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Universidad Católica del Norte
Antofagasta - Chile

A variant of the quadratic functional equation on semigroups

B. Fadli
IBN Tofail University, Morocco
D. Zeglami
Moulay ISMAIL University, Morocco
and
S. Kabbaj
IBN Tofail University, Morocco
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Abstract

Let S be a semigroup, let H be an abelian group which is uniquely 2-divisible, and let σ be an involutive automorphism of S. We express the solutions $f : S \to H$ of the following variant of the quadratic functional equation

$$f(xy) + f(\sigma(y)x) = 2f(x) + 2f(y), \quad x, y \in S,$$

in terms of bi-additive maps and solutions of the symmetrized additive Cauchy equation.

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1. Set up, notation and terminology

Throughout the paper we work in the following framework and with the following notation and terminology. We use it without explicit mentioning.

S is a *semigroup* [a set equipped with an associative composition rule $(x,y) \mapsto xy$], $\sigma : S \to S$ is an homomorphism satisfying $\sigma \circ \sigma = id$, and $(H,+)$ denotes an abelian group which is *uniquely 2-divisible*, i.e., for any $h \in H$ the equation $2x = h$ has exactly one solution $x \in H$.

A function $a : S \to H$ is said to be *additive* if

$$a(xy) = a(x) + a(y) \text{ for all } x, y \in S. \tag{1.1}$$

A function $f : S \to H$ is *abelian*, if

$$f(x_{\pi(1)}x_{\pi(2)} \cdots x_{\pi(k)}) = f(x_1 x_2 \cdots x_k)$$

for all $x_1, x_2, \ldots, x_k \in S$, all permutations π of k elements and all $k = 2, 3, \ldots$. Any abelian function f is *central*, meaning $f(xy) = f(yx)$ for all $x, y \in S$.

By $\mathcal{N}(S, H, \sigma)$ we mean the set of the solutions $\theta : S \to H$ of the homogeneous equation

$$\theta(xy) - \theta(\sigma(x)y) = 0, \quad x, y \in S.$$

We recall that the *Cauchy difference* Cf of a function $f : S \to H$ is defined by

$$Cf(x,y) := f(xy) - f(x) - f(y), \quad x, y \in S.$$

If $f : S \to H$ is a function, then $J_f : S \to H$ and $\varphi_f : S \to H$ are defined by

$$2f(x) := J_f(x) + Cf(x,x) \quad \text{and} \quad \varphi_f(x) := f(\sigma(x)x)$$

for all $x \in S$.

2. Introduction

The purpose of the present paper is to solve the following *variant of the quadratic functional equation*

$$f(xy) + f(\sigma(y)x) = 2f(x) + 2f(y), \quad x, y \in S, \tag{2.1}$$

where $f : S \to H$ is the unknown function. The difference between (2.1) and the quadratic standard functional equation

$$f(xy) + f(x\sigma(y)) = 2f(x) + 2f(y), \quad x, y \in S, \tag{2.2}$$

is that the new equation (2.1) has, on the second term, $f(\sigma(y)x)$ while the old one (2.2) has $f(x\sigma(y))$. On abelian semigroups the functional Eqs. (2.1) and (2.2) coincide, and their solutions are known see e.g. [10] and [13,Chapter 13], so the contributions of the present paper to the theory of quadratic functional equations lie in the non-abelian case.

A special case of (2.1) is the *symmetrized additive Cauchy equation*

$$f(xy) + f(yx) = 2f(x) + 2f(y), \quad x, y \in S. \tag{2.3}$$

Eq. (2.3) is a non-commutative version of the additive equation (1.1), because it reduces to (1.1) if S is abelian. On groups the solutions of (2.3) are according to [13, Proposition 2.17] the same as those solutions f of *Jensen's functional equation* $f(xy) + f(xy^{-1}) = 2f(x)$ for which $f(e) = 0$. Example 12.4 in [13] present a non-abelian solution of (2.3) on the 3-dimensional Heisenberg group $H_3(\mathbf{R})$. Therefore the functional equation (2.1) has in general non-abelian solutions.

