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Fine spectrum of the upper triangular matrix $U(r, 0, 0, s)$ over the sequence spaces c_0 and c

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Abstract

Fine spectra of various matrices have been examined by several authors. In this article we have determined the fine spectrum of the upper triangular matrix $U(r, 0, 0, s)$ on the sequence spaces c_0 and c.

Key Words : *Spectrum of an operator; matrix mapping; sequence space; upper triangular matrix; fine spectrum.*

AMS Classification : *47A10; 47B37; 40C05; 40C15; 40D20; 40H05.*

1. Introduction

The study of spectrum and fine spectrum for various operators are made by various authors. Okutoyi [15] determined the spectrum of the Cesàro operator C_1 on the sequence space bv_0. The fine spectra of the Cesàro operator C_1 over the sequence space bv_p, $(1 \leq p < \infty)$ was determined by Akhmedov and Başar [2]. Altay and Başar [3, 4] determined the fine spectrum of the difference operator Δ and the generalized difference operator $B(r, s)$ on the sequence spaces c_0 and c. The spectrum and fine spectrum of the Zweier Matrix on the sequence spaces ℓ_1 and bv were studied by Altay and Karakuş [5]. Altun [6, 7] determined the fine spectra of triangular Toeplitz operators and tridiagonal symmetric matrices over some sequence spaces. Fine spectra of operator $B(r, s, t)$ over the sequence spaces ℓ_1 and bv and generalized difference operator $B(r, s)$ over the sequence spaces ℓ_p and bv_p, $(1 \leq p < \infty)$ were studied by Bilgiç and Furkan [9, 10]. Akhmedov and El-Shabrawy [1] determined the fine spectrum of the operator $\Delta_{a,b}$ on the sequence space c. Panigrahi and Srivastava [16, 17] studied the spectrum and fine spectrum of the second order difference operator Δ^2_{uv} on the sequence space c_0 and generalized second order forward difference operator Δ^2_{uvw} on the sequence space ℓ_1. Fine spectrum of the generalized difference operator Δ_v on the sequence space ℓ_1 was investigated by Srivastava and Kumar [19]. Fine spectra of upper triangular double-band matrix $U(r, s)$ over the sequence spaces c_0 and c were studied by Karakaya and Altun [14]. The spectra of some matrix classes has been investigated recently by Rhoades [18], Tripathy and Das [20, 21, 22], Tripathy and Pal [23, 24, 25, 26, 27] and Tripathy and Saikia [28].

In this paper, we shall determine the spectrum and fine spectrum of the upper triangular matrix $U(r, 0, 0, s)$ over the sequence spaces c_0 and c, where $U(r, 0, 0, s) =$

$$u_{nk} \text{ such that } u_n k = \begin{cases} r, & \text{if} \quad n = k \\ s, & \text{if} \quad n + 3 = k \\ 0, & \text{otherwise} \end{cases} \quad \text{for all } n, k \in \mathbf{N}_0 \text{ and } s \neq 0.$$

2. Preliminaries and Background

Let X and Y be Banach spaces and $T : X \to Y$ be a bounded linear operator. By $R(T)$, we denote the range of T, i.e.

$$R(T) = \{y \in Y : y = Tx, x \in X\}.$$

By $B(X)$,we denote the set of all bounded linear operators on X into itself. If $T \in B(X)$, then the adjoint T^* of T is a bounded linear operator on the dual X^* of X defined by $(T^*f)(x) = f(Tx)$, for all $f \in X^*$ and $x \in X$. Let $X \neq \{\theta\}$ be a complex normed linear space, where θ is the zero element and $T : D(T) \to X$ be a linear operator with domain $D(T) \subseteq X$. With T, we associate the operator

$$T_\lambda = T - \lambda I,$$

where λ is a complex number and I is the identity operator on $D(T)$. If T_λ has an inverse which is linear, we denote it by T_λ^{-1}, that is

$$T_\lambda^{-1} = (T - \lambda I)^{-1},$$

and call it the *resolvent* operator of T.

A *regular value* λ of T is a complex number such that

(R1): T_λ^{-1} exists,

(R2): T_λ^{-1} is bounded

(R3): T_λ^{-1} is defined on a set which is dense in X i.e. $\overline{R(T_\lambda)} = X$.

