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Vertex graceful labeling of some classes of graphs

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Abstract

A connected graph $G = (V, E)$ of order atleast two, with order p and size q is called vertex-graceful if there exists a bijection $f : V \rightarrow \{1, 2, 3, \cdots p\}$ such that the induced function $f^* : E \rightarrow \{0, 1, 2, \cdots q - 1\}$ defined by $f^*(uv) = (f(u) + f(v))(mod\ q)$ is a bijection. The bijection f is called a vertex-graceful labeling of G. A subset S of the set of natural numbers N is called consecutive if S consists of consecutive integers. For any set X, a mapping $f : X \rightarrow N$ is said to be consecutive if $f(X)$ is consecutive. A vertex-graceful labeling f is said to be strong if the function $f_1 : E \rightarrow N$ defined by $f_1(e) = f(u) + f(v)$ for all edges $e = uv$ in E forms a consecutive set. It is proved that one vertex union of odd number of copies of isomorphic caterpillars is vertex-graceful and any caterpillar is strong vertex-graceful. It is proved that a spider with even number of legs (paths) of equal length appended to each vertex of an odd cycle is vertex-graceful. It is also proved that the graph $lA(m_j, n)$ is vertex-graceful for both n and l odd, $0 \leq i \leq n - 1$, $1 \leq j \leq m_i$. Further, it is proved that the graph $A(m_j, n)$ is strong vertex-graceful for n odd, $0 \leq i \leq n - 1$, $1 \leq j \leq m_i$.

Key words: *Caterpillar, one vertex union graphs, regular spider, actinia graph, vertex-graceful labeling, strong vertex-graceful labeling.*

2010 Mathematics Subject Classification: *05C78.*

1. Introduction

By a graph $G = (V, E)$, we mean a finite undirected graph without loops or multiple edges. The order and size of G are denoted by p and q, respectively. For basic graph theoretic terminology, we refer to [5, 8]. A *caterpillar* is a tree, the removal of whose end vertices, gives a path. The resulting path is called a *spine*. Denote by $C(n, m_i)$, a caterpillar, where n is the number of vertices on the spine and m_i is the number of end vertices adjacent to each vertex of the spine. Let $G_i (1 \leq i \leq l)$ be a connected graph with vertex set $V_i = \{v_1^i, v_2^i, v_3^i, \cdots, v_{n_l}^i\}$. Then the *one vertex union* of $G_i (1 \leq i \leq l)$ at v_1^1 is obtained by identifying all the vertices $v_1^1, v_1^2, v_1^3, \cdots, v_1^l$ at v_1^1 and is denoted by $G^{(l)}$.

A *spider* is a tree that has at most one vertex (called the center) of degree greater than 2. A *regular spider* is the one vertex union of paths of equal length. A *bamboo tree* is a rooted tree consisting of branches of equal length, the end points of which are identified with the end points of stars of different sizes. A bamboo tree is called *regular* if the sizes of the stars considered are equal. The bamboo trees were introduced and studied in [11]. The graph $A(m_i, n)$ obtained by identifying $m_i, (1 \leq i \leq n)$ pendant edges to the vertices of a cycle C_n is called *Actinia graph*. The Actinia graphs were introduced and studied in [19].

A *graph labeling* is an assignment of numbers to the vertices or edges, or both vertices and edges subject to certain conditions. A graph G is called *graceful graph* if there exists an injection $f : V \to \{0, 1, 2, \cdots, q\}$ such that the induced function $f^* : E \to \{1, 2, \cdots, q\}$ defined by $f^*(uv) = |f(u) - f(v)|$ is a bijection. The injection f is called a *graceful labeling* of graph G. The values $f(u)$ and $f^*(uv)$ are called the*graceful labels* of the vertex u and the edge uv, respectively. A graph G is called *vertex-graceful* if there exists a bijection $f : V \to \{0, 1, 2, \cdots, p\}$ such that the induced function $f^* : E \to \{0, 1, 2, \cdots, q - 1\}$ by $f^*(uv) = (f(u) + f(v))(mod\ q)$ is a bijection. The bijection f is called a *vertex-graceful labeling* of G . Let N denote the set of natural numbers. A subset S of N is called *consecutive* if S consists of consecutive integers. For any set X, a mapping $f : X \to N$ is said to be *consecutive* if $f(X)$ is consecutive. A vertex-graceful labeling f is said to be *strong* if the function by $f_1 : E \to N$ defined by $f_1(e) = f(u) + f(v)$ for all edges $e = uv$ in E forms a consecutive set.

