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Generalized Drazin-type spectra of Operator matrices

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Abstract

In this paper, we investigate the limit points set of surjective and approximate point spectra of upper triangular operator matrices $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$. *We prove that* $\sigma_*(M_C) \cup W = \sigma_*(A) \cup \sigma_*(B)$ *where* W *is the union of certain holes in* $\sigma_*(M_C)$, *which happen to be subsets of* $\sigma_{lgD}(B) \cap \sigma_{rgD}(A)$, $\sigma_* \in \{\sigma_{lgD}, \sigma_{rgD}\}$ *are the limit points set of surjective and approximate point spectra. Furthermore, several sufficient conditions for* $\sigma_*(M_C) = \sigma_*(A) \cup \sigma_*(B)$ *holds for every* $C \in \mathcal{B}(Y, X)$ *are given.*

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1. Introduction and Preliminaries

Let X, Y be Banach space and $\mathcal{B}(X,Y)$ denote the algebra of all bounded linear operators from X to Y. For $Y = X$ we write $\mathcal{B}(X,X) = \mathcal{B}(X)$. For $T \in \mathcal{B}(X)$, we denote by T^*, $N(T)$, $R(T)$, $R^\infty(T) = \bigcap_{n\geq 0} R(T^n)$, $\sigma_{ap}(T)$, $\sigma_{su}(T)$, $\sigma_p(T)$, $\rho(T)$ and $\sigma(T)$, respectively the adjoint, the null space, the range, the hyper-range, the approximate point spectrum, the surjectivity spectrum, the point spectrum, the resolvent set and the spectrum of T.

Next, let $T \in \mathcal{B}(X)$, T has the single valued extension property at $\lambda_0 \in \mathbf{C}$ (SVEP) if for every open neighborhood $U \subseteq \mathbf{C}$ of λ_0, the only analytic function $f : U \longrightarrow X$ which satisfies the equation $(T - zI)f(z) = 0$ for all $z \in U$ is the function $f \equiv 0$. T is said to have the SVEP if T has the SVEP for every $\lambda \in \mathbf{C}$. Obviously, every operator $T \in \mathcal{B}(X)$ has the SVEP at every $\lambda \in \rho(T)$, then T and T^* have the SVEP at every point of the boundary $\partial(\sigma(T))$ of the spectrum. In particular, T and T^* have the SVEP at every isolated point of the spectrum. We denote by $S(T)$ the open set of $\lambda \in \mathbf{C}$ where T fails to have SVEP at λ, and we say that T has SVEP if $S(T) = \emptyset$. Note that $S(T) \subset \sigma_p(T)$ and $\sigma(T) = S(T) \cup \sigma_{su}(T)$ see [1, 4].

For a compact subset K of $\mathbf{C}$, let $accK$, $intK$, $isoK$, ∂K and $\eta(K)$ be the set of all points of accumulation of K, the interior of K, the isolated points of K, the boundary of K and the polynomially convex hull of K respectively.

Let $T \in \mathcal{B}(X)$, T is said to be Drazin invertible if there exists a positive integer k and an operator $S \in \mathcal{B}(X)$ such that

$$ST = TS, \;\; T^{k+1}S = T^k \;\; and \; S^2T = S.$$

Which is also equivalent to the fact that $T = T_1 \oplus T_2$; where T_1 is invertible and T_2 is nilpotent. The concept of Drazin invertible operators has been generalized by Koliha [5]. In fact, $T \in \mathcal{B}(X)$ is generalized Drazin invertible if and only if $0 \notin acc(\sigma(T))$, which is also equivalent to the fact that $T = T_1 \oplus T_2$ where T_1 is invertible and T_2 is quasi-nilpotent. The generalized Drazin invertible spectrum is defined by

$$\sigma_{gD}(T) = \{\lambda \in \mathbf{C} : \; T - \lambda I \text{ is not generalized Drazin invertible}\}$$

Now, set

$$\sigma_{lgD}(T) = acc(\sigma_{ap}(T)).$$

$$\sigma_{rgD}(T) = acc(\sigma_{su}(T)).$$

Hence $\sigma_{gD}(T) = \sigma_{lgD}(T) \cup \sigma_{rgD}(T)$.
For $A \in \mathcal{B}(X)$, $B \in \mathcal{B}(Y)$, we denote by $M_C \in \mathcal{B}(X \oplus Y)$ the operator defined on $X \oplus Y$ by

$$\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$$

It is well known that, in the case of infinite dimensional, the inclusion $\sigma(M_C) \subset \sigma(A) \cup \sigma(B)$, may be strict. This motivates serval authors to study the defect set $(\sigma_*(A) \cup \sigma_*(B)) \setminus \sigma_*(M_C)$ where σ_* runs different type spectra [2], [3], [6], [7], [9].
The following questions arise naturally:

1. Under which conditions on A and B does $\sigma_*(M_C) = \sigma_*(A) \cup \sigma_*(B)$ for arbitrary $C \in \mathcal{B}(Y, X)$?
2. Given A and B, for which operators $C \in \mathcal{B}(Y, X)$ does $\sigma_*(M_C) = \sigma_*(A) \cup \sigma_*(B)$?
3. How to describe the passage of $\sigma_*(M_0)$ to $\sigma_*(M_C)$?

