Proyecciones Journal of Mathematics
Vol. 37, N^o 1, pp. 153-169, March 2018.
Universidad Católica del Norte
Antofagasta - Chile

Sequentially spaces and the finest locally K-convex of topologies having the same onvergent sequences

A. El Amrani
University Sidi Mohamed Ben Abdellah, Morocco
Received : June 2017. Accepted : November 2017

Abstract

The present paper is concerned with the concept of sequentially topologies in non-archimedean analysis. We give characterizations of such topologies.

Keywords : *Non-archimedean topological space, sequentially spaces, convergent sequence in non-archimedean space.*

MSC2010 : *11F85 - 46A03 - 46A45.*

1. Introduction

In 1962 Venkataramen, in [19], posed the following problem:
Characterize "the class of topological spaces which can be specified completely by the knowledge of their convergent sequences".
Several authors then agreed to provide a solution, based on the concept of sequential spaces, in particular: In [9] and [10] Franklin gave some properties of sequential spaces, examples, and a relationship with the Frechet spaces; after Snipes in [17], has studied a new class of spaces called $T-$ sequential space and relationships with sequential spaces; in [2], Boone and Siwiec gave a characterization of sequential spaces by sequential quotient mappings; in [4], Cueva and Vinagre have studied the $\mathbf{K}-c-$Sequential spaces and the $\mathbf{K}-s-$bornological spaces and adapted the results established by Snipes using linear mappings; thereafter Katsaras and Benekas, in [13], starting with a topological vector space $(t.v.s.)$ (E,τ), have built up, the finest of topologies on E having the same convergent sequences as τ; and the thinnest of topologies on E having the same precompact as τ; using the concept of String (this study is a generalization of the study led by Weeb [21], on 1968, in case of locally convex spaces $l.c.s.$); in [8], Ferrer, Morales and Ruiz, have reproduced previous work by introducing the concept of maximal sequentially topology. Goreham, in [11], has conducted a study linking sequentiality and countable subsets in a topological space by considering the five classes of spaces: spaces of countable first case, sequential spaces, Frechet spaces, spaces of "C.T." type and perfect spaces.

In this work, we will study, in the non-archimedean $(n.a)$ case, for a locally $\mathbf{K}-$convex space E the finest sequential locally $\mathbf{K}-$convex topology on E having the same convergent sequences as the original topology.

2. Preliminaries

Throughout this paper $\mathbf{K}$ is a $(n.a)$ non trivially valued complete field with the valuation $|.|$, and the valuation ring is $B\left(0{,}1\right) : = \{\lambda \in \mathbf{K} : |\lambda| \leq 1\}$. There exists $\varrho \in \mathbf{R}$ such that $\varrho > 1$ and for all $n \in \mathbf{Z}$ there exists $\lambda_n \in \mathbf{K}$ verifing $\mid \lambda_n \mid = \varrho^n$ see $[18], p.251$.
The field $\mathbf{K}$ is spherically complete if any decreasing sequence of closed balls in $\mathbf{K}$ has a non-empty intersection.
For the basic notions and properties concerning locally $\mathbf{K}-$convex spaces we refer to [14] or [18] if $\mathbf{K}$ is spherically complete and to [15] if $\mathbf{K}$ is not

spherically complete. However we recall the following:

Let E be a $\mathbf{K}-$vector space, a nonempty subset A of E is called $\mathbf{K}-$convex if $\lambda x+\mu y+\gamma z\in A$ whenever $x,y,z\in A$, $\lambda,\mu,\gamma\in\mathbf{K}$, $|\lambda|\leq 1$, $|\mu|\leq 1,|\gamma|\leq 1$ and $\lambda+\mu+\gamma=1$. A is said to be absolutely $\mathbf{K}-$convex if $\lambda x+\mu y\in A$ whenever $x,y\in A$, $\lambda,\mu\in\mathbf{K}$, $|\lambda|\leq 1$, $|\mu|\leq 1$. For a nonempty set $A\subset E$ its absolutely $\mathbf{K}-$convex hull $c_0\left(A\right)$ is the smallest absolutely $\mathbf{K}-$convex set that contains A. If A is a finite set $\{x_1,...,x_n\}$ we sometimes write $c_0\left(x_1,...,x_n\right)$ instead of $c_0\left(A\right)$.

A topology on a vector space E over $\mathbf{K}$ is said to be locally $\mathbf{K}-$convex ($l\mathbf{K}cs$) if there exists in E a fundamental system of zero-neighbourhoods consisting of absolutely $\mathbf{K}-$convex subsets of E.

If E is a $l\mathbf{K}cs$, $E^{'}$ and E^{*} denote its topological and algebraic dual, respectively, and $\sigma(E,E^{'})$ and $\sigma(E^{'},E)$ the weak topology of E and $E^{'}$, respectively.
If (E,τ) is a locally $\mathbf{K}-$convex space with topology τ we denote by $\mathcal{P}_E$, (or by $\mathcal{P}$ if no confusion is possible) a family of semi-norms determining the topology τ. We always assume that (E,τ) is a Hausdorff space.
If A is a subset of E we denote by $[A]$ the vector space spanned by A. Remark that, if A is absolutely $\mathbf{K}-$convex $[A]=\mathbf{K}A$. For an absolutely $\mathbf{K}-$convex subset A of E we denote by p_A the Minkowski functional on $[A]$, i.e for $x\in[A]$, $p_A(x)=inf\left\{\mid\lambda\mid: x\in\lambda A\right\}$. If A is bounded then p_A is a norm on $[A]$. We then denote by E_A the space $[A]$ normed by p_A.