In 1967, Rosa [12] introduced the labeling method called *β-valuation*. A function f is called β-valuation of a graph G with q edges if f is an injection from the vertices of G to the set $\{0, 1, 2, \cdots, q\}$ such that, when each edge xy is assigned the label $|f(x) - f(y)|$, the resulting edge labels are distinct. Golomb [7] called such labelings *graceful* and this term is followed presently. The book edited by Acharia, Arumugam and Rosa [1] includes a variety of labeling methods. Hsu and Keedwell [9, 10] introduced and studied the extension of graceful labeling of directed graphs. The relationship between graceful directed graphs and a variety of algebraic structures including cyclic difference sets, sequenceable groups, generalized complete mappings, near-complete mapping and neofield is discussed in [3] and [4]. Bahl, Lake and Wertheim [2] proved that spiders for which the lengths of every path from the center to a leaf differ by at most one are graceful and spiders for which the lengths of every path from the center to a leaf has the same length and there is an odd number of such paths there is a family of graceful labelings.

Graph labeling, where the vertices are assigned values subject to certain conditions have often been motivated by practical problems. Labeled graphs serve as useful mathematical models for a broad range of applications such as coding theory, including the design of good radar type codes, synch-set codes, missile guidance codes and convolution codes with optimal autocorrelation properties. The notion of vertex-graceful graph of order p and size $p + 1$ was introduced and studied in [18]. The vertex graceful labeling of many classes of graphs were studied in [13, 14, 15, 16, 17]. For detailed study of labeling of graphs, we refer to [6].

2. Vertex Gracefulness of One Vertex Union of Caterpillars

In this section, it is proved that one vertex union of odd number of copies of isomorphic caterpillars is vertex-graceful and also proved that any caterpillar is strong vertex graceful.

Let $G = C^{(l)}(n, m_i)$ be the one vertex union of l copies of isomorphic caterpillars with n vertices in the spine and each of its vertices are appended with $m_i (1 \leq i \leq n)$ pendant vertices. Let $v_1^k, v_2^k, v_3^k, \cdots, v_n^k$ be the vertices of the spine of the k^{th} copy of the isomorphic caterpillars. Let $m_1, m_2, m_3, \cdots, m_n$ be the number of pendant vertices, which are appended to each vertex of the spine of the caterpillar. Let $v_1^1, v_1^2, v_1^3, \cdots, v_1^l$ be the initial vertex of each spine of l isomorphic caterpillars. The one vertex union of l copies of isomorphic caterpillars at v_1^1 is obtained by identifying all the vertices $v_1^1, v_1^2, v_1^3, \cdots, v_1^l$ at v_1^1. Let v_{ij}^k be the pendant vertices of G

for $1 \leq i \leq n, 1 \leq j \leq m_i, 1 \leq k \leq l$. This graph is shown in Figure 2.1.

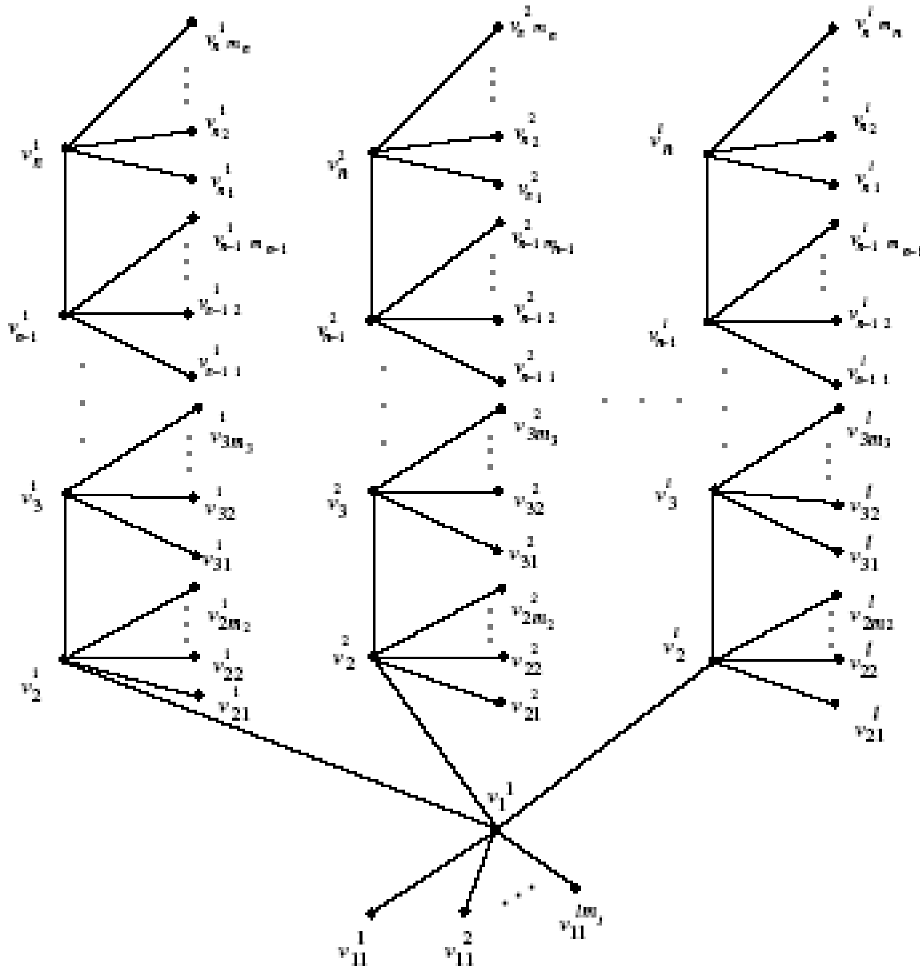

Figure 2.1: One vertex union of l copies of isomorphic caterpillars

Theorem 2.1. For $1 \leq i \leq n, 1 \leq j \leq m_i$ and l odd, the graph $G = C^{(l)}(n, m_i)$ is vertex-graceful.