Where $\sigma_* \in \{\sigma_{lgD}, \sigma_{rgD}\}$.

In this paper, we motivated by the relationship between $\sigma_*(M_C)$ and $\sigma_*(A) \cup \sigma_*(B)$, where $\sigma_* \in \{\sigma_{lgD}, \sigma_{rgD}\}$. In addition, we show that the passage from $\sigma_*(M_0)$ to $\sigma_*(M_C)$ can be described as follows:

$$\sigma_*(M_C) \cup W = \sigma_*(M_0) = \sigma_*(A) \cup \sigma_*(B)$$

where W is the union of certain holes in $\sigma_*(M_C)$, which happen to be subsets of $\sigma_{lgD}(B) \cap \sigma_{rgD}(A)$, $\sigma_* \in \{\sigma_{lgD}, \sigma_{rgD}\}$.

2. Main results

We start this section by the following proposition.

Proposition 2.1. *Let $(A, B) \in \mathcal{B}(X) \times \mathcal{B}(Y)$ and $C \in \mathcal{B}(Y, X)$. Then :*

$$\sigma_{lgD}(M_0) = \sigma_{lgD}(A) \cup \sigma_{lgD}(B)$$

$$\sigma_{rgD}(M_0) = \sigma_{rgD}(A) \cup \sigma_{rgD}(B)$$

Proof. $\lambda \in \sigma_{lgD}(M_0)$ if and only if $\lambda \in acc(\sigma_{ap}(M_0))$ if and only if $\lambda \in acc(\sigma_{ap}(A) \cup \sigma_{ap}(B)) = acc(\sigma_{ap}(A)) \cup acc(\sigma_{ap}(B))$ if and only if $\lambda \in \sigma_{lgD}(A) \cup \sigma_{lgD}(B)$.

By duality, we have: $\sigma_{rgD}(M_0) = \sigma_{rgD}(A) \cup \sigma_{rgD}(B)$ □

As a straightforward consequence, we have the result of H.Zariouh and H. Zguitti [6].

Corollary 2.1. *[6] Let $(A, B) \in \mathcal{B}(X) \times \mathcal{B}(Y)$ and $C \in \mathcal{B}(Y, X)$. Then :*

$$\sigma_{gD}(M_0) = \sigma_{gD}(A) \cup \sigma_{gD}(B)$$

Theorem 2.1. *Let $(A, B) \in \mathcal{B}(X) \times \mathcal{B}(Y)$ and $C \in \mathcal{B}(Y, X)$. Then :*

$$\sigma_{lgD}(A) \subseteq \sigma_{lgD}(M_C) \subseteq \sigma_{lgD}(A) \cup \sigma_{lgD}(B) \subseteq \sigma_{lgD}(M_C) \cup \sigma_{lgD}(B)$$

$$\sigma_{rgD}(B) \subseteq \sigma_{rgD}(M_C) \subseteq \sigma_{rgD}(A) \cup \sigma_{rgD}(B) \subseteq \sigma_{rgD}(M_C) \cup \sigma_{rgD}(A)$$