Let $\langle,\rangle$ be a duality between E and F where E and F are two vectors spaces over $\mathbf{K}$ (see [1] for general results), if A is a subset of E, the polar of A is a subset of F defined by $A^{\circ}=\left\{y\in F\ /\ \forall x\in A\,|\langle x,y\rangle|\leq 1\right\}$.

We define also the polar of a subset B of F in the same way. A subset A of E is said to be a polar set if $A^{\circ\circ}=A$ ($A^{\circ\circ}$ is the bipolar of A)

A continuous semi-norm p on E is called a polar seminorm if the corresponding zero-neighbourhood $A=\{x\in E: p(x)\leq 1\}$ is a polar set. The space E is called strongly polar if every continuous semi-norm on E is polar, and it is called polar if $\exists\mathcal{P}_E$ such that every $p\in\mathcal{P}_E$ is polar. (see [15]). Obviously:

E strongly polar $\Longrightarrow$ E polar.

If E is a polar space then the weak topology $\sigma(E, E')$ is Hausdorff. ([15] prop. 5.6). In that case we have a dual pair (E, E'). The value of the bilinear form on $E \times E'$ (and similarly on $E' \times E$) is denoted by $\langle x, a \rangle$, $x \in E$, $a \in E'$. If E is a polar space and p is a continuous semi-norm on E we denote by E_p the vector space $E/Ker(p)$ and by π_p the canonical surjection $\pi_p : E \longrightarrow E_p$. The space E_p is normed by $\| \pi_p(x) \|_p = p(x)$. Its unit ball is $\pi_p(U)$, with $U = \{x \in E : p(x) \leq 1\}$. Its completion is denoted by $\widehat{E_p}$.

3. Sequential spaces in non-Archimedean analysis

3.1. Definitions and properties

Definitions 1. *1. Let E a locally $\mathbf{K}-$convex space and V a subset of E.*
V is called a sequential neighborhood ($S-neighborhood$) of 0 if every null sequence in E lies eventually in V, that is to say:

$$(\forall (x_n)_n \in C_0)\ (\exists N_0 \in \mathbf{N}) : (\forall n \geq N_0),\ x_n \in V.$$

2. A locally $\mathbf{K}-$convex space E is called sequential space if every convex sequential neighborhood of 0 is a neighborhood of 0.

Remark 1. *Every sequential neighborhood of 0 is absorbent and contains 0.*

Proposition 1. *If V is absolutely $\mathbf{K}-$convex and absorbent subset of a locally $\mathbf{K}-$convex space E, the following are equivalent:*

(i) V is a $S-neighborhood$ of 0;

(ii) p_V is sequentially continuous. Where p_V is the Minkowski functional associated to V.

Proof. $(i) \Rightarrow (ii)$ Suppose that V is a sequentially neighbourhood of 0; and let $(x_n)_n \in C_0(E)$, let us show that $P_V(x_n) \overset{n\to+\infty}{\longrightarrow} 0$.

Let $\varepsilon > 0$. Let us consider $\lambda \in \mathbf{K}$ such that $0 < |\lambda| \leq \varepsilon$, then $\left(\dfrac{x_n}{\lambda}\right)_n \in C_0(E)$, from where there exists $N \in \mathbf{N}$ such that $(\forall n \geq N)$, $\dfrac{x_n}{\lambda} \in V$, which implies that $\forall n \geq N,\ p_V\left(\frac{x_n}{\lambda}\right) \leq 1$, or $(\forall n \geq N),\ p_V(x_n) \leq\mid \lambda \mid\leq \varepsilon$. Thus the result follows.

Reciprocally, suppose that p_V is sequentially continuous over E. Let $(x_n)_n \in C_0(E)$, so $P_V(x_n) \stackrel{n\to+\infty}{\longrightarrow} 0$, therefore there exists $N \in \mathbf{N}$ such that $(\forall n \geq N),\ p_V(x_n) < 1$, and so $(\forall n \geq N),\ x_n \in V$. ∎

Proposition 2. *For a locally* **K**−*convex space E the following are equivalent:*

(i) E is a sequential space;

(ii) Every sequentially continuous seminorm on E is continuous;

(iii) For every locally **K**−*convex space F, every sequentially continuous linear map from E to F is continuous;*

(iv) For every Banach space F, every sequentially continuous linear map from E to F is continuous.

Proof. $(i) \Rightarrow (ii)$. Suppose that E is sequential and let p a seminorm sequentially continuous on E. Let: $V = \{x \in E : p(x) \leq 1\}$.

V is a sequential neighborhood of 0 and so V is a neighborhood of 0 and consequently p is continuous: $\forall \varepsilon > 0,$ *let* $\lambda \in \mathbf{K}$ *such that* $0 <\mid \lambda \mid< \varepsilon$. Then:

$$U = \lambda V \text{ is a neighborhood of 0 and we have } p(U) \subset B(0, \varepsilon).$$

$(ii) \Rightarrow (i)$. Let V a convex subset of E which is a sequential neighborhood of 0. V is absorbent and contains 0, therefore it's absolutely **K**−convex (**K**−convex and contains 0). Then, by Proposition 1, p_V is sequetially continuous, then continuous, and so V is a neighborhood of 0. Therefore E is sequential.