Proof. It is clear from the definition of G that the number of vertices p and edges q are $p = l\left(n-1+\sum_{a=1}^{n} m_a\right)+1$ and $q = p-1$. Let $\alpha = n-1+\sum_{a=1}^{n-1} m_{a+1}$. The required vertex-graceful labeling for vertices is defined as follows:
Let $f : V \to \{1, 2, \cdots, p\}$.

Case 1. n even.

Let $\beta = \sum_{a=1}^{\frac{n-2}{2}} m_{2a}$

Define $f(v_1^1) = 1$,

$$\mathrm{f}(\mathrm{v}_i^k) = \begin{cases} \alpha(k-1)2+\frac{i+1}{2}+\sum_{a=1}^{\frac{i-1}{2}} m_{2a}, & 3 \le i \le n, i \text{ odd}; 1 \le k \le l, k \text{ odd}. \\ \frac{\alpha(l+k-2)}{2}+\beta+lm_1+m_n+\frac{n+i}{2}+\sum_{a=1}^{\frac{i-2}{2}} m_{2a+1}, & 2 \le i \le n, i \text{ even}; 1 \le k \le l, k \text{ odd}. \\ \frac{\alpha(k-2)}{2}+\beta+m_n+\frac{n+i}{2}+\sum_{a=1}^{\frac{i-2}{2}} m_{2a+1}, & 2 \le i \le n, i \text{ even}; 1 \le k \le l, k \text{ even}. \\ \frac{\alpha(l+k-1)}{2}+lm_1+\frac{i+1}{2}+\sum_{a=1}^{\frac{i-1}{2}} m_{2a}, & 2 \le i \le n, i \text{ odd}; 1 \le k \le l, k \text{ even}. \end{cases}$$

The vertex labeling of pendant vertices at $v_1^k, v_2^k, v_3^k, \cdots, v_n^k$ is given by

$$\mathrm{f}(\mathrm{v}_{ij}^k) = \begin{cases} \alpha(l-1)2+\beta+\frac{n}{2}+m_n+k, & \text{for } i=j=1;\ 1 \le k \le lm_1. \\ \frac{\alpha(l+k-2)}{2}+\beta+lm_1+m_n+\frac{n+i-1}{2}+j+\sum_{a=1}^{\frac{i-3}{2}} m_{2a+1}, & 2 \le i \le n,\ i \text{ odd}; 1 \le j \le m_i; 1 \le k \le l, k \text{ odd}. \\ \frac{\alpha(k-1)}{2}+\frac{i}{2}+j+\sum_{a=1}^{\frac{i-2}{2}} m_{2a}, & 2 \le i \le n, i \text{ even}; 1 \le j \le m_i; 1 \le k \le l, k \text{ odd}. \\ \frac{\alpha(k-2)}{2}+\beta+m_n+\frac{n+i-1}{2}+j+\sum_{a=1}^{\frac{i-3}{2}} m_{2a+1}, & 2 \le i \le n, i \text{ odd}; 1 \le j \le m_i; 1 \le k \le l, k \text{ even}. \end{cases}$$

$\frac{\alpha(l+k-1)}{2} + lm_1 + \frac{i}{2} + j + \sum_{a=1}^{\frac{i-2}{2}} m_{2a}$,
$2 \le i \le n, i$ even; $1 \le j \le m_i; 1 \le k \le l, k$ even.
The corresponding edge labeling is defined as follows:

Let $A = \left\{ f^*(e_{11}^k) : f^*(e_{11}^k) = \left(\frac{\alpha(l-1)}{2} + \frac{n}{2} + m_n + k + 1 + \beta \right) (mod\ q) \right\}$,
where $e_{11}^k = v_1^1 v_{11}^k,\ 1 \le k \le lm_1$.

Let $\delta = \frac{n}{2} + lm_1 + m_n + \beta$,
$B = \left\{ f^*(e_i^k) : f^*(e_i^k) = \left(\frac{\alpha(l+2k-3)}{2} + 1 + \delta + i + \sum_{a=1}^{i-1} m_{a+1} \right) (mod\ q) \right\}$,
where $e_i^k = v_i^k v_{i+1}^k,\ 2 \le i \le n;\ 1 \le k \le l$;

$$C = \left\{ f^*(e_{ij}^k) : f^*(e_{ij}^k) = \left(\frac{\alpha(l+2k-3)}{2} + \delta + i + j + \sum_{a=1}^{i-2} m_{a+1} \right) (mod\ q) \right\},$$
where $e_{ij}^k = v_i^k v_{ij}^k,\ 2 \le i \le n;\ 1 \le j \le m_i;\ 1 \le k \le l$;