Proof. Without loss of generality, let $\mu = 0 \notin \sigma_{lgD}(A) \cup \sigma_{lgD}(B)$, then $0 \notin acc(\sigma_{ap}(A)) \cup acc(\sigma_{ap}(B))$. Thus there exists $\varepsilon > 0$ such that for any λ, $0 < |\lambda| < \varepsilon$, we have $A - \lambda I$ and $B - \lambda I$ are bounded below. According to [7, Theorem 3.5], we have $M_C - \lambda I$ is bounded below for any λ, $0 < |\lambda| < \varepsilon$, thus $0 \notin acc(\sigma_{ap}(M_C)) = \sigma_{lgD}(M_C)$. Therefore $\sigma_{lgD}(M_C) \subseteq \sigma_{lgD}(A) \cup \sigma_{lgD}(B)$.
If $0 \notin \sigma_{lgD}(M_C)$, then $0 \notin acc(\sigma_{ap}(M_C))$, as a result, there exists $\varepsilon > 0$ such that for any λ, $0 < |\lambda| < \varepsilon$, we have $M_C - \lambda I$ is bounded below, then $A - \lambda I$ is bounded below for any λ, $0 < |\lambda| < \varepsilon$ by [7, Theorem 3.5], thus $0 \notin \sigma_{lgD}(A)$. Therefore $\sigma_{lgD}(A) \subseteq \sigma_{lgD}(M_C)$
Let $\mu = 0 \notin \sigma_{lgD}(M_C) \cup \sigma_{lgD}(B)$, then $0 \notin acc(\sigma_{ap}(M_C)) \cup acc(\sigma_{ap}(B))$. Thus there exists $\varepsilon > 0$ such that for any λ, $0 < |\lambda| < \varepsilon$, we have $M_C - \lambda I$ and $B - \lambda I$ are bounded below. According to [7, Theorem 3.5], we have $A - \lambda I$ is bounded below for any λ, $0 < |\lambda| < \varepsilon$, thus $0 \notin acc(\sigma_{ap}(A)) = \sigma_{lgD}(A)$. Therefore $\sigma_{lgD}(A) \cup \sigma_{lgD}(B) \subseteq \sigma_{lgD}(M_C) \cup \sigma_{lgD}(B)$.

By duality, we have:

$$\sigma_{rgD}(B) \subseteq \sigma_{rgD}(M_C) \subseteq \sigma_{rgD}(M_0) = \sigma_{rgD}(A) \cup \sigma_{rgD}(B) \subseteq \sigma_{rgD}(M_C) \cup \sigma_{rgD}(A)$$

□

The inclusion, $\sigma_{rgD}(M_C) \subseteq \sigma_{rgD}(A) \cup \sigma_{rgD}(B)$, may be strict as we can see in the following example.

Example 1. *Let $A, B, C \in \mathcal{B}(l^2)$ defined by:*

$$Ae_n = e_{n+1}.$$

$$B = A^*.$$

$$C = e_0 \otimes e_0.$$

where $\{e_n\}_{n\in\mathbf{N}}$ is the unit vector basis of l^2. We have $\sigma_{su}(A) = \{\lambda \in \mathbf{C}; |\lambda| \leq 1\}$, then $\sigma_{rgD}(A) = \{\lambda \in \mathbf{C}; |\lambda| \leq 1\}$. Since M_C is unitary, then $\sigma_{rgD}(M_C) \subseteq \{\lambda \in \mathbf{C}; |\lambda| = 1\}$. So $0 \notin \sigma_{rgD}(M_C)$, but $0 \in \sigma_{rgD}(A) \cup \sigma_{rgD}(B)$. Notes that $A^* = B$ has not the SVEP. Also, we can show that the inclusion $\sigma_{lgD}(M_C) \subset \sigma_{lgD}(A) \cup \sigma_{lgD}(B)$ is strict. This will lead us to a necessary condition that ensures the equality desired.

The following proposition will be needed in the sequel.

Proposition 2.2. *Let $(A, B) \in \mathcal{B}(X) \times \mathcal{B}(Y)$ and $C \in \mathcal{B}(Y, X)$.*

1. *If A is invertible, then $0 \notin \sigma_{lgD}(M_C)$ if and only if $0 \notin \sigma_{lgD}(B)$.*
2. *If B is invertible, then $0 \notin \sigma_{rgD}(M_C)$ if and only if $0 \notin \sigma_{rgD}(A)$.*

Proof. 1) Suppose that $0 \notin \sigma_{lgD}(M_C)$, $0 \notin acc(\sigma_{ap}(M_C))$, then there exists $\varepsilon > 0$ such that $M_C - \lambda I$ is bounded below for every $\lambda, 0 < |\lambda| < \varepsilon$. Since A is invertible, then there exists $\beta > 0$ such that $A - \lambda I$ is invertible for every $\lambda, |\lambda| < \beta$. Let $\eta = min(\varepsilon, \beta)$, $A - \lambda I$ is invertible for every $\lambda, |\lambda| < \eta$ and $M_C - \lambda I$ is bounded below for every $\lambda, 0 < |\lambda| < \eta$. Hence $B - \lambda I$ is bounded below for every $\lambda, 0 < |\lambda| < \eta$, by [9, Lemma 2.7], the converse is similar.
By duality, we have 2). □

Theorem 2.2. *Let $(A, B) \in \mathcal{B}(X) \times \mathcal{B}(Y)$ and $C \in \mathcal{B}(Y, X)$. Then :*

$$\sigma_{lgD}(M_C) \cup S(A^*) = \sigma_{lgD}(A) \cup \sigma_{lgD}(B) \cup S(A^*)$$

