle \leq 0$$

for every $u \in C_c^\infty(\mathbb{R}^d;\mathbb{C})$.

Remark. Notice that, under the assumptions of Theorem A.1, $P(t)u = T(t)u$ for every $u \in L^2(\mathbb{R}^d;\mathbb{C}) \cap C_0^0(\mathbb{R}^d;\mathbb{R})$ and $t \geq 0$. In fact $P(t)$ and $T(t)$ are bounded operators coinciding on the dense (in $L^2(\mathbb{R}^d;\mathbb{C})$ and $C_0^0(\mathbb{R}^d;\mathbb{R})$) subset $C_c^\infty(\mathbb{R}^d;\mathbb{R})$.

We refer to [43] for the proofs of the above results.

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