$$D = \left\{ f^*(e_1^k) : f^*(e_1^k) = \left(\frac{\alpha(l+k-2)}{2} + \delta + 2 \right) (mod\ q) \right\},$$
where $e_1^k = v_1^1 v_2^k,\ 1 \le k \le l,\ k$ odd;

$$E = \left\{ f^*(e_1^k) : f^*(e_1^k) = \left(\frac{\alpha(k-2)}{2} + \beta + m_n + \frac{n+4}{2} \right) (mod\ q) \right\}$$
where $e_1^k = v_1^1 v_2^k,\ 1 \le k \le l,\ k$ even .

It is clear that all the vertex labels are distinct. Also it is easily verified that the edge label sets A, B, C, D and E are mutually disjoint and $A \cup B \cup C \cup D \cup E = \{0, 1, 2, \cdots, q-1\}$. Therefore, it follows that the induced mapping $f^* : E \to \{0, 1, 2, \cdots, q-1\}$ is a bijection. Hence the graph G is a vertex-graceful when n is even.

Case 2. n odd.

Let $\gamma = \sum_{a=1}^{\frac{n-1}{2}} m_{2a}$ and $\mu = \gamma + lm_1 + \frac{n+1}{2}$.
Define $f(v_1^1) = 1$,

$\mathrm{f}(\mathrm{v}_i^k) = \Big\{ \ \alpha(k-1)2 + \frac{i+1}{2} + \sum_{a=1}^{\frac{i-1}{2}} m_{2a}, \ \ 2 \le i \le n, i \text{ odd}; 1 \le k \le l, k \text{ odd}.$

$\frac{\alpha(l+k-2)}{2} + \gamma + lm_1 + \frac{n+i+1}{2} + \sum_{a=1}^{\frac{i-2}{2}} m_{2a+1}, \ \ 2 \le i \le n, i \text{ even};$

$1 \le k \le l, k \text{ odd}.$

$\frac{\alpha(k-2)}{2} + \gamma + \frac{n+i+1}{2} + \sum_{a=1}^{\frac{i-2}{2}} m_{2a+1}, \ \ 2 \le i \le n, i \text{ even}; 1 \le k \le l, k \text{ even}.$

$\frac{\alpha(l+k-1)}{2} + lm_1 + \frac{i+1}{2} + \sum_{a=1}^{\frac{i-1}{2}} m_{2a}, \ \ 2 \le i \le n, i \text{ odd}; 1 \le k \le l, k \text{ even}.$

The vertex labeling of pendant vertices at $v_1^k, v_2^k, v_3^k, \cdots, v_n^k$ is given by
$\mathrm{f}(\mathrm{v}_{ij}^k) = \Big\{ \ \alpha(l-1)2 + \frac{n+1}{2} + k + \gamma$, for $i = j = 1; \ 1 \le k \le lm_1.$

$\frac{\alpha(k-1)}{2} + \frac{i}{2} + j + \sum_{a=1}^{\frac{i-2}{2}} m_{2a},$

$2 \le i \le n, \ i \text{ even}; \ 1 \le j \le m_i; \ 1 \le k \le l, \ k \text{ odd}.$

$\frac{\alpha(l+k-2)}{2} + \gamma + lm_1 + \frac{n+i}{2} + j + \sum_{a=1}^{\frac{i-3}{2}} m_{2a+1},$

$2 \le i \le n, \ i \text{ odd}; \ 1 \le j \le m_i; \ 1 \le k \le l, \ k \text{ odd}.$

$\frac{\alpha(k-2)}{2} + \gamma + \frac{n+i}{2} + j + \sum_{a=1}^{\frac{i-3}{2}} m_{2a+1},$

$2 \le i \le n, \ i \text{ odd}; \ 1 \le j \le m_i; \ 1 \le k \le l, \ k \text{ even}.$

$\frac{\alpha(l+k-1)}{2} + lm_1 + \frac{i}{2} + j + \sum_{a=1}^{\frac{i-2}{2}} m_{2a},$

$2 \le i \le n, \quad i \text{ even}; 1 \le j \le m_i; 1 \le k \le l, \ k \text{ even}.$

The corresponding edge labeling is defined as follows:
Let $A = \left\{ f^*(e_1^k) : f^*(e_1^k) = \left(\frac{\alpha(l+k-2)}{2} + \gamma + lm_1 + \frac{n+5}{2} \right) (mod\ q) \right\},$
where $e_1^k = v_1^1 v_2^k, \ 1 \le k \le l, \ k$ odd ;

$B = \left\{ f^*(e_1^k) : f^*(e_1^k) = \left(\frac{\alpha(k-2)}{2} + \gamma + \frac{n+5}{2} \right) (mod\ q) \right\},$
where $e_1^k = v_1^k v_2^k, \ 1 \le k \le l, k$ even;