Proof. Since $\sigma_{lgD}(M_C) \subseteq \sigma_{lgD}(A) \cup \sigma_{lgD}(B)$, then $\sigma_{lgD}(M_C) \cup S(A^*) \subseteq \sigma_{lgD}(A) \cup \sigma_{lgD}(B) \cup S(A^*)$. Conversely, let $\lambda \in (\sigma_{lgD}(A) \cup \sigma_{lgD}(B)) \setminus \sigma_{lgD}(M_C)$, we can assume without loss of generality that $\lambda = 0$. Then $0 \notin acc(\sigma_{ap}(M_C))$, since $\sigma_{lgD}(A) \subseteq \sigma_{lgD}(M_C)$, then $0 \notin acc(\sigma_{ap}(A))$. Suppose that $0 \notin S(A^*)$:

- If $0 \in \sigma(A)$, since $\sigma(A) = \sigma_{ap}(A) \cup S(A^*)$ then $0 \in \sigma_{ap}(A)$. As $0 \notin acc(\sigma_{ap}(A))$ then $0 \in iso(\sigma_{ap}(A))$, therefore $0 \in iso(\sigma(A))$, which implies that there exists $\varepsilon > 0$ such that $\lambda I - A$ is invertible for all λ, $0 < |\lambda| < \varepsilon$, since $0 \notin acc(\sigma_{ap}(M_C))$, then there exists $\beta > 0$ such that $\lambda - M_C$ is bounded below for all $0 < |\lambda| < \beta$. Let $\alpha = min(\beta, \varepsilon)$, then $\lambda I - A$ is invertible and $\lambda I - M_C$ is bounded below for all $0 < |\lambda| < \alpha$, by [9, Lemma 2.7], we have $\lambda I - B$ is bounded below for all λ, $0 < |\lambda| < \alpha$, hence $0 \notin acc(\sigma_{ap}(B))$, thus $0 \notin \sigma_{lgD}(B)$. Then we have $0 \notin \sigma_{lgD}(A) \cup \sigma_{lgD}(B)$, contradiction.

- If $0 \notin \sigma(A)$ then A is invertible and since $0 \notin \sigma_{lgD}(M_C)$, according to Proposition 2.2, we have $0 \notin \sigma_{lgD}(B)$, thus $0 \notin \sigma_{lgD}(A) \cup \sigma_{lgD}(B)$, contradiction.
Then $(\sigma_{lgD}(A) \cup \sigma_{lgD}(B)) \backslash \sigma_{lgD}(M_C) \subseteq S(A^*)$, this finishes the proof.
□

Theorem 2.3. *Let $(A, B) \in \mathcal{B}(X) \times \mathcal{B}(Y)$ and $C \in \mathcal{B}(Y, X)$. Then :*

$$\sigma_{rgD}(M_C) \cup S(B) = \sigma_{rgD}(A) \cup \sigma_{rgD}(B) \cup S(B)$$

Proof. Since $\sigma_{rgD}(M_C) \subseteq \sigma_{rgD}(A) \cup \sigma_{rgD}(B)$, then $\sigma_{rgD}(M_C) \cup S(B) \subseteq \sigma_{rgD}(A) \cup \sigma_{rgD}(B) \cup S(B)$. Conversely, let $\lambda \in (\sigma_{rgD}(A) \cup \sigma_{rgD}(B)) \backslash \sigma_{rgD}(M_C)$, we can assume without loss of generality that $\lambda = 0$. Then $0 \notin acc(\sigma_{su}(M_C))$, since $\sigma_{rgD}(B) \subseteq \sigma_{rgD}(M_C)$, then $0 \notin acc(\sigma_{su}(B))$. Suppose that $0 \notin S(B)$:

- If $0 \in \sigma(B)$, since $\sigma(B) = \sigma_{su}(B) \cup S(B)$, then $0 \in \sigma_{su}(B)$. As $0 \notin acc(\sigma_{su}(B))$ then $0 \in iso(\sigma_{su}(B))$, therefore $0 \in iso(\sigma(B))$, which implies that there exists $\varepsilon > 0$ such that $\lambda I - B$ is invertible for all λ, $0 < |\lambda| < \varepsilon$, since $0 \notin acc(\sigma_{su}(M_C))$, then there exists $\beta > 0$ such that $\lambda I - M_C$ is surjective for all λ, $0 < |\lambda| < \beta$. Let $\alpha = min(\beta, \varepsilon)$, then $\lambda I - B$ is invertible and $\lambda I - M_C$ is surjective for all λ, $0 < |\lambda| < \alpha$, by [9, Lemma 2.7], we have $\lambda I - A$ is surjective for all λ, $0 < |\lambda| < \alpha$, hence $0 \notin acc(\sigma_{su}(A))$, thus $0 \notin \sigma_{rgD}(A)$. Then we have $0 \notin \sigma_{rgD}(A) \cup \sigma_{rgD}(B)$, contradiction.