The *resolvent set* of T, denoted by $\rho(T, X)$, is the set of all regular values λ of T. Its complement $\sigma(T, X) = \mathbf{C} - \rho(T, X)$ in the complex plane $\mathbf{C}$ is called the *spectrum* of T. Furthermore, the spectrum $\sigma(T, X)$ is partitioned into three disjoint sets as follows:

The *point(discrete) spectrum* $\sigma_p(T, X)$ is the set such that T_λ^{-1} does not exist. Any such $\lambda \in \sigma_p(T, X)$ is called an eigenvalue of T.

The *continuous spectrum* $\sigma_c(T, X)$ is the set such that T_λ^{-1} exists and satisfies $(R3)$, but not $(R2)$, that is, T_λ^{-1} is unbounded.

The *residual spectrum* $\sigma_r(T, X)$ is the set such that T_λ^{-1} exists (and may be bounded or not), but does not satisfy $(R3)$, that is, the domain of T_λ^{-1} is not dense in X.

From Goldberg [13], if X is a Banach space and $T \in B(X)$, then there are three possibilities for $R(T)$ and T^{-1}:

(I) $R(T) = X$,

(II) $R(T) \neq \overline{R(T)} = X$

(III) $\overline{R(T)} \neq X$

and

(1) T^{-1} exists and is continuous,

(2) T^{-1} exists but is discontinuous,

(3) T^{-1} does not exist.

Applying Goldberg [13] classification to T_λ , we have three possibilities for T_λ and T_λ^{-1};

(I) T_λ is surjective,

(II) $R(T_\lambda) \neq \overline{R(T_\lambda)} = X$,

(III) $\overline{R(T_\lambda)} \neq X$,

and

(1) T_λ is injective and T_λ^{-1} is continuous,

(2) T_λ is injective but T_λ^{-1} is discontinuous,

(3) T_λ is not injective.

If these possibilities are combined in all possible ways, nine different states are created which may be shown as in the Table 2.1.

	I	II	III
1	$\rho(T, X)$	$\rho(T, X)$	$\sigma_r(T, X)$
2	$\cdots$	$\sigma_c(T, X)$	$\sigma_r(T, X)$
3	$\sigma_p(T, X)$	$\sigma_p(T, X)$	$\sigma_p(T, X)$

Table 2.1: Subdivisions of spectrum of a linear operator

These are labeled by: I_1,I_2,I_3,II_1, II_2, II_3, III_1, III_2 and III_3 . If λ is a complex number such that $T_\lambda \in I_1$ or $T_\lambda \in I_2$,then λ is in the resolvent set $\rho(T,X)$ of T . The further classification gives rise to the fine spectrum of T . If an operator is in state II_2 for example, then $R(T) \neq \overline{R(T)} = X$ and T^{-1} exists but is discontinuous and we write $\lambda \in II_2\sigma(T,X)$.

By w, we denote the space of all real or complex valued sequences. Any vector subspace of w is called a sequence space. Throughout the paper c, c_0, ℓ_1, ℓ_∞ represent the spaces of all convergent, null, absolutely summable and bounded sequences respectively.

Let λ and μ be two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real or complex numbers a_{nk}, where $n, k \in \mathbf{N}_0 = \{0, 1, 2, ...\}$. Then, we say that A defines a matrix mapping from λ into μ, and we denote it by $A : \lambda \to \mu$, if for every sequence $x = (x_k) \in \lambda$, the sequence $Ax = \{(Ax)_n\}$, the A-transform of x, is in μ, where

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk}x_k, n \in \mathbf{N}_0. \tag{2.1}$$

By $(\lambda : \mu)$, we denote the class of all matrices such that $A : \lambda \to \mu$. Thus, $A \in (\lambda : \mu)$ if and only if the series on the right hand side of equation (2.1) converges for each $n \in \mathbf{N}_0$ and every $x \in \lambda$ and we have $Ax = \{(Ax)_n\}_{n\in\mathbf{N}_0} \in \mu$ for all $x \in \lambda$.

The upper triangular matrix $U(r,0,0,s)$ is an infinite matrix of the form

$$U(r,0,0,s) = \left\{ \begin{array}{ccccccc} r & 0 & 0 & s & 0 & 0 & \cdots \\ 0 & r & 0 & 0 & s & 0 & \cdots \\ 0 & 0 & r & 0 & 0 & s & \cdots \\ 0 & 0 & 0 & r & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array} \right.$$

where $s \neq 0$.