$C = \left\{ f^*(e_i^k) : f^*(e_i^k) = \left(\frac{\alpha(l+2k-3)}{2} + i + 1 + \mu + \sum_{a=1}^{i-1} m_{a+1} \right) (mod\ q) \right\},$
where $e_i^k = v_i^k v_{i+1}^k, \ 2 \le i \le n-1; \ 1 \le k \le l;$

$$D = \left\{ f^*(e_{ij}^k) : f^*(e_{ij}^k) = \left(\frac{\alpha(l+2k-3)}{2} + i + j + \mu + \sum_{a=1}^{i-2} m_{a+1} \right) (mod\ q) \right\},$$
where $e_{ij}^k = v_i^k v_{ij}^k, 2 \leq i \leq n;\ 1 \leq j \leq m_i;\ 1 \leq k \leq l$;

$$E = \left\{ f^*(e_{11}^k) : f^*(e_{11}^k) = \left(\frac{\alpha(l-1)}{2} + \frac{n+1}{2} + k + \gamma + 1 \right) (mod\ q) \right\},$$
where $e_{11}^k = v_1^k v_{11}^k,\ 1 \leq k \leq lm_1$.

It is clear that all the vertex labels are distinct. Also it is easily verified that the edge label sets A, B, C, D and E are mutually disjoint and $A \cup B \cup C \cup D \cup E = \{0, 1, 2, \cdots, q-1\}$. Therefore, it follows that the induced mapping $f^* : E \rightarrow \{0, 1, 2, \cdots, q-1\}$ is a bijection. Hence the graph G is a vertex-graceful when n is odd. Thus the graph G is vertex-graceful. □

Illustrative examples of the labeling of the graph G in Theorm 2.1 are given in Figures 2.2 and 2.3.

We leave the following problem as an open question.

Problem 2.2. Is it true that for $1 \leq i \leq n, 1 \leq j \leq m_i$ and l even, the graph $G = C^{(l)}(n, m_i)$ is vertex-graceful?

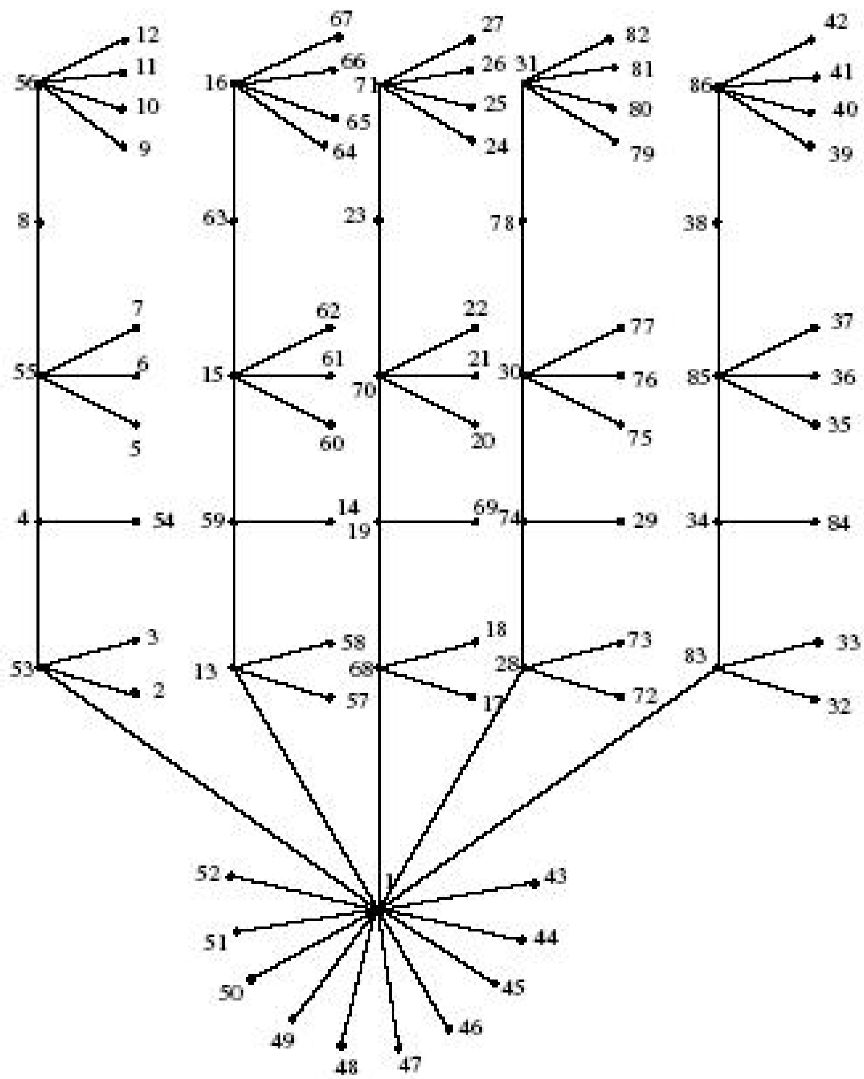

Figure 2.2: Vertex-graceful labeled graph of $C^6(6, m_i), 1 \le i \le 6$, $m_1 = 2, m_2 = 2, m_3 = 1, m_4 = 3, m_5 = 0, m_6 = 4$ (case 1)