- If $0 \notin \sigma(B)$ then B is invertible and since $0 \notin \sigma_{rgD}(M_C)$, according to Proposition 2.2, we have $0 \notin \sigma_{rgD}(A)$, thus $0 \notin \sigma_{rgD}(A) \cup \sigma_{rgD}(B)$, contradiction. □

Corollary 2.2. *1) Let $A \in \mathcal{B}(X)$. If $S(A^*) = \emptyset$, then for every $B \in \mathcal{B}(Y)$ and $C \in \mathcal{B}(Y, X)$, we have:*

$$\sigma_{lgD}(M_C) = \sigma_{lgD}(A) \cup \sigma_{lgD}(B)$$

2) Let $B \in \mathcal{B}(Y)$. If $S(B) = \emptyset$ then for every $A \in \mathcal{B}(X)$ and $C \in \mathcal{B}(Y, X)$, we have:

$$\sigma_{rgD}(M_C) = \sigma_{rgD}(A) \cup \sigma_{rgD}(B)$$

Lemma 2.1. *Let $T \in \mathcal{B}(X)$. Then:*

$$S(T) \subset \sigma_{lgD}(T) \text{ and } S(T^*) \subset \sigma_{rgD}(T).$$

Corollary 2.3. *Let $(A, B) \in \mathcal{B}(X) \times \mathcal{B}(Y)$ and $C \in \mathcal{B}(Y, X)$. If one of the following conditions holds:*

1. *$int(\sigma_p(A^*)) = \emptyset$,*
2. *$int(\sigma_{rgD}(A)) = \emptyset$,*
3. *$\sigma_{lgD}(B) \cap \sigma_{rgD}(A) = \emptyset$.*

then we have

$$\sigma_{lgD}(M_C) = \sigma_{lgD}(A) \cup \sigma_{lgD}(B)$$

Proof. Note that $S(A^*) \subseteq \sigma_p(A^*)$. If $int(\sigma_p(A^*)) = \emptyset$, by Corollary 2.2, we have $\sigma_{lgD}(M_C) = \sigma_{lgD}(A) \cup \sigma_{lgD}(B)$.
If $int(\sigma_{rgD}(A)) = \emptyset$, as $S(A^*) \subseteq \sigma_{rgD}(A)$, by Corollary 2.2 we have the result.
According to Theorem 2.1, we have $(\sigma_{lgD}(A) \cup \sigma_{lgD}(B)) \setminus \sigma_{lgD}(M_C) \subseteq \sigma_{lgD}(B)$ and from Theorem 2.2 $(\sigma_{lgD}(A) \cup \sigma_{lgD}(B)) \setminus \sigma_{lgD}(M_C) \subseteq S(A^*)$, since $S(A^*) \subseteq \sigma_{rgD}(A)$, then $(\sigma_{lgD}(A) \cup \sigma_{lgD}(B)) \setminus \sigma_{lgD}(M_C) \subseteq \sigma_{lgD}(B) \cap \sigma_{rgD}(A)$. If $\sigma_{lgD}(B) \cap \sigma_{rgD}(A) = \emptyset$, then $\sigma_{lgD}(M_C) = \sigma_{lgD}(A) \cup \sigma_{lgD}(B)$. □

Corollary 2.4. *Let $(A, B) \in \mathcal{B}(X) \times \mathcal{B}(Y)$ and $C \in \mathcal{B}(Y, X)$. If one of the following conditions holds:*

1. *$int(\sigma_p(B)) = \emptyset$,*
2. *$int(\sigma_{lgD}(B)) = \emptyset$,*
3. *$\sigma_{lgD}(B) \cap \sigma_{rgD}(A) = \emptyset$.*

then we have

$$\sigma_{rgD}(M_C) = \sigma_{rgD}(A) \cup \sigma_{rgD}(B)$$

Proof. Note that $S(B) \subseteq \sigma_p(B)$. If $int(\sigma_p(B)) = \emptyset$, by Corollary 2.2, we have $\sigma_{rgD}(M_C) = \sigma_{rgD}(A) \cup \sigma_{rgD}(B)$.
If $int(\sigma_{lgD}(B)) = \emptyset$, as $S(B) \subseteq \sigma_{lgD}(B)$, by Corollary 2.2 we have the result.
From Theorem 2.1, we have $(\sigma_{rgD}(A) \cup \sigma_{rgD}(B)) \setminus \sigma_{rgD}(M_C) \subseteq \sigma_{rgD}(A)$ and from Theorem 2.3 $(\sigma_{rgD}(A) \cup \sigma_{rgD}(B)) \setminus \sigma_{rgD}(M_C) \subseteq S(B)$, since $S(B) \subseteq \sigma_{lgD}(B)$, then $(\sigma_{rgD}(A) \cup \sigma_{rgD}(B)) \setminus \sigma_{rgD}(M_C) \subseteq \sigma_{lgD}(B) \cap \sigma_{rgD}(A)$. If $\sigma_{lgD}(B) \cap \sigma_{rgD}(A) = \emptyset$, then $\sigma_{lgD}(M_C) = \sigma_{lgD}(A) \cup \sigma_{lgD}(B)$. □