The following results will be used in order to establish the results of this article.

Lemma 2.1. [Wilansky [29] Theorem 1.3.6, Page 6] *The matrix $A = (a_{nk})$ gives rise to a bounded linear operator $T \in B(c)$ from c to itself if and only if:*

(i) the rows of A are in ℓ_1 and their ℓ_1 norms are bounded,

(ii) *the columns of A are in c ,*

(iii) *the sequence of row sums of A is in c .*

The operator norm T is the supremum of ℓ_1 norms of the rows.

Corollary 2.1. *$U(r,0,0,s) : c \to c$ is a bounded linear operator and $||U(r,0,0,s)||_{(c:c)} = |r| + |s|$.*

Lemma 2.2. [Wilansky [29] Example 8.4.5 A, Page 129] *The matrix $A = (a_{nk})$ gives rise to a bounded linear operator $T \in B(c_0)$ from c_0 to itself if and only if:*

(i) *the rows of A are in ℓ_1 and their ℓ_1 norms are bounded,*

(ii) *the columns of A are in c .*

The operator norm T is the supremum of ℓ_1 norms of the rows.

Corollary 2.2. *$U(r,0,0,s) : c_0 \to c_0$ is a bounded linear operator and $||U(r,0,0,s)||_{(c_0:c_0)} = |r| + |s|$.*

Lemma 2.3. [Goldberg [13], Page 59] *T has a dense range if and only if T^* is one to one.*

Lemma 2.4. [Goldberg [13], Page 60] *T has a bounded inverse if and only if T^* is onto.*

3. Fine spectrum of the operator $U(r,0,0,s)$ on the sequence space c_0

From now onwards we denote the matrix $U(r,0,0,s)$ by U.
The following result will be used for establishing some results of this section.

Lemma 3.1. [Akhmedov and El-Shabrawy [1], Lemma 2.1] *Let (c_n) and (d_n) be two sequences of complex numbers such that $\lim_{n\to\infty} c_n = c$ and $|c| < 1$. Define the sequence (z_n) of complex numbers such that $z_{n+1} = c_{n+1}z_n + d_{n+1}$ for all $n \in \mathbf{N}_0$.Then*

(i) *if (d_n) is bounded, then (z_n) is bounded.*

(ii) *if (d_n) is convergent then (z_n) is convergent.*

(iii) if (d_n) *is a a null sequence, then* (z_n) *is a null sequence.*

In view of Lemma 3.1 we can formulate the following result:

Lemma 3.2. *Let* (c_n) *and* (d_n) *be two sequences of complex numbers such that* $\lim_{n\to\infty} c_n = c$ *and* $|c| < 1$. *Define the sequence* (z_n) *of complex numbers such that* $z_{n+k} = c_n z_n + d_n$ *for all* $n \in \mathbf{N}_0$.*Then*

(i) if (d_n) *is bounded, then* (z_n) *is bounded.*

(ii) if (d_n) *is convergent then* (z_n) *is convergent.*

(iii) if (d_n) *is a a null sequence, then* (z_n) *is a null sequence.*

Theorem 3.1. *The point spectrum of the operator* U *over* c_0 *is given by*

$$\sigma_p(U, c_0) = \{\lambda \in \mathbf{C} : |\lambda - r| < |s|\}.$$

Proof. Let λ be an eigenvalue of the operator U . Then there exists $x \neq \theta = (0,0,0,0,\ldots)$ in c_0 such that $Ux = \lambda x$. Then, we have

$$\begin{aligned}
rx_0 + sx_3 &= \lambda x_0 \\
rx_1 + sx_4 &= \lambda x_1 \\
rx_2 + sx_5 &= \lambda x_2 \\
&\cdots \\
rx_n + sx_{n+3} &= \lambda x_n, \quad n \geq 0
\end{aligned}$$

Without loss of any generality we may assume that $x_0 \neq 0$. Then, by using the recurrence relation $rx_n + sx_{n+3} = \lambda x_n, \quad n \geq 0$, we have

$$\begin{aligned}
x_3 &= \frac{\lambda - r}{s} x_0 \\
x_6 &= \frac{\lambda - r}{s} x_3 = \left(\frac{\lambda - r}{s}\right)^2 x_0 \\
x_9 &= \frac{\lambda - r}{s} x_6 = \left(\frac{\lambda - r}{s}\right)^3 x_0 \\
&\cdots \\
x_{3n} &= \left(\frac{\lambda - r}{s}\right)^n x_0, \quad n \geq 0
\end{aligned}$$

Since, $x = (x_n) \in c_0$, so the subsequence (x_{3n}) also converges to 0. But (x_{3n}) converges to 0 if and only if $|\lambda - r| < |s|$.