$(ii) \Rightarrow (iii)$. Let F a locally **K**−convex space and $f : E \longrightarrow F$ a sequentially-continuous linear mapping .

Let V a **K**−convex neighborhood of 0 in F, $f^{-1}(V)$ is a sequential **K**−convex neighborhood of 0 in E, and so $f^{-1}(V)$ is a neighborhood of 0 in E, (E is sequential and $(ii) \Leftrightarrow (i)$). And then f is continuous.

$(iv) \Rightarrow (ii)$. Let p a sequentially-continuous semi-norm on E; consider the Banach space $\widehat{E_p}$ the completion of $E_p = E/Ker(p)$. The canonical mapping $\pi_p : E \longrightarrow \widehat{E_p}$ is sequentially continuous, because: for all $(x_n)_n \subset E$ such that $x_n \overset{n\to+\infty}{\longrightarrow} 0$, we have:

$$\begin{aligned} x_n \overset{n\to+\infty}{\longrightarrow} 0 \quad &\Rightarrow p(x_n) \overset{n\to+\infty}{\longrightarrow} 0 \\ &\Rightarrow \widehat{p}(\widehat{x_n}) \overset{n\to+\infty}{\longrightarrow} 0 \\ &\Rightarrow \pi_p(x_n) \overset{n\to+\infty}{\longrightarrow} 0. \end{aligned}$$

Then π_p is continuous, and so p is continuous:

$$(\forall \varepsilon > 0) \quad U = \pi_p^{-1}\left(B_{\widehat{p}}(0,\varepsilon)\right) \textit{ is a neighborhood of } 0 \textit{ in } E$$

and we have $\quad \mathrm{p}(U) \subset B(0,\varepsilon)$.

(iii) $\Rightarrow (iv)$ Obvious.

3.2. The sequential topology

Let (E,τ) a locally $\mathbf{K}-$convex space. Consider $\mathcal{U}$ the set of all sequentially $\mathbf{K}-$convex neighborhood of 0 and let $\mathcal{P}_s$ the family of all sequentially $\tau-$continuous $n.a.$ semi-norm on E .

- $\mathcal{U}$ is a base of neighborhood of 0 for a locally $\mathbf{K}-$convex topology on E which is denoted τ^s [16, 1.2. p.14]. Since every neighborhood of 0 is a sequential neighborhood of 0, then $\tau \leq \tau^s$.
- $\mathcal{P}_s$ define a locally $\mathbf{K}-$convex topology on E which is denoted T^s. A base of neighborhood of 0 for T^s is formed by the balls $B_p(0,\varepsilon)$ where $\varepsilon > 0$ and p is a $n.a.$ sequentially $\tau-$continuous semi-norm. $B_p(0,\varepsilon)$ is sequential neighborhood of 0, because for all sequence $(x_n)_n$ converging to zero in (E,τ), there exists $n_0 \in \mathbf{N}$ such that for all $(n \geq n_0)$, $p(x_n) < \varepsilon$ $\left(p(x_n) \overset{n\to+\infty}{\longrightarrow} 0\right)$.

Remark 2. *The topology T^s is sequential.*

Proposition 3. *τ^s is the coarset of all sequential locally $\mathbf{K}-$convex topologies on E finer than τ.*

Proof. τ^s is sequential and $\tau^s \geq \tau$.

Let ϱ a sequential locally $\mathbf{K}-$convex topology on E finer than τ. Let $U \in \mathcal{U}$; U sequential neighborhood of 0 for τ, and so U is a sequential neighborhood of 0 for ϱ $(\varrho \geq \tau)$ and then U is a neighborhood of 0 for ϱ (ϱ is sequential). Therefore $\varrho \geq \tau^s$. Which proves the proposition. ∎

3.2.1. Characterization of sequential locally K− convex spaces

Proposition 4. *τ is sequential if, and only if, $\tau = \tau^s$.*

Proof. $\Longleftarrow$ Obvious.
Suppose that τ is sequential and let $U \in \mathcal{U}$; U is a sequential $\mathbf{K}-$convex neighborhood of 0 for τ, so U is a neighborhood of 0 for τ and then $\tau \geq \tau^s$. Finally $\tau = \tau^s$. ∎

Lemma 1. *For all sequence $(x_n)_n$ of (E, τ) we have:*

$$\left(x_n \overset{n\to+\infty}{\to} 0 \ for \ \tau\right) \Leftrightarrow \left(x_n \overset{n\to+\infty}{\to} 0 \ for \ \tau^s\right).$$

Proof. $\Rightarrow$] Let $U \in \mathcal{U}$, there exists $N \in \mathbf{N}$ such that: $\forall n \geq N \ \ x_n \in U$, hence $x_n \overset{n\to+\infty}{\to} 0$ for τ^s.
The converse follows by $\tau \leq \tau^s$. ∎

Lemma 2. *Let ϱ a locally $\mathbf{K}-$convex topology on E such that for all null sequence for τ is a null sequence for ϱ. Then $\tau^s \geq \varrho$.*

Proof. Consider $i : (E, \tau) \longrightarrow (E, \varrho)$ the canonical injection. Then for every sequence $(x_n)_n$ in E we have:
$x_n \xrightarrow{\tau^s} 0 \overset{Lemma\ 1}{\Longrightarrow} x_n \xrightarrow{\tau} 0$
$\Longrightarrow x_n \xrightarrow{\varrho} 0$