For $(A, B) \in \mathcal{B}(X) \times \mathcal{B}(Y)$, let L_A (resp. R_B) be the left (resp. right) multiplication operator given by $L_A(X) = AX$; (resp. $R_B(X) = XB$), and let $\delta_{A,B}(X) = AX - XB = L_A(X) - R_B(X)$ be the usual generalized derivation associated with A and B. When A = B, we simply write $\delta_{A,A} = \delta_A$. $N^\infty(A) = \bigcup_{n\geq 1} N(A^n)$ the generalized kernel of A.

The following theorem gives an answer to Question 2.

Theorem 2.4. *Let $A \in \mathcal{B}(X)$ and $B \in \mathcal{B}(Y)$. If C is in the closure of the set*

$$R(\delta_{A,B}) + N(\delta_{A,B}) + \bigcup_{\lambda,\mu\in\mathbf{C}} [N^\infty(L_{A-\lambda}) \cap N^\infty(R_{B-\mu})]$$

then :

$$\sigma_{lgD}(M_0) = \sigma_{lgD}(M_C)$$
$$\sigma_{rgD}(M_0) = \sigma_{rgD}(M_C)$$

Proof. If C is in the closure of the set

$$R(\delta_{A,B}) + N(\delta_{A,B}) + \bigcup_{\lambda,\mu\in\mathbf{C}} [N^\infty(L_{A-\lambda}) \cap N^\infty(R_{B-\mu})]$$

then, $\sigma_{ap}(M_C)\setminus\{0\} = \sigma_{ap}(M_0)\setminus\{0\}$ and $\sigma_{su}(M_C)\setminus\{0\} = \sigma_{su}(M_0)\setminus\{0\}$, see [2, Theorem 2.2], it remains to discuss the cas of the origin. Let $C \in N^\infty(L_{A-\lambda}) \cap N^\infty(R_{B-\mu})$, because of translation stability of spectra, we may assume $\lambda = \mu = 0$. If $C \in N(L_A^n)$ is a nonzero operator, then $0 \in \sigma_p(A) \subseteq \sigma_{ap}(A) \subseteq \sigma_{ap}(M_C) \cap \sigma_{ap}(M_0)$. On the other hand by duality, we use the assumption $C \in N(R_B^n)$ to obtain $0 \in \sigma_{su}(B) \subseteq \sigma_{su}(M_C) \cap \sigma_{su}(M_0)$. Finally $\sigma_{ap}(M_C) = \sigma_{ap}(M_0)$ and $\sigma_{su}(M_C) = \sigma_{su}(M_0)$. □

Remark 1. 1) *Let T defined on $l^2(\mathbf{N})$ by $T(x_1, x_2, ...) = (0, x_1, x_2, ...)$. Let $A = T$, $B = T^*$ and $C = A - B$, then $C \in R(\delta_{A,B})$, according to theorem 2.4, we have $\sigma_{lgD}(M_0) = \sigma_{lgD}(M_C)$ and $\sigma_{rgD}(M_0) = \sigma_{rgD}(M_C)$. But, $S(A^*) = S(B) = \{\lambda \in \mathbf{C}, |\lambda| < 1\}$. We conclude that in general there is no* *definite relation between the condition considered in corollary 2.2 and the condition considered in the above theorem.*

2) The closure of the set $R(\delta_{A,B})+N(\delta_{A,B})+\bigcup_{\lambda,\mu\in\mathbf{C}}[N^{\infty}(L_{A-\lambda})\cap N^{\infty}(R_{B-\mu})]$ *is not the largest class for which* $\sigma_{lgD}(M_0)=\sigma_{lgD}(M_C)$ *and* $\sigma_{rgD}(M_0)=\sigma_{rgD}(M_C)$ *hold. Indeed: Let* $A\in\mathcal{B}(X)$ *such that* $A^3=0\neq A^2$ *and* $A=B$. *For every* $C\in\mathcal{B}(X)$, *we have* $\sigma_{lgD}(M_0)=\sigma_{lgD}(M_C)$ *and* $\sigma_{rgD}(M_0)=\sigma_{rgD}(M_C)$. *A simple computation shows that* $\delta_A^5=0$ *then* $R(\delta_A)\subsetneq N(\delta_A^4)$. *If* $\lambda\in\mathbf{C}$, $C\in N(L_{(A-\lambda)^2})$, *then* $(\lambda^2-2\lambda A+A^2)C=0$. *Since* $A^3=0$, *we conclude that* $A^2C=0$ *and so* $C\in N(\delta_A^4)$. *Then*

$$\bigcup_{\lambda,\mu\in\mathbf{C}}[N^{\infty}(L_{A-\lambda})\cap N^{\infty}(R_{B-\mu})]\subseteq N(\delta_A^4)$$