Hence, $\sigma_p(U, c_0) = \{\lambda \in \mathbf{C} : |\lambda - r| < |s|\}$. □

If $T : c_0 \to c_0$ is a bounded linear operator represented by a matrix A, then it is known that the adjoint operator $T^* : c_0^* \to c_0^*$ is defined by the transpose A^t of the matrix A. It should be noted that the dual space c_0^* of c_0 is isometrically isomorphic to the Banach space ℓ_1 of all absolutely summable sequences normed by $||x|| = \sum\limits_{n=0}^{\infty} |x_n|$.

Theorem 3.2. *The point spectrum of the operator U^* over c_0^* is given by*

$$\sigma_p(U^*, c_0^* \cong \ell_1) = \emptyset.$$

Proof. Let λ be an eigenvalue of the operator U^* . Then there exists $x \neq \theta = (0, 0, 0, ...)$ in ℓ_1 such that $U^*x = \lambda x$. Then, we have

$$\begin{aligned} rx_0 &= \lambda x_0 \\ rx_1 &= \lambda x_1 \\ rx_2 &= \lambda x_2 \\ sx_0 + rx_3 &= \lambda x_3 \\ sx_1 + rx_4 &= \lambda x_4 \\ sx_2 + rx_5 &= \lambda x_5 \\ \cdots & \\ sx_{n-3} + rx_n &= \lambda x_n, \quad n \geq 3 \end{aligned}$$

If x_k is the first non-zero entry of the sequence (x_n) , then $\lambda = r$. Then from the relation $sx_k + rx_{k+3} = \lambda x_{k+3}$, we have $sx_k = 0$. But $s \neq 0$ and hence, $x_k = 0$, a contradiction. Hence, $\sigma_p(U^*, c_0^* \cong \ell_1) = \emptyset$. □

Theorem 3.3. *For any $\lambda \in \mathbf{C}$, $U - \lambda I$ has a dense range.*

Proof. By Theorem 3.2, $\sigma_p(U^*, c_0^* \cong \ell_1) = \emptyset$.
Hence, $U^* - \lambda I$ i.e. $(U - \lambda I)^*$ is one to one for all $\lambda \in \mathbf{C}$. So, by applying Lemma 2.3, we get the required result. □

Corollary 3.1. *The residual spectrum of the operator U over c_0 is given by $\sigma_r(U, c_0) = \emptyset$.*

Proof. Since, $U-\lambda I$ has a dense range for all $\lambda \in \mathbf{C}$, so $\sigma_r(U,c_0)=\emptyset$. $\square$

Theorem 3.4. *The continuous spectrum and the spectrum of the operator U over c_0 are respectively given by $\sigma_c(U,c_0)=\{\lambda \in \mathbf{C} : |\lambda - r| = |s|\}$ and $\sigma(U,c_0)=\{\lambda \in \mathbf{C} : |\lambda - r| \le |s|\}$.*

Proof. Let $y=(y_n) \in \ell_1$ be such that $U_\lambda^* x = y$ for some $x=(x_n)$. Then we have following system of linear equations:

$$
\begin{aligned}
(r-\lambda)x_0 &= y_0 \\
(r-\lambda)x_1 &= y_1 \\
(r-\lambda)x_2 &= y_2 \\
sx_0+(r-\lambda)x_3 &= y_3 \\
sx_1+(r-\lambda)x_4 &= y_4 \\
&\cdots \\
sx_{n-3}+(r-\lambda)x_n &= y_n, \quad n \ge 3
\end{aligned}
$$