Then, i is sequentially continuous, and since (E, τ^s) is sequential, i is continuous (Proposition 2). Hence $\tau \leq \tau^s$. ∎

Proposition 5. *τ^s is the finest locally $\mathbf{K}-$convex topology on E having the same convergent suequences as τ.*

Proof. By *Lemma* 1 before, τ^s and τ has the same convergent sequences. Let ϱ a locally $\mathbf{K}-$convex topology on E having the same convergent sequence as τ and let $(x_n)_n$ a sequence of E converging to 0 for τ, then $x_n \longrightarrow 0$ for ϱ, hence, by *Lemma* 3, $\tau^s \geq \varrho$. ∎

Remark 3. *τ^s is also the finer topology on E having the same null sequences as τ.*

Lemma 3. *Let (E, τ) a locally $\mathbf{K}-$convex space and A a subset of E, then: A is $\tau-$bounded if, and only if, for all null sequence $(\lambda_n)_n$ in $\mathbf{K}$ and all sequence $(x_n)_n$ in A; the sequence $(\lambda_n x_n)_n$ converges to zero in (E, τ) that is to say $(\lambda_n x_n)_n$ is a null sequence in (E, τ).*

Proof. Suppose that A be bounded in (E, τ).
Let $(\lambda_n)_n \in C_0(\mathbf{K})$ and $(x_n)_n$ a sequence in A.
Let V a $\mathbf{K}-$convex neighborhood of zero in E, then there exists λ in $\mathbf{K}^*$ such that $\lambda A \subset V$ and there exists $N \in \mathbf{N}^*$ such that $(\forall n \geq N) \quad \mid \lambda_n \mid \leq \mid \lambda \mid$; but

$$(\forall n \geq N) \quad \lambda_n x_n \begin{array}{l} = \frac{\lambda_n}{\lambda}\lambda x_n \\ \in \frac{\lambda_n}{\lambda}\lambda A \\ \subset \frac{\lambda_n}{\lambda}V \\ \subset V. \end{array}$$

Then the sequence $(\lambda_n x_n)_n$ converges to zero in (E, τ).
Reciprocally, if A is no $\tau-$bounded, then there exists U a $\mathbf{K}-$convex neighborhood of zero such that $\forall n \in \mathbf{N} \ \ A \not\subset \dfrac{1}{\lambda_n}U$ where $(\lambda_n)_n$ is the sequence of general term $|\lambda_n| = \varrho^n$ and ϱ is the real number defined in the preliminary. For all $n \in \mathbf{N}$, let x_n the element of A such that $x_n \notin \dfrac{1}{\lambda_n}U$, then, $(\forall n \in \mathbf{N}) \quad \lambda_n x_n \notin U$ that is to say that the sequence $(\lambda_n x_n)_n$ does not converge to zero, and we have: $(x_n)_n \subset A$ and $(\lambda_n)_n \in C_0(\mathbf{K})$. ∎

Proposition 6. *Let (E, τ) a locally $\mathbf{K}-$convex space, then:*
τ and τ^s have the same bounded subsets.

Proof. Let A a subset of E.
If A is τ^s-bounded, A is $\tau-$bounded, because $\tau^s \geq \tau$.
If A is $\tau-$bounded, let $(x_n)_n \subset A$ and $(\lambda_n)_n \in C_0(\mathbf{K})$, then, according to the previous Lemma, the sequence $(\lambda_n x_n)_n$ converges to zero in (E, τ) and therefore it converges to zero in (E, τ^s) (Lemma 1). So A is τ^s-bounded. ∎

Proposition 7. *Let $\left(F, \tau'\right)$ a locally $\mathbf{K}-$convex space and $f : E \longrightarrow F$ a linear mapping, then:*
f is τ^s-continuous if, and only if, f is sequentially $\tau-$continuous.

Proof. Suppose that f be τ^s-continuous, and let $(x_n)_n$ a converging sequence to zero in (E, τ) and let $V \in \mathcal{V}_F(0)$, there exists $U \in \mathcal{U}$ such that $f(U) \subset V$. U being a sequential neighborhood of zero, so there exists $n_0 \in \mathbf{N}$ such that $(\forall n \geq n_0) \quad x_n \in U$ and consequently $(\forall n \geq n_0)$ $f(x_n) \in f(U) \ (\subset V)$. Therefore the sequence $(f(x_n))_n$ converges to zero in F.

Conversely, suppose that f is sequentially $\tau-$continuous; let us show that $f:(E,\tau^s)\longrightarrow F$ is continuous. According to Proposition 2, it suffices to show that f is sequentially τ^s-continuous. Let then $(x_n)_n$ a converging sequence to zero in (E,τ^s), then it converges to zero in (E,τ) (Lemma 1) and consequently $(f(x_n))_n$ is converging to zero in F. ∎

3.3. Comparison of topologies τ^s and T^s

Lemma 4. *For every $U\in\mathcal{U}, p_U$ is a n.a. sequentially $\tau-$continuous seminorm.*

Proof. Let $U\in\mathcal{U}$; then for all $(x_n)_n\in C_0(E)$, all $\varepsilon>0$ and all $\lambda\in\mathbf{K}^*$ such that $0<\mid\lambda\mid\leq\varepsilon$ we have: $(\lambda^{-1}x_n)_n\in C_0(E)$ from where it exists $n_0\in\mathbf{N}:(\forall n\geq n_0)\quad\lambda^{-1}x_n\in U$ and then:

$(\forall n\geq n_0)\quad p_U(\lambda^{-1}x_n)\leq 1\Rightarrow(\forall n\geq n_0)\quad p_U(x_n)\leq\mid\lambda\mid\leq\varepsilon$. Therefore the sequence $(p_U(x_n))_n$ converges to zero in $\mathbf{R}^+$ and consequently p_U is sequentially $\tau-$continuous. ∎

Proposition 8. $\tau^s=T^s$

Proof. T^s being a sequential locally $\mathbf{K}-$convex topology $(\textit{Remark } 2)$, whence $\tau^s\geq T^s$.
For the other inequality, it suffices to show that $i:(E,T^s)\to(E,\tau^s)$ is continuous, and by $\textit{Proposition}2$, it suffices to show that the mapping i is sequentially T^s-continuous.
Let $(x_n)_n$ a sequence that tends towards zero in (E,T^s). Then for any $U\in\mathcal{U}$, p_U is sequentially $\tau-$continuous, therefore the sequence $(p_U(x_n))_n$ converges to zero in $\mathbf{R}^+$, from where it exists $n_0\in\mathbf{N}:(\forall n\geq n_0)\quad p_U(x_n)<1$, or $(\forall n\geq n_0)\quad x_n\in U$. Therefore the sequence $(x_n)_n$ tends to zero in (E,τ^s). From where i is T^s-sequentially continuous. And consequently $T^s\geq\tau^s$. ∎

Remark 4. *We can show otherwise the previous Proposition: Since any n.a. $\tau-$continuous seminorm on E is sequentially $\tau-$continuous, $T^s\geq\tau$. But T^s is sequential and τ^s is the coarset sequential locally* **K**−*convex topology finer than τ, then $T^s\geq\tau^s$.*

3.4. The sequential polar topology

Let $\mathcal{V}$ the family of all $\mathbf{K}-$convex, subsets A of E which are polar and sequential neighborhood of 0 in (E,τ). $\mathcal{V}$ is a base of neighborhood of 0 of a locally $\mathbf{K}-$convex topology on E which we noted τ^{ps} [16, 1.2., p. 14]. τ^{ps} is a polar topology on E and $\tau^{s} \geq \tau^{ps} \quad (\mathcal{V} \subset \mathcal{U})$.

Remark 5. *Since, if $V \in \mathcal{V}$, then $\overline{V}^{\tau} \in \mathcal{V}$, we can suppose that all $V \in \mathcal{V}$, V is $\tau-$closed.*

Lemma 5. *Suppose that $\mathbf{K}$ is spherically complete, and let A a subset of E absolutely $\mathbf{K}-$convex and $\tau-$closed, then:*

1. *If $\mathbf{K}$ is discrete, $A^{\circ\circ} = A$.*
2. *If $\mathbf{K}$ is dense, $\forall \alpha \in \mathbf{K} :\mid \alpha \mid> 1 \ A^{\circ\circ} \subset \alpha A$.*

Where $A^{\circ\circ}$ is the bipolar of A with respect the duality $\left\langle E, E' \right\rangle$.

Proof. See $[18, \ Theorems \ 4.14, \ 4.15, \ p.280 - 281]$. ∎

Lemma 6. *If $\mathbf{K}$ is spherically complete, then τ^{ps} is a sequential topology.*

Proof. Let U a subset of E which is $\mathbf{K}-$convex, $\tau-$closed and sequential neighborhood of 0 on (E, τ). Let us show that U is a neighborhood of 0 of τ^{ps}. By Lemma 5, before $U^{\circ\circ} \subset \alpha U$ for $\alpha = 1$ if $\mathbf{K}$ is discrete and $\mid \alpha \mid> 1$ is $\mathbf{K}$ dense. We pose $V = U^{\circ\circ}$, then V is $\mathbf{K}-$convex, polar and sequential neighborhood of 0 on (E, τ) $(U \subset V)$, then V is a neighborhood of 0 for τ^{ps} and therefore U is a neighborhood of 0 for τ^{ps} $\left(\frac{1}{\alpha} V \subset U\right)$. Then τ^{ps} is sequential. ∎

Proposition 9. *If $\mathbf{K}$ is spherically complete, then $\tau^{ps} \geq \tau$.*

Proof. It is a matter of showing that $i : (E, \tau^{ps}) \longrightarrow (E, \tau)$ is continuous, and since τ^{ps} is sequential (Lemma 6), it suffices to show that i is sequentially $\tau^{ps}-$continuous.
Let $(x_n)_n$ a sequence of E which is converging to zero on (E, τ^{ps}) and let U an absolutely $\mathbf{K}-$convex and $\tau-$closed neighborhood of zero on (E, τ), then $U^{\circ\circ} \subset \alpha U$ where $\alpha = 1$ if $\mathbf{K}$ is discrete and $\mid \alpha \mid> 1$ if $\mathbf{K}$ is dense (Lemma 5). The sequence $(\alpha x_n)_n$ converges to zero on (E, τ^{ps}) and $U^{\circ\circ} \in \mathcal{V}$ hence there exists $n_0 \in \mathbf{N} : (\forall n \geq n_0) \quad \alpha x_n \in U^{\circ\circ}$ and so $(\forall n \geq n_0) \quad x_n \in \dfrac{1}{\alpha} U^{\circ\circ} \subset U$. Then the sequence $(x_n)_n$ converges to zero on (E, τ). ∎

Remark 6. *If* $\mathbf{K}$ *is spherically complete, then* $\tau^{ps} \geq \tau$; *but* τ^{s} *is the coarset of all sequential locally* $\mathbf{K}-$*convex topologies finest than* τ *and since* τ^{ps} *is sequential, then* $\tau^{ps} \geq \tau^{s}$, *and so* $\tau^{ps} = \tau^{s}$.