Similar functional equations that have also been studied are

$$f(xy) + f(\sigma(y)x) = 2f(x)f(y), \quad x, y \in S, \tag{2.4}$$

$$f(xy) + f(\sigma(y)x) = 2f(x)g(y), \quad x, y \in S, \tag{2.5}$$

$$f(xy) + f(\sigma(y)x) = 2f(x), \quad x, y \in S. \tag{2.6}$$

The complex-valued solutions of (2.4) were determined by Stetkær in [14], while the complex-valued solutions (f, g), where S is a possibly non-abelian group or monoid, of (2.5) and the solutions $f : S \to H$ of (2.6) were obtained by the authors in [3] and [4], respectively. It turns out that, like on abelian groups, only multiplicative and additive functions occur in the solution formulas of (2.4), (2.5) and (2.6). We will prove that these contrast the solutions of the functional equation (2.1), where the non-abelian phenomena like solutions of (2.3) may occur. For other similar functional equations we refer to [1,2,5,7-9,11,12,16].

One of the main results is that the solutions for the variant (2.1) of the quadratic functional equation can be expressed in terms of bi-additive maps and solutions of the symmetrized additive Cauchy equation (Theorem 5.4), so that the form of the solutions generalizes the case where S is abelian, see e.g. [10,Theorem 3] and [13, Theorem 13.6]. Thus the contribution by our paper of new knowledge is an extension of earlier results from the abelian to the non-abelian case because (2.3) becomes (1.1) if S is abelian.

As applications, two important results (Corollaries 5.3 and 5.6 about Drygas' type equation

$$f(xy) + f(\sigma(y)x) = 2f(x) + f(y) + f(\sigma(y)), \quad x, y \in S,$$

are presented. Our solution formulas contain the abelian ones as special cases.

Finally, we note that the results about Whitehead's functional equation

$$f(xyz) = f(xy) + f(xz) + f(yz) - f(x) - f(y) - f(z), \quad x, y, z \in S,$$

(2.7)

given in [15] play an important role in finding solutions to the functional equation (2.1).

3. Results about Whitehead's functional equation (2.7)

The following lemma lists pertinent basic properties of any solution $f : S \to H$ of (2.7). For the notation J_f, see Section 1.

Theorem 3.1. *Let $f : S \to H$ be a solution of (2.7). In that case*

(a) $Cf : S \times S \to H$ is bi-additive.

(b) $J_f : S \to H$ satisfies (2.3).

(c) If f is central, then J_f is additive.

(d) Let $s \in Hom(S, S)$. If $f \circ s = f$, then $J_f \circ s = J_f$.

Proof. (a) and (b) can be found in [15]

(c) Let f be central. To get that J_f is central it suffices to prove that so is $x \mapsto Cf(x, x)$. That is an easy task. The rest follows from (b).

(d) By the definition of J_f it suffices to prove that $Cf(s(x), s(x)) = Cf(x, x)$ for all $x \in S$, and that follows from the definition of Cf. $\square$

4. Connections between (2.1) and (2.7)

Lemma 4.1 below derives one connection between (2.1) and (2.7), viz.,
Lemma 4.1. If $f : S \to H$ satisfies (2.1), then it also satisfies (2.7).

Proof. Making the substitutions (xy, z), $(\sigma(z)x, y)$, and (x, yz) in (2.1), we get respectively

$$\begin{aligned} f(xyz) + f(\sigma(z)xy) &= 2f(xy) + 2f(z), \\ f(\sigma(z)xy)) + f(\sigma(yz)x) &= 2f(\sigma(z)x) + 2f(y), \\ f(xyz) + f(\sigma(yz)x) &= 2f(x) + 2f(yz). \end{aligned}$$

Subtracting the middle identity from the sum of the other two we find that

$$2f(xyz) = 2f(xy) + 2f(yz) + 2f(x) + 2f(z) - 2f(\sigma(z)x) - 2f(y).$$

Replacing here $f(\sigma(z)x)$ by $2f(x) + 2f(z) - f(xz)$ and using the fact that H is *uniquely 2-divisible*, we get (2.7). □

In the following lemma, we derive another connection between (2.1) and (2.7).