Solving these equations we get,

$$
\begin{aligned}
x_0 &= \frac{1}{r-\lambda}y_0 \\
x_1 &= \frac{1}{r-\lambda}y_1 \\
x_2 &= \frac{1}{r-\lambda}y_2 \\
x_3 &= \frac{1}{r-\lambda}(y_3 - sx_0) = \frac{1}{r-\lambda}y_3 - \frac{s}{(r-\lambda)^2}y_0 \\
x_4 &= \frac{1}{r-\lambda}(y_4 - sx_1) = \frac{1}{r-\lambda}y_4 - \frac{s}{(r-\lambda)^2}y_1 \\
x_5 &= \frac{1}{r-\lambda}(y_5 - sx_2) = \frac{1}{r-\lambda}y_5 - \frac{s}{(r-\lambda)^2}y_2 \\
x_6 &= \frac{1}{r-\lambda}(y_6 - sx_3) = \frac{1}{r-\lambda}y_6 - \frac{s}{(r-\lambda)^2}y_3 + \frac{s^2}{(r-\lambda)^3}y_0 \\
x_7 &= \frac{1}{r-\lambda}(y_7 - sx_4) = \frac{1}{r-\lambda}y_7 - \frac{s}{(r-\lambda)^2}y_4 + \frac{s^2}{(r-\lambda)^3}y_1 \\
x_8 &= \frac{1}{r-\lambda}(y_8 - sx_5) = \frac{1}{r-\lambda}y_8 - \frac{s}{(r-\lambda)^2}y_5 + \frac{s^2}{(r-\lambda)^3}y_2
\end{aligned}
$$

$\cdots$

Now,

$$\begin{aligned}\sum_{n=0}^{\infty}|x_n| &\leq \frac{1}{|r-\lambda|}\sum_{n=0}^{\infty}|y_n| + \frac{|s|}{|r-\lambda|^2}\sum_{n=0}^{\infty}|y_n| + \frac{|s|^2}{|r-\lambda|^3}\sum_{n=0}^{\infty}|y_n| + \cdots \\ &= \frac{1}{|r-\lambda|}\left(1+\frac{|s|}{|r-\lambda|}+\frac{|s|^2}{|r-\lambda|^2}+\cdots\right)||y||\end{aligned}$$

as $y=(y_n)\in\ell_1$.

Let $\lambda\in\mathbf{C}$ be such that $|r-\lambda|>|s|$. Then $\sum_{n=0}^{\infty}|x_n|\leq\frac{1}{|r-\lambda|-|s|}||y||$. Therefore, $x=(x_n)\in\ell_1$ and hence, for $|r-\lambda|>|s|$ the operator $U_\lambda^*=(U-\lambda I)^*$ is onto. So, by Lemma 2.4, $U_\lambda=U-\lambda I$ has a bounded inverse for all $\lambda\in\mathbf{C}$ be such that $|r-\lambda|>|s|$.
Therefore, $\lambda\notin\{\lambda\in\mathbf{C}:|r-\lambda|\leq|s|\}\Rightarrow\lambda\notin\sigma_c(U,c_0)$ and so,
$\sigma_c(U,c_0)\subseteq\{\lambda\in\mathbf{C}:|r-\lambda|\leq|s|\}$.

Now,

$$\sigma(U,c_0)=\sigma_p(U,c_0)\cup\sigma_r(U,c_0)\cup\sigma_c(U,c_0)\subseteq\{\lambda\in\mathbf{C}:|r-\lambda|\leq|s|\}.$$

By Theorem 3.1, we have

$$\{\lambda\in\mathbf{C}:|r-\lambda|<|s|\}=\sigma_p(U,c_0)\subset\sigma(U,c_0)$$

Since $\sigma(U,c_0)$ is a compact set, so it is closed and hence,

$$\overline{\{\lambda\in\mathbf{C}:|r-\lambda|<|s|\}}\subset\overline{\sigma(U,c_0)}=\sigma(U,c_0)$$

and therefore,$\{\lambda\in\mathbf{C}:|r-\lambda|\leq|s|\}\subset\sigma(U,c_0)$.

Hence, $\sigma(U,c_0)=\{\lambda\in\mathbf{C}:|\lambda-r|\leq|s|\}$.
Since $\sigma(U,c_0)$ is disjoint union of $\sigma_p(U,c_0)$,$\sigma_r(U,c_0)$ and $\sigma_c(U,c_0)$, therefore $\sigma_c(U,c_0)=\{\lambda\in\mathbf{C}:|\lambda-r|=|s|\}$.
□

Theorem 3.5. *If $|\lambda-r|<|s|$, then $\lambda\in I_3\sigma(U,c_0)$.*

Proof. Let $|\lambda - r| < |s|$. Then by Theorem 3.1, $\lambda \in \sigma_p(U, c_0)$.So λ satisfies Goldberg's condition 3.