Proposition 10. *Let* (E,τ) *a locally* $\mathbf{K}-$*convex space. Then* τ^{ps} *is the finer of all polar locally* $\mathbf{K}-$*convex topologies* δ *on* E *such that all sequence on* E *which is* $\tau-$*convergent is* $\delta-$*convergent.*

Proof. τ^{ps} is a locally $\mathbf{K}-$convex polar topology on E.
Let $(x_n)_n$ a converging sequence to zero on (E,τ), then for all $V \in \mathcal{V}$, there exists $n_0 \in \mathbf{N} : (\forall n \geq n_0)\ \ x_n \in V$, then $(x_n)_n$ converges to zero on (E,τ^{ps}).

Let δ a locally $\mathbf{K}-$convex polar topology on E such that all sequence on E which is $\tau-$convergent is $\delta-$convergent; showing that $\tau^{ps} \geq \delta$. Let U a $\mathbf{K}-$convex and polar neighborhood of zero for δ, and let $(x_n)_n$ a sequence which converges to zero on (E,τ), then it's convergent to zero for δ; hence there exists $n_0 \in \mathbf{N} : (\forall n \geq n_0)\ \ x_n \in U$. Then U is a sequential neighborhood of zero and so $U \in \mathcal{V}$. And then $\tau^{ps} \geq \delta$. ∎

Corollary 1. *If* τ *is polar, then* $\tau^{ps} \geq \tau$ *and* τ^{ps} *and* τ *have the same convergent sequences.*

Proof. $\tau^{ps} \geq \tau$ follows immediately of the proposition before and we have all $\tau^{ps}-$convergent sequence is $\tau-$convergent. And we have already all $\tau-$convergent sequence is $\tau^{ps}-$convergent; then τ^{ps} and τ have the same convergent sequences.

Or equivalentely the two topologies have the same null sequences .

Lemma 7. *Let* p *a seminorm n.a. over* E. *And let:*

$$A = \{x \in E : p(x) < 1\} \quad \text{and} \quad B = \{x \in E : p(x) \leq 1\}.$$

Then $A^{\circ} = B^{\circ}$.

Proof. If $\mathbf{K}$ is discrete, $A = B$, then we can suppose that $\mathbf{K}$ is dense.

A is a subset of B, then $B^{\circ} \subset A^{\circ}$.

Let $f \in E^{*}$ such that $f \notin B^{\circ}$, then there exists $y \in B$ such that $\mid f(y) \mid > 1$. Suppose that $f \in A^{\circ}$, that is to say that $(\forall x \in A) \quad \mid f(x) \mid \leq 1$; then, since $\mathbf{K}$ is dense, there exists $\lambda \in \mathbf{K}$ such that $1 < \mid \lambda \mid < \mid f(y) \mid$ so:

$$\begin{aligned} 1 < \mid f\left(\tfrac{y}{\lambda}\right) \mid \quad & \Longrightarrow \tfrac{y}{\lambda} \notin A \\ & \Longrightarrow p\left(\tfrac{y}{\lambda}\right) \geq 1 \\ & \Longrightarrow p(y) \geq \mid \lambda \mid \\ & \Longrightarrow p(y) > 1 \\ & \Longrightarrow y \notin B \end{aligned}$$

Which is absurd. ∎

Proposition 11. *τ^{ps} coincides with the locally $\mathbf{K}-$convex topology generated by all n.a. polar and sequentially $\tau-$continuous semi-norms.*

Proof. Let T^{ps} the locally $\mathbf{K}-$convex topology generated by $\mathcal{S}_p$ the familly of all *n.a.* polar and sequentially $\tau-$continuous semi-norms. Then T^{ps} admits a basis $\mathcal{B}$ of neighborhoods of zero formed by polar balls $B_p(0,\varepsilon)$ where $p \in \mathcal{S}_p$ and $\varepsilon > 0$.

Let us show that $i : (E, \tau^{ps}) \longrightarrow (E, T^{ps})$ is bicontinuous.

Let $V = B_p(0,\varepsilon)$ an element of $\mathcal{B}$, then V is $\mathbf{K}-$convex. Let $(x_n)_n$ a sequence of elements of E which converges to zero in (E,τ), then $\lim\limits_{n\to+\infty} p(x_n) = 0$ (p is sequetially $\tau-$continuous), hence there is $n_0 \in \mathbf{N}$: $(\forall n \geq n_0) \quad p(x_n) < \varepsilon$, or $(\forall n \geq n_0) \quad x_n \in V$ which implies that V is a sequentially neighborhood of zero, hence $V \in \mathcal{V}$ and so $T^{ps} \leq \tau^{ps}$.