Lemma 4.1. If $f : S \to H$ satisfies (2.1), then $\varphi_f = J_f$.

Proof. The proof is a small computation, based on (2.1).

$$\begin{aligned} \varphi_f(x) - J_f(x) &= f(\sigma(x)x) - [2f(x) - Cf(x, x)] \\ &= f(\sigma(x)x) + f(xx) - f(x^2) - [2f(x) - f(x^2) + 2f(x)] \\ &= 2f(x) + 2f(x) - 4f(x) = 0. \end{aligned}$$

□

5. Results about (2.1)

We start with Lemma 5.1, in which we derive some properties of the solutions of (2.1).

Lemma 5.1. If $f : S \to H$ satisfies (2.1), then

a) $f \circ \sigma = f$.

b) $Cf(x, \sigma(y)) = -Cf(y, x)$ for all $x, y \in S$.

c) φ_f is a solution of (2.1).

d) $\varphi_f \in \mathcal{N}(S, H, \sigma)$, and $\varphi_f \circ \sigma = \varphi_f$.

Proof. (a) Let $x, y \in S$ be arbitrary. Using (2.7) and (2.1), we obtain

$$\begin{aligned} f(\sigma(y)xy) &= f(\sigma(y)x) + f(\sigma(y)y) + f(xy) - f(\sigma(y)) - f(x) - f(y) \\ &= f(\sigma(y)y) + [f(xy) + f(\sigma(y)x)] - f(\sigma(y)) - f(x) - f(y) \\ &= f(\sigma(y)y) + f(x) + f(y) - f(\sigma(y)). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} f(\sigma(y)xy) &= 2f(xy) + 2f(y) - f(xy^2) \\ &= 2f(xy) + 2f(y) - [2f(xy) + f(y^2) - f(x) - 2f(y)] \\ &= 4f(y) - f(y^2) + f(x) \\ &= f(\sigma(y)y) + f(x). \end{aligned}$$

So $f(y) - f(\sigma(y)) = 0$ for all $y \in S$, i.e., $f \circ \sigma = f$.
(b) Let $x, y \in S$ be arbitrary. By help of (a) and (2.1), we get that

$$\begin{aligned} Cf(x, \sigma(y)) &= f(x\sigma(y)) - f(x) - f(\sigma(y)) \\ &= f(\sigma(x)y) - f(x) - f(y) \\ &= 2f(y) + 2f(x) - f(yx) - f(x) - f(y) \\ &= f(y) + f(x) - f(yx) \\ &= -Cf(y, x). \end{aligned}$$

In (c) and (d) we use a couple of times that $\varphi_f = J_f$ (Lemma 4.2).

(c) Recalling the definition of J_f (Section 1) we get that $2f(x) = Cf(x, x) + \varphi_f(x)$, so by linearity it suffices to show that the function $x \mapsto Cf(x, x)$ is a solution of (2.1). And that is a simple computation, based on the bi-additivity of Cf and (b).

(d) We read from Theorem 3.1(b) that $\varphi_f(xy) + \varphi_f(yx) = 2\varphi_f(x) + 2\varphi_f(y)$ for all $x, y \in S$. Comparing this with (c) gives the first statement of (d). That $\varphi_f \circ \sigma = \varphi_f$ is a special instance of Theorem 3.1(d). $\square$

In the following theorem, we determine the central solutions $f : S \to H$ of the functional equation (2.1). For abelian case it generalizes many results (see, e.g., [10,Theorem 3] and [13, Theorem 13.6]).

Theorem 5.2. The central solutions $f : S \to H$ of (2.1) are the functions of the form

$$f(x) = Q(x, x) + a(x),$$

where $Q : S \times S \to H$ is an arbitrary symmetric, bi-additive map such that $Q(x, \sigma(y)) = -Q(x, y)$ for all $x, y \in S$, and where $a : S \to H$ is an arbitrary additive map such that $a \circ \sigma = a$.