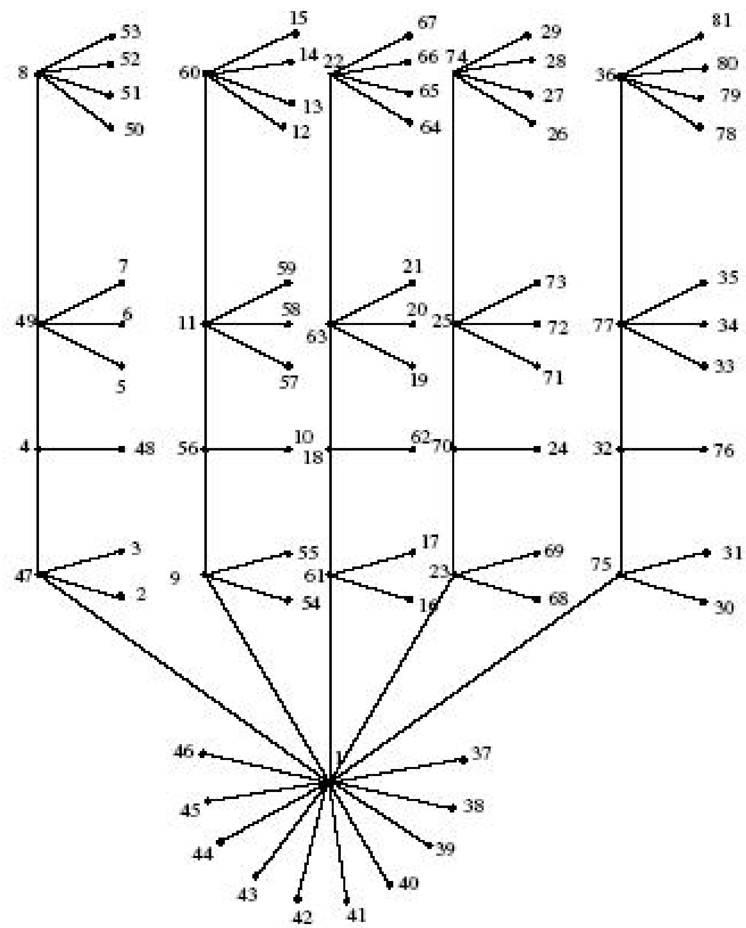

Figure 2.3: Vertex-graceful labeled graph of $C^5(5, m_i)$, $m_1 = 2, m_2 = 2, m_3 = 1, m_4 = 3, m_5 = 4$ (case 2)

Remark 2.3. If $l = 1$ in Figure 1, then the graph G represents a caterpillar. Hence it follows from Theorem 2.1 that any caterpillar is vertex-graceful. Also if $m_i = 0$ for $i = 1, 2, \cdots, n-1$ and $m_n \neq 0$ in Figure 2.1, then the graph G represents a regular bamboo tree. Hence it follows from Theorem 2.1 that a regular bamboo tree is vertex-graceful.

Theorem 2.4. Any caterpillar G is strong vertex graceful.

Proof. Let the caterpillar G be a tree with n vertices $v_1, v_2, \cdots, v_n$ on the spine and each v_i is attached with m_i pendant edges $(1 \leq i \leq n)$. The graph is given in Figure 2.4.

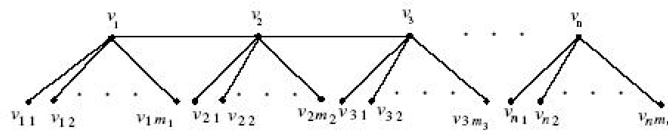

Figure 2.4: G

Then $p = n + \sum_{a=1}^{n} m_a$ and $q = n - 1 + \sum_{a=1}^{n} m_a$. The required strong vertex-graceful labeling for vertices is defined as follows:

Let $f : V \rightarrow \{1, 2, \cdots, p\}$ be such that

Case 1. n even.

Let $\beta = \sum_{a=1}^{\frac{n-2}{2}} m_{2a}$

$$\text{f(v}_i) = \Big\{ \;\; i + 12 + \sum_{a=1}^{\frac{i-1}{2}} m_{2a}, 1 \leq i \leq n, \; i \text{ odd}.$$

$$\beta + m_1 + m_n + \tfrac{n+i}{2} + \sum_{a=1}^{\frac{i-2}{2}} m_{2a+1}, 1 \leq i \leq n, \; i \text{ even}.$$

The vertex labeling of pendant vertices at $v_1, v_2, v_3, \cdots, v_n$ is given by

$$\text{f(v}_{ij}) = \{ \, \beta + m_n + \tfrac{n+i-1}{2} + j + \sum_{a=1}^{\frac{i-1}{2}} m_{2a-1}, 1 \leq i \leq n, \; i \text{ odd } \; 1 \leq j \leq m_i.$$

$$\tfrac{i}{2} + j + \sum_{a=1}^{\frac{i-2}{2}} m_{2a}, 1 \leq i \leq n, \;\; i \text{ even } 1 \leq j \leq m_i.$$