To get the result we need to show that $U - \lambda I$ is surjective when $|\lambda - r| < |s|$.
Let $y = (y_n) \in c_0$ be such that $U_\lambda x = y$ for some $x = (x_n)$.

Then

$$\begin{aligned}
(r-\lambda)x_{3n} + sx_{3n+3} &= y_{3n}; \quad n \geq 0 \\
(r-\lambda)x_{3n+1} + sx_{3n+4} &= y_{3n+1}; \quad n \geq 0 \\
(r-\lambda)x_{3n} + sx_{3n+3} &= y_{3n+2}; \quad n \geq 0
\end{aligned}$$

Now,

$$\begin{aligned}
(r-\lambda)x_{3n} + sx_{3n+3} &= y_{3n} \\
\Rightarrow x_{3n+3} &= -\frac{(r-\lambda)}{s}x_{3n} + \frac{1}{s}y_{3n}
\end{aligned}$$

Let $x_{3n} = z_n$, $x_{3n+3} = x_{3(n+1)} = z_{n+1}$, $c_n = -\frac{(r-\lambda)}{s}$, $d_{n+1} = \frac{1}{s}y_{3n}$. Then $z_{n+1} = c_{n+1}z_n + d_{n+1}$. Now, $\lim\limits_{n\to\infty} c_n = -\frac{(r-\lambda)}{s} = c$ and $|c| = |\frac{r-\lambda}{s}| < 1$.

Also, as $y = (y_n) \in c_0$, so $(d_{n+1}) \in c_0$.
Hence, by Lemma 3.1(iii), $(z_n) = (x_{3n}) \in c_0$.

Similarly we can show that (x_{3n+1}), $(x_{3n+2}) \in c_0$. Therefore, $(x_n) \in c_0$ if and only if $|\lambda - r| < |s|$. Therefore, $U - \lambda I$ is onto if $|\lambda - r| < |s|$. So, λ satisfies Goldberg's condition I .Hence the result.
□

4. Fine spectrum of the operator $U(r,0,0,s)$ on the sequence space c

Theorem 4.1. *The point spectrum of the operator* U *over* c *is given by*

$$\sigma_p(U, c) = \{\lambda \in \mathbf{C} : |\lambda - r| < |s|\} \cup \{r + s\}.$$

Proof. Let λ be an eigenvalue of the operator U . Then there exists $x \neq \theta = (0,0,0,0,\dots)$ in c such that $Ux = \lambda x$. Then, we have

$$\begin{aligned} rx_0 + sx_3 &= \lambda x_0 \\ rx_1 + sx_4 &= \lambda x_1 \\ rx_2 + sx_5 &= \lambda x_2 \\ &\dots \\ rx_n + sx_{n+3} &= \lambda x_n, \quad n \geq 0 \end{aligned}$$

If $x = (x_n) \in c$ is a constant sequence, then $\lambda = r + s$. Then $\lambda = r + s$ is an eigen value of the operator U.
If $x = (x_n) \in c$ is not a constant sequence, then proceeding exactly as Theorem 3.1 we get a subsequence of $x = (x_n)$ which is also convergent if and only if $|\lambda - r| < |s|$.

Hence, $\sigma_p(U, c) = \{\lambda \in \mathbf{C} : |\lambda - r| < |s|\} \cup \{r + s\}$. $\square$

If $T : c \to c$ is a bounded linear operator represented by a matrix A, then $T^* : c^* \to c^*$ acting on $\mathbf{C} \oplus \ell_1$ has a matrix representation of the form $\begin{pmatrix} \chi & 0 \\ b & A^t \end{pmatrix}$, where is χ the limit of the sequence of row sums of A minus the sum of the limit of the columns of A, and b is the column vector whose k^{th} entry is the limit of the k^{th} column of A for each $n \in \mathbf{N}_0$.