Proof. Assume that $f : S \to H$ is a central solution of (2.1). Since f is central, then Cf is symmetric and φ_f is additive (Theorem 3.1 (c)). So f has the desired form by the decomposition $2f(x) = Cf(x, x) + \varphi_f(x)$. Take $Q = \frac{1}{2}Cf$ and $a = \frac{1}{2}\varphi_f$.

The other direction of the proof is trivial to verify. $\square$

As a consequence of Theorem 5.2, we have the following result on the central solutions of the functional equation

$$f(xy) + f(\sigma(y)x) = 2f(x) + f(y) + f(\sigma(y)), \quad x, y \in S, \tag{5.1}$$

which reveals a connection between (5.1) and (2.1) and contains the solution of Drygas' equation on commutative semigroups.

Corollary 5.3. The central solutions $f : S \to H$ of (5.1) are the functions of the form

$$f(x) = Q(x, x) + a(x), \tag{5.2}$$

where $Q : S \times S \to H$ is an arbitrary symmetric, bi-additive map such that $Q(x, \sigma(y)) = -Q(x, y)$ for all $x, y \in S$, and where $a : S \to H$ is an arbitrary additive map.

Proof. It is easy to check that any function f of the form (5.2) is central and satisfies (5.1). Conversely, assume that f is a central solution of (5.1). Let f_e and f_o denote the σ-even and the σ-odd parts of f, i.e.,

$$f_e = \frac{f + f \circ \sigma}{2} \quad \text{and} \quad f_o = \frac{f - f \circ \sigma}{2}.$$

Simple computations show that f_e is a central solution of (5.1). Hence f_e is a central solution of (2.1). From Theorem 5.2, we see that there exist a symmetric, bi-additive map $Q : S \times S \to H$ with $Q(x, \sigma(y)) = -Q(x, y)$ for all $x, y \in S$, and an additive map $a_1 : S \to H$ with $a_1 \circ \sigma = a_1$ such that

$$f_e(x) = Q(x, x) + a_1(x), \quad x \in S.$$

On the other hand, since $f_o = f - f_e$, f_o is also a solution of (5.1), so that f_o is a solution of the variant (2.6) of Jensen's functional equation. According to [4, Theorem 3.2], we see that f_o is additive. Therefore $f = f_e + f_o$ has the required form with $a = a_1 + f_o$ and this completes the proof. □

Now we treat the general case where the solution $f : S \to H$ need not be central.

Theorem 5.4. The general solution $f : S \to H$ of (2.1) is

$$f(x) = Q(x, x) + \psi(x), \quad x \in S, \tag{5.3}$$

where $Q : S \times S \to H$ is an arbitrary bi-additive map such that $Q(x, \sigma(y)) = -Q(y, x)$ for all $x, y \in S$, and where $\psi : S \to H$ is an arbitrary solution of the symmetrized additive Cauchy equation such that $\psi \circ \sigma = \psi$ and $\psi \in \mathcal{N}(S, H, \sigma)$.

Proof. Let $f : S \to H$ be a solution of (2.1). Using the decomposition $2f(x) = Cf(x, x) + \varphi_f(x)$, Theorem 3.1 and Lemma 5.1, we see that f has the desired form. Take $Q = \frac{1}{2}Cf$ and $\psi = \frac{1}{2}\varphi_f$.

The other direction of the proof is trivial to verify. □

Remark 5.1. *Theorem 5.4 is a non-abelian version of e.g. [10,Theorem 3] and [13, Theorem 13.6]. Indeed, if S is abelian, then any solution of the symmetrized additive Cauchy equation reduces to an additive function and so the condition $\psi \in \mathcal{N}(S, H, \sigma)$ becomes $\psi = \psi \circ \sigma$.*

In view of Theorem 5.4, we obtain the following result about solution, that need not be central, of Drygas' type equation (5.1) on any semigroup.