Consequently, we have

$$cl(R(\delta_A)+N(\delta_A)+\bigcup_{\lambda,\mu\in\mathbf{C}}[N^{\infty}(L_{A-\lambda})\cap N^{\infty}(R_{B-\mu})])\subseteq N(\delta_A^4)\neq\mathcal{B}(X)$$

In [9], it was shown that the passage from $\sigma_{gD}(M_0)$ to $\sigma_{gD}(M_C)$ is accomplished removing certain open subsets of $\sigma_{gD}(A)\cap\sigma_{gD}(B)$ from the former, that is, there is equality

$$\sigma_{gD}(A)\cup\sigma_{gD}(B)=\sigma_{gD}(M_0)=\sigma_{gD}(M_C)\cup W$$

where W is the union of certain of the holes in $\sigma_{gD}(M_C)$ which happen to be subsets of $\sigma_{gD}(A)\cap\sigma_{gD}(B)$. The passage from $\sigma_{lgD}(M_0)$ (resp. $\sigma_{rgD}(M_0)$) to $\sigma_{lgD}(M_C)$ (resp. $\sigma_{rgD}(M_C)$) is more delicate.

Theorem 2.5. *Let* $(A,B)\in\mathcal{B}(X)\times\mathcal{B}(Y)$ *and* $C\in\mathcal{B}(Y,X)$. *Then*

$$\sigma_{lgD}(M_C)\cup W=\sigma_{lgD}(A)\cup\sigma_{lgD}(B)$$

where W *is the union of certain holes in* $\sigma_{lgD}(M_C)$, *which happen to be subsets of* $\sigma_{lgD}(B)\cap\sigma_{rgD}(A)$.

And

$$\sigma_{rgD}(M_C)\cup W'=\sigma_{rgD}(A)\cup\sigma_{rgD}(B)$$

where W' *is the union of certain holes in* $\sigma_{rgD}(M_C)$, *which happen to be subsets of* $\sigma_{lgD}(B)\cap\sigma_{rgD}(A)$.

Proof. First, we can claim that, for every $C \in \mathcal{B}(Y,X)$.

$$(\sigma_{lgD}(A) \cup \sigma_{lgD}(B)) \setminus \sigma_{lgD}(M_C) \subseteq \sigma_{lgD}(B) \cap \sigma_{rgD}(A) \quad (1)$$

$$\sigma_{lgD}(M_C) \subseteq \sigma_{lgD}(A) \cup \sigma_{lgD}(B)$$

Indeed, the second inclusion follows from Theorem 2.1. For the first inclusion, according to Theorem 2.1, we have $(\sigma_{lgD}(A) \cup \sigma_{lgD}(B)) \setminus \sigma_{lgD}(M_C) \subseteq \sigma_{lgD}(B)$ and from Theorem 2.2 $(\sigma_{lgD}(A) \cup \sigma_{lgD}(B)) \setminus \sigma_{lgD}(M_C) \subseteq S(A^*)$, since $S(A^*) \subseteq \sigma_{rgD}(A)$, then $(\sigma_{lgD}(A) \cup \sigma_{lgD}(B)) \setminus \sigma_{lgD}(M_C) \subseteq \sigma_{lgD}(B) \cap \sigma_{rgD}(A)$.
Next we claim that, for every $T \in \mathcal{B}(X)$, we have

$$\eta(\sigma_{lgD}(T)) = \eta(\sigma_{gD}(T)) \quad (2)$$

where $\eta(K)$ denotes the polynomially convex hull of the compact subset K of $\mathbf{C}$.

Since $\sigma_{lgD}(T) \subseteq \sigma_{gD}(T)$, we need to prove $\partial\sigma_{gD}(T) \subseteq \partial\sigma_{lgD}(T)$. But since $int(\sigma_{lgD}(T)) \subseteq int(\sigma_{gD}(T))$, it suffices to show that $\partial\sigma_{gD}(T) \subseteq \sigma_{lgD}(T)$. Without loss of generality, suppose $0 \in \partial\sigma_{gD}(T)$. There are two cases to consider.