For $U = U(r, 0, 0, s) : c \to c$, the matrix of the operator $U^* = U(r, 0, 0, s)^* \in B(\ell_1)$ is of the form

$$U^* = U(r,0,0,s)^* = \begin{pmatrix} r+s & 0 \\ 0 & U(r,0,0,s)^t \end{pmatrix} = \begin{pmatrix} r+s & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & r & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & r & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & r & 0 & 0 & \cdots \\ 0 & s & 0 & 0 & r & 0 & \cdots \\ 0 & 0 & s & 0 & 0 & r & \cdots \\ 0 & 0 & 0 & s & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Theorem 4.2. *The point spectrum of the operator U^* over c^* is given by*

$$\sigma_p(U^*, c^* \cong \mathbf{C} \oplus \ell_1) = \{r + s\}.$$

Proof. Let λ be an eigenvalue of the opertaor U^*. Then there exists $x \neq \theta = (0,0,0,\cdots)$ in ℓ_1 such that $U^*x = \lambda x$.
Then, we have

$$\begin{aligned}
(r+s)x_0 &= \lambda x_0 \\
rx_1 &= \lambda x_1 \\
rx_2 &= \lambda x_2 \\
rx_3 &= \lambda x_3 \\
sx_1 + rx_4 &= \lambda x_4 \\
sx_2 + rx_5 &= \lambda x_5 \\
\cdots\cdots & \\
sx_{n-3} + rx_n &= \lambda x_n, \qquad n \geq 4
\end{aligned}$$

If $x_0 \neq 0$, then $\lambda = r+s$. Thus, $\lambda = r+s$ is an eigenvalue of U^* corresponding to the eigenvector $(x_0,0,0,\cdots)$ with $x_0 \neq 0$.
Let $\lambda \neq r+s$. Then, $x_0 = 0$.
Let x_k be the first non-zero entry of the sequence $x = (x_n)$. Then from the relation $sx_{k-3} + rx_k = \lambda x_k$, we get $\lambda = r$. Now from the relation $sx_k + rx_{k+3} = \lambda x_{k+3}$, we get $sx_k = 0$. Since, $s \neq 0$ so we have $x_k = 0$, a contradiction. So, U^* does not have any other eigenvalue other than $\lambda = r+s$.
Hence, $\sigma_p(U^*, c^* \cong \mathbf{C} \oplus \ell_1) = \{r+s\}$. $\square$

Theorem 4.3. *$U_\lambda : c \to c$ has a dense range if and only if $\lambda \neq r+s$.*

Proof. By Theorem 4.2, $\sigma_p(U^*, c^* \cong \mathbf{C} \oplus \ell_1) = \{r+s\}$.
Hence, $U^* - \lambda I$ i.e. $U^*_\lambda = (U - \lambda I)^*$ is one to one for all $\lambda \in \mathbf{C} \setminus \{r+s\}$.
So, by applying Lemma 2.3, we get the result. $\square$

Corollary 4.1. *The residual spectrum of the operator U over c is given by*
$\sigma_r(U,c) = \emptyset$.

Proof. Let $\lambda \in \sigma_r(U,c)$. Then, U_λ does not have a dense range.
By Theorem 4.3, we get $\lambda = r+s$. Therefore, $\sigma_r(U,c) = \{r+s\}$ and hence, $r+s \in \sigma_p(U,c) \cap \sigma_r(U,c)$, a contradiction. Hence, the result. $\square$

Theorem 4.4. *The continuous spectrum and the spectrum of the operator U over c are respectively given by $\sigma_c(U,c) = \{\lambda \in \mathbf{C} : |\lambda - r| = |s|\} \setminus \{r+s\}$ and*
$\sigma(U,c) = \{\lambda \in \mathbf{C} : |\lambda - r| \leq |s|\}$.

Proof. Proceeding exactly as in Theorem 3.4, we can show that $\sigma(U,c) = \{\lambda \in \mathbf{C} : |\lambda - r| \leq |s|\}$. Since, $\sigma(U,c)$ is disjoint union of $\sigma_p(U,c)$, $\sigma_r(U,c)$ and $\sigma_c(U,c)$, thefore using Theorem 4.1 and Corollary 4.1, we get $\sigma_c(U,c) = \{\lambda \in \mathbf{C} : |\lambda - r| = |s|\} \setminus \{r+s\}$. $\square$

It is known from Cartlidge [11] that, if a matrix operator A is bounded in c, then $\sigma(A,c) = \sigma(A,\ell_\infty)$. Therefore, we have the following result:

Corollary 4.2. *The spectrum of the operator U over ℓ_∞ is given by* $\sigma(U,\ell_\infty) = \{\lambda \in \mathbf{C} : |\lambda - r| \leq |s|\}$.

Theorem 4.5. *If $|\lambda - r| < |s|$, then $\lambda \in I_3\sigma(U,c)$.*

Proof. The proof of the theorem is analogous to the proof of the Theorem 3.5. $\square$

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