Corollary 5.2. *The general solution $f : S \to H$ of (5.1) is*

$$f(x) = Q(x,x) + \psi(x) + a(x), \quad x \in S,$$

where $Q : S \times S \to H$ is an arbitrary bi-additive map such that $Q(x, \sigma(y)) = -Q(y,x)$ for all $x, y \in S$, $\psi : S \to H$ is an arbitrary solution of the symmetrized additive Cauchy equation such that $\psi \circ \sigma = \psi$ and $\psi \in \mathcal{N}(S, H, \sigma)$, and where $a : S \to H$ is an additive function such that $a \circ \sigma = -a$.

Proof. As the proof of Corollary 5.3 with the necessary changes. □

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References

[1] H. Drygas, Quasi-inner products and their applications, in: Advances in Multivariate Statistical Analysis (ed. A. K. Gupta), D. Reidel Publishing Co., pp. 13-30, (1987).

[2] B. Fadli, D. Zeglami and S. Kabbaj, On a Gajda's type quadratic equation on a locally compact abelian group, Indagationes Math. **26** (4), pp. 660-668, (2015).

[3] B. Fadli, D. Zeglami and S. Kabbaj, A variant of Wilson's functional equation, Publ. Math. Debrecen **87** (3-4), pp. 415-427, (2015).

[4] B. Fadli, D. Zeglami and S. Kabbaj, A variant of Jensen's functional equation on semigroups, Demonstratio Math. **49** (4), pp. 413-420, (2016).

[5] B. Fadli, A. Chahbi, Iz. EL-Fassi and S. Kabbaj, On Jensen's and the quadratic functional equations with involutions, Proyecciones (Antofagasta) **35** (2), pp. 213-223, (2016).

[6] V. A. Faĭziev and P. K. Sahoo, Solution of Whitehead equation on groups, Math. Bohem. **138**(2), pp. 171-180, (2013).

[7] PL. Kannappan, Functional equations and inequalities with applications, Springer, New York, (2009).

[8] C.T. Ng, Jensen's functional equation on groups, III, Aequationes Math. **62** (1-2), pp. 143-159, (2001).

[9] P. de Place Friis and H. Stetkær, On the quadratic functional equation on groups, Publ. Math. Debrecen **69** (1-2), pp. 65-93, (2006).

[10] P. Sinopoulos, Functional equations on semigroups, Aequationes Math. **59**(3), pp. 255-261, (2000).

[11] H. Stetkær, Functional equations on abelian groups with involution, Aequationes Math. **54** (1-2), pp. 144-172, (1997).

[12] H. Stetkær, On a variant of Wilson's functional equation on groups, Aequationes Math. **68** (3), pp. 160-176, (2004).

[13] H. Stetkær, Functional Equations on Groups, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, (2013).

[14] H. Stetkær, A variant of d'Alembert's functional equation, Aequationes Math. **89** (3), pp. 657-662, (2015).

[15] H. Stetkær, The kernel of the second order Cauchy difference on semigroups, Aequationes Math. (2016). DOI 10.1007/s00010-016-0453-8.

[16] D. Zeglami, B. Fadli and S. Kabbaj, On a variant of μ-Wilson's functional equation on a locally compact group, Aequationes Math. **89** (5), pp. 1265-1280, (2015).

B. Fadli
Department of Mathematics,
Faculty of Sciences,
IBN TOFAIL University,
B. P. : 14000. Kenitra,
Morocco
e-mail : himfadli@gmail.com

D. Zeglami
Department of Mathematics,
E. N. S. A. M,
Moulay ISMAIL University,
B. P: 15290 Al Mansour,
Meknes,
Morocco
e-mail : zeglamidriss@yahoo.fr

and

S. Kabbaj
Department of Mathematics,
Faculty of Sciences,
IBN TOFAIL University,
B. P. : 14000. Kenitra,
Morocco
e-mail : samkabbaj@yahoo.fr