Case 1: If $0 \in acc(\partial\sigma_{gD}(T))$, then there exists $(\lambda_n) \subseteq \partial(\sigma_{gD}(T))$, such that $\lim_{n\to\infty} \lambda_n = 0$, since

$$\partial(\sigma_{gD}(T)) = \partial(acc\sigma(T)) \subseteq acc\sigma(T)\setminus int\sigma(T) \subseteq \partial(\sigma(T)) \subseteq \sigma_{ap}(T)$$

we have, $\lambda_n \in \sigma_{ap}(T)$, $n = 1, 2, ...$, hence $0 \in acc(\sigma_{ap}(T)) = \sigma_{lgD}(T)$.

Case 2: If $0 \in iso(\partial\sigma_{gD}(T))$, since $\sigma_{gD}(T)$ is closed, then $iso(\partial\sigma_{gD}(T)) = iso(\sigma_{gD}(T))$. $0 \in iso(\sigma_{gD}(T)) = iso(acc\sigma(T))$, then $0 \in acc\sigma(T)$ and there exists $\varepsilon > 0$ such that $\lambda \notin acc(\sigma(T))$ for every λ, $0 < |\lambda| < \varepsilon$. Since $0 \in acc\sigma(T)$, there exists $(\mu_n) \subseteq \sigma(T)$ such that $\lim_{n\to\infty} \mu_n = 0$, $\mu_n \neq 0$ for all n, thus there exists certain positive integer N such that $0 < |\mu_n| < \varepsilon$ for any $n \geq N$. Let $\lambda_n = \mu_{N+1+n}$, then $\lambda_n \in iso(\sigma(T))$ $n = 1, 2, ..$ and $\lim_{n\to\infty} \lambda_n = 0$. Since $\sigma(T)$ is closed, then

$$iso(\sigma(T)) \subseteq \partial(\sigma(T)) \subseteq \sigma_{ap}(T)$$

Then, $\lambda_n \in iso(\sigma(T)) \subseteq \sigma_{ap}(T)$ $n = 1, 2, ...$ Since $\lim\limits_{n\to\infty} \lambda_n = 0$, then $0 \in acc(\sigma_{ap}(T))$. So $0 \in \sigma_{lgD}(T)$.
Then $\partial\sigma_{gD}(T) \subseteq \sigma_{lgD}(T)$. This proves (2). Similarly, for every $T \in \mathcal{B}(X)$ and $S \in \mathcal{B}(Y)$, $\eta(\sigma_{lgD}(T) \cup \sigma_{lgD}(S)) = \eta(\sigma_{gD}(T) \cup \sigma_{gD}(S))$. From [8], if $(A, B) \in \mathcal{B}(X) \times \mathcal{B}(Y)$ and $C \in \mathcal{B}(Y, X)$, we have

$$\eta(\sigma_{gD}(M_C)) = \eta(\sigma_{gD}(A) \cup \sigma_{gD}(B))$$

Then

$$\eta(\sigma_{lgD}(M_C)) = \eta(\sigma_{gD}(M_C)) = \eta(\sigma_{gD}(A) \cup \sigma_{gD}(B)) = \eta(\sigma_{lgD}(A) \cup \sigma_{lgD}(B))$$

Hence

$$\eta(\sigma_{lgD}(M_C)) = \eta(\sigma_{lgD}(A) \cup \sigma_{lgD}(B)) \quad (3)$$

(3) says that the passage from $\sigma_{lgD}(M_C)$ to $\sigma_{lgD}(A) \cup \sigma_{lgD}(B)$ is the filling in certain of the holes in $\sigma_{lgD}(M_C)$. But since $(\sigma_{lgD}(A) \cup \sigma_{lgD}(B)) \setminus \sigma_{lgD}(M_C)$ is contained in $\sigma_{lgD}(B) \cap \sigma_{rgD}(A)$, it follows that the filling in certain of the holes in $\sigma_{lgD}(M_C)$ should occur in $\sigma_{lgD}(B) \cap \sigma_{rgD}(A)$.

Similarly we have 2) □

Corollary 2.5. *Let $(A, B) \in \mathcal{B}(X) \times \mathcal{B}(Y)$. If $\sigma_{lgD}(B) \cap \sigma_{rgD}(A)$ has no interior points, then for every $C \in \mathcal{B}(Y, X)$, we have*

1. $\sigma_{lgD}(M_C) = \sigma_{lgD}(A) \cup \sigma_{lgD}(B)$
2. $\sigma_{rgD}(M_C) = \sigma_{rgD}(A) \cup \sigma_{rgD}(B)$
3. $\sigma_{gD}(M_C) = \sigma_{gD}(A) \cup \sigma_{gD}(B)$

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