Conversely, either $V \in \mathcal{V}$, then it's sequentially $\mathbf{K}-$convex neighborhood of zero. Whe have:

$$\{x \in E : p_V(x) < 1\} \subset V \subset \{x \in E : p_V(x) \leq 1\}.$$

And by the previous *Lemma* 7:

$$A^{\circ\circ} = B^{\circ\circ} = V^{\circ\circ} = V,$$

where $A = \{x \in E : p_V(x) < 1\}$ and $B = \{x \in E : p_V(x) \leq 1\}$; from where $B^{\circ\circ} = B$, and consequently p is polar or p_V is polar [15, *Proposition* 3.4, *p.* 195]. Let us show that p_V is sequentially $\tau-$continuous. Let $(x_n)_n$ a sequence of elements of E which converges to zero in (E,τ) and let $\varepsilon > 0$; let us consider $\lambda \in \mathbf{K}$ such that $0 < \mid \lambda \mid < \varepsilon$, then the sequence $\left(\lambda^{-1}x_n\right)_n$ converges to zero in (E,τ) and V being a sequential neighborhood of 0, then there exists $n_0 \in \mathbf{N} : (\forall n \geq n_0) \quad \lambda^{-1}x_n \in V$, from where $(\forall n \geq n_0) \quad p_V\left(\lambda^{-1}x_n\right) \leq 1$ or $(\forall n \geq n_0) \quad p_V\left(x_n\right) \leq \mid \lambda^{-1} \mid < \varepsilon$; from where p_V is sequentially $\tau-$continuous and consequently $p_V \in \mathcal{S}_p$; then $T^{ps} \geq \tau^{ps}$. So what $T^{ps} = \tau^{ps}$. ∎

Proposition 12. *τ^{ps} is the finer of all polar locally* **K***−convex topologies which are less fine than τ^s.*

Proof. τ^{ps} is a polar locally **K**−convex topology and $\tau^{ps} \leq \tau^s$. Let ϱ a polar locally **K**−convex topology such that $\varrho \leq \tau^s$, and let V a polar **K**−convex neighborhood of 0 for ϱ, then there exists $U \in \mathcal{U}$ such that $U \subset V \quad (\varrho \leq \tau^s)$, from where V is a sequential neighborhood of zero, consequently it is a sequential neighborhood of zero for τ^{ps}. Therefore$\tau^{ps} \geq \varrho$. ∎

3.4.1. Continuity of linear mappings

Lemma 8. *Let E and F be two locally* **K**− *convex spaces and $f : E \longrightarrow F$ a continuous linear mapping, then for any subset V of F.*

If V is polar in F, $f^{-1}(V)$ is polar in E.

Proof. Let $V \subset F$, putting $U = f^{-1}(V)$. Suppose that V is sequential.

Let $x \in U^{\circ\circ}$; let us show that $x \in U$. By absurd, suppose that $x \notin U$, and let $y = f(x)$ then $y \notin V$ from where $y \notin V^{\circ\circ}$ $(V^{\circ\circ} = V)$ then there exists $\varphi \in V^\circ : \mid \varphi(y) \mid > 1$. But $\forall t \in U, \quad f(t) \in V$ from where

$\forall t \in U \quad \mid \varphi(f(t)) \mid \leq 1$ and consequently $\varphi \circ f \in U^\circ$ and so $\mid \varphi(f(x)) \mid \leq 1$, therefore $\mid \varphi(y) \mid \leq 1$; which is absurd. ∎

Proposition 13. *Let (E,τ) and (F,τ_1) two locally* **K***−convex spaces .*

If $f : (E,\tau) \longrightarrow (F,\tau_1)$ is a continuous linear mapping, then f is (τ^s,τ_1^s) −continuous and (τ^{ps},τ_1^{ps}) −continuous.

Proof. Let us show that $f : (E,\tau^s) \longrightarrow (F,\tau_1^s)$ is continuous. For this it suffices to show that for every sequential neighborhood V of zero for

τ_1, $f^{-1}(V)$ is a sequential neighborhood V of zero for τ.

Let V a sequential neighborhood of zero for τ_1 and let $(x_n)_n$ a sequence of E which converges towards zero in (E,τ); then the sequence $(f(x_n))_n$ converges towards zero in (F,τ_1), from where there exists $n_0 \in \mathbf{N} : (\forall n \geq n_0)\, f(x_n) \in V$, then $(\forall n \geq n_0)\, x_n \in f^{-1}(V)$.

Let us show that $f : (E,\tau^{ps}) \longrightarrow (F,\tau_1^{ps})$ is continuous. For this it suffices to show that for every polar and sequential neighborhood V of zero for τ_1, $f^{-1}(V)$ is a polar and sequential neighborhood V of zero for τ.

Let V a polar and sequential neighborhood of zero for τ_1, then by Lemma 8, $f^{-1}(V)$ is polar for τ. In the other hand, for all sequence $(x_n)_n$ of E which converges towards zero in (E,τ), the sequence $(f(x_n))_n$ converges to zero in (F,τ_1), from where there exists $n_0 \in \mathbf{N} : (\forall n \geq n_0)\, f(x_n) \in V$, therefore $(\forall n \geq n_0)\, x_n \in f^{-1}(V)$. ∎

Proposition 14. *Let* $(E,\tau) = \prod_{k=1}^{n} (E_k,\tau_k)$, *then:*

(i) $\tau^s = \prod_{k=1}^{n} \tau_k^s$;

(ii) $\tau^{ps} = \prod_{k=1}^{n} \tau_k^{ps}$.