The corresponding edge labeling is defined as follows:

$$\text{Let } A = \left\{ f_1(e_{ij}) : f_1(e_{ij}) = \left(\tfrac{n}{2} + m_n + \beta + i + j + \sum_{a=1}^{i-1} m_a \right) \right\},$$

where $e_{ij} = v_i v_{ij}, 1 \leq i \leq n; 1 \leq j \leq m_i$.

$$B = \left\{ f_1(e_i) : f_1(e_i) = \left(\tfrac{n}{2} + m_n + \beta + 1 + i + \sum_{a=1}^{i} m_a \right) \right\},$$
where $e_i = v_i v_{i+1}, 1 \leq i \leq n-1.$

It is clear that the vertex labels are distinct. Also it is easily verified that the edge labels sets are mutually disjoint and $A \cup B = \{\frac{n}{2} + \beta + m_n + 2, \frac{n}{2} + \beta + m_n + 3, \cdots, \frac{n}{2} + \beta + m_n + p\}$. Therefore, the induced edge labels of G have q consecutive values. Hence the graph G is strong vertex-graceful when n is even.

Case 2. n odd.

Let $\gamma = \sum_{a=1}^{\frac{n-1}{2}} m_{2a}$

$$\text{f}(\text{v}_i) = \Big\{ \;\; i + 12 + \sum_{a=1}^{\frac{i-1}{2}} m_{2a}, 1 \leq i \leq n,\ i \text{ odd.}$$

$$\gamma + m_1 + \frac{n+i+1}{2} + \sum_{a=1}^{\frac{i-2}{2}} m_{2a+1}, 1 \leq i \leq n,\ i \text{ even.}$$

The vertex labeling of pendant vertices at $v_1, v_2, v_3, \cdots, v_n$ is given by

$$\text{f}(\text{v}_{ij}) = \{\, \gamma + \tfrac{n+i}{2} + j + \sum_{a=1}^{\frac{i-1}{2}} m_{2a-1}, 1 \leq i \leq n,\ i \;\text{ odd }\; 1 \leq j \leq m_i.$$

$$\tfrac{i}{2} + j + \sum_{a=1}^{\frac{i-2}{2}} m_{2a}, 1 \leq i \leq n,\ i \text{ even } 1 \leq j \leq m_i.$$

The corresponding edge labeling is defined as follows:

Let $A = \left\{ f_1(e_{ij}) : f_1(e_{ij}) = \left(\frac{n+1}{2} + \gamma + i + j + \sum_{a=1}^{i-1} m_a \right) \right\},$
where $e_{ij} = v_i v_{ij}, 1 \leq i \leq n; 1 \leq j \leq m_i.$

$$B = \left\{ f_1(e_i) : f_1(e_i) = \left(\tfrac{n+1}{2} + \gamma + 1 + i + \sum_{a=1}^{i} m_a \right) \right\},$$
where $e_i = v_i v_{i+1}, 1 \leq i \leq n-1.$

It is clear that the vertex labels are distinct. Also it is easily verified that the edge labels sets are mutually disjoint and $A \cup B = \{\frac{n+1}{2} + \gamma + 2, \frac{n+1}{2} + \gamma + 3, \cdots, \frac{n+1}{2} + \gamma + p\}$. Therefore the induced edge labels of G have q consecutive values. Hence the graph G is strong vertex-graceful when n

is odd. Thus the graph G is strong- vertex graceful. $\square$

3. Vertex-graceful Labeling of Cycle with Regular Spider

In this section, it is proved that a spider with even number of legs (paths) of equal length appended to each vertex of an odd cycle is vertex-graceful.

We start with the following definitions :

Definition 3.1. A subdivision of an edge uv in a graph is obtained by removing edge uv, adding a new vertex w and adding edges uw and vw. A (wounded) spider is the graph formed by subdividing at most $t-1$ of the edges of a star $K_{1,t}$ for $t \geq 0$.

Examples of wounded spiders include K_1, the star $K_{1,n-1}$, and the graph shown in Figure 3.1.

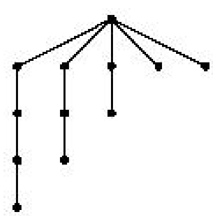

Figure 3.1: A wounded spider

Definition 3.2. A regular spider is the one vertex union of paths of equal length. Denote by $C_nP_n^{(l)}$ a cycle $C_m (m \geq 3)$ attached at each of its vertices a regular spider $P_n^{(l)}$, where l denotes the number of paths P_n in the spider (Refer Figure 3.2).

Example 3.3.