Proof. Let us show that $i : (E,\tau^s) \longrightarrow \left(E, \prod_{k=1}^{n} \tau_k^s\right)$ is continuous. Let us show that V is neighborhood of zero for (E,τ^s), where $V = (U_k)_{1\leq k\leq n}$ is a K-convex neighborhood of zero for the arrival space. Let $(y_p)_p = \left(\prod_{k=1}^{n} x_k^p\right)_p$ a sequence of E which converges to zero in (E,τ), then for all $k \in \mathbf{N},\ 1 \leq k \leq n$, the sequence $(x_k^p)_p$ converges to zero in (E_k,τ_k), from where there exists $p_k \in \mathbf{N}$: $(\forall p \geq p_k)\quad x_k^p \in U_k$. Let $p_0 = \max_{1\leq k\leq n} p_k$, so $(\forall p \geq p_0)\ \forall k \in \mathbf{N},\ 1 \leq k \leq n,\ \ x_k^p \in U_k$, from where $(\forall p \geq p_0)\quad y_p \in V$. Therefore V is a sequential neighborhood of zero in (E,τ); V being $\mathbf{K}-$convex, therefore V is a neighborhood of zero in (E,τ^s).

Let us show that $i : \left(E, \prod_{k=1}^{n} \tau_k^s\right) \longrightarrow (E, \tau^s)$ is continuous. Let V a sequential $\mathbf{K}-$convex neighborhood of zero in (E,τ). For all $k,\ 1 \leq k \leq n$, let $j_k : E_k \to E$ the canonical injection and pose $V_k = j_k^{-1}(V)$, so V_k is a sequential neighborhood of zero in (E_k, τ_k), from where V_k is a neighborhood of zero in (E_k, τ_k^s), and consequently $U = \prod_{k=1}^{n} V_k$ is a neighborhood of zero in $\left(E, \prod_{k=1}^{n} \tau_k^s\right)$. But $U \subset V$ (V *is absolutely* $\mathbf{K}-$ *convex*); therefore V is a neighborhood of zero in $\prod_{k=1}^{n} \tau_k^s$. ∎

References

[1] R. Ameziane Hassani and M. Babahmed, Topologies polaires compatibles avec une dualité séparante sur un corps valué non-archimédien, Proyecciones. Vol. 20, No. 2, pp. 217-241, (2001).

[2] J. R. Boone and F.Siwiec, Sequentially quotient mappings, Czechoslovak Mathematical Journal, Vol. 26, No. 2, pp. 174-182, (1976).

[3] J. Boos and T. Leiger, Duals pairs of sequence spaces, IJMMS **28**, No.1, pp. 9-23, (2001).

[4] M. C. Cueva and C. T. Maia Vinagre, Sequences in non-archimedean locally convex spaces, Indian J. pure appl. Math., 25(9): pp. 955-962, (1994).

[5] N. De Grande-De Kimpe, Perfect locally K-convex sequence spaces, Indag. Math. (33), pp. 371-482, (1971).

[6] A. El amrani, R. Ameziane Hassani and M. Babahmed, Topologies on sequence spaces in non-archimedean analysis, J. of Mathematical Sciences: Advances and Applications **6**, No. 2, pp. 193-214, (2010).

[7] A. El Amrani, R. Ameziane Hassani and M. Babahmed, Polar topologies in sequence spaces in non-archimedean analysis. Universidad Catolica del Norte Antofagasta-Chile. Proyecciones Journal of Mathematics **31**, No. 2, pp. 103-123, (2012).

[8] J. R. Ferrer, I. Morales, L. M. Sanchez Ruiz, Sequential convergence in topological vector spaces, Topology and its applications 108, pp. 1-6, (2000).

[9] S. P. Franklin, Spaces in which sequences suffice," Fundamenta Mathematicae 67, pp. 107-116, (1965).

[10] S. P. Franklin, "Spaces of which sequences suffice, II," Fund. Math. 61, pp. 51-56, (1967).

[11] A. Goreham, Sequential convergence in topological spaces, arXiv:math/0412558v2-[math.GN] 10 April, (2016).

[12] D.A. Gregory, Vector sequence spaces, Ph.D thesis, University of Machigan, (1967).

[13] A. K. Katsaras and V. Benekas, Sequential convergence in topological vector spaces, Georgian mathematical journal: Vol. 2, No. 2, pp. 151-164 , (1995).

[14] A. F. Monna, Analyse non- archimédienne, Springer-Verlag, band 56, (1970).

[15] W. H. Schikhof, Locally convex spaces over nonspherically complete valued field, I, II, Bull. Soc. Math. Belg. sér. B 38, pp. 187-224 (1986).

[16] H. H. Schaefer, Topological vector spaces, Springer-Verlag New-York, herdlberg Berlin, (1971).

[17] R. F. Snipes, T-sequential topological spaces, Fundamenta Mathematicae, T. LXXVII, pp. 95-98, (1970).

[18] J. Van-tiel, Espaces localement K-convexes, I-III. Proc. Kon. Ned. Akad. van Wetensch. A 68, pp. 249-289, (1965).

[19] M. Venkataramen, Directed sets in topology, Math. student 30, pp. 99-100, (1962).

[20] S. Warner, Topological Fields, North-Holland Mathematics studies 157, (1989).

[21] J. H. Webb, Sequential convergence in locally convex spaces, Proc. Cambridge Philos. Soc. 64, pp. 341-364, (1968).

A. El Amrani
Department of mathematics and computer science
Faculty of Sciences Dhar El Mahraz
B. P. 1796 Atlas Fès,
Morocco
e-mail: abdelkhalek.elamrani@usmba.ac.ma