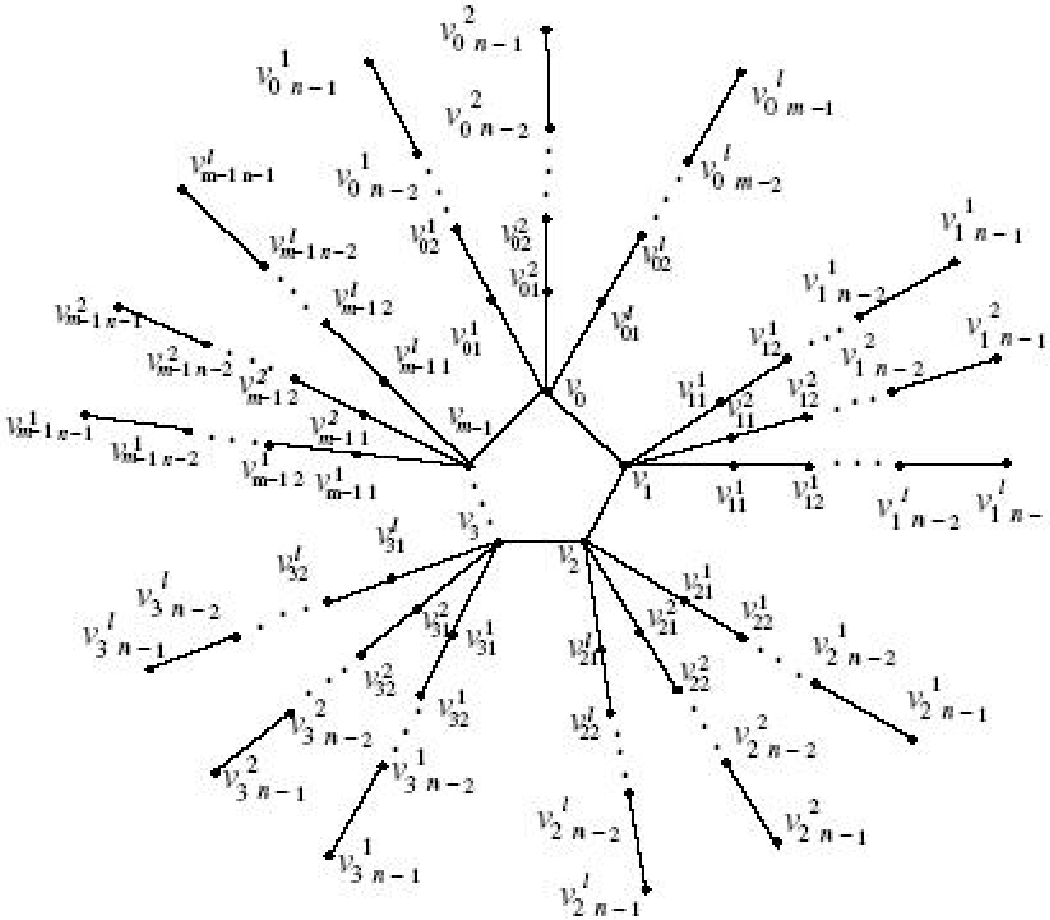

Figure 3.2: The graph $C_n \cdot P_n^{(l)}$

Theorem 3.4. For any n, the graph $C_nP_n^{(l)}$ is vertex-graceful where m odd and l even.

Proof. Let $v_0, v_1, \cdots, v_{m-1}$ be the vertices of the cycle C_m. Now, let $C_nP_n^{(l)}$ be the graph as shown in Figure 6. It is easily verified that G has $p = m[l(n-1)+1]$ vertices and $q = p$ edges. Let $\alpha = l(n-1)+1$. The required vertex-graceful labeling for vertices is defined as follows :
Let $f : V \to \{1, 2, \cdots, p\}$ be such that

Case 1. n even.

Define $f(v_i) = \alpha i + 1, 0 \leq i \leq m-1$,

f(v_{ij}^k) = { $\alpha\ \ i + \frac{(n-1)(k-1)}{2} + \frac{j+2}{2}$,
$0 \leq i \leq m-1; 1 \leq j \leq n-1, j$ even; $1 \leq k \leq l,\ k$ odd.
$\alpha\, i + \frac{(n-1)(k-1)}{2} + \beta + \frac{j+3}{2}$,
$0 \leq i \leq m-1; 1 \leq j \leq n-1,\ j$ odd; $1 \leq k \leq l,\ k$ odd.
$\alpha\, i + \frac{k(n-1)}{2} + \frac{3-j}{2}$,
$0 \leq i \leq m-1; 1 \leq j \leq n-1,\ j$ odd; $1 \leq k \leq l, k$ even.
$\alpha\, i + \beta + \frac{k(n-1)}{2} + 2 - \frac{j}{2}$,
$0 \leq i \leq m-1; 1 \leq j \leq n-1,\ j$ even; $1 \leq k \leq l,\ k$ even.

The corresponding edge labeling is defined as follows:
Let $A = \{f^*(e_i) : f^*(e_i) = (\alpha(2i+1)+2)\,(mod\ q)\}$,
where $e_i = v_i v_{(i+1) mod\ m},\ 0 \leq i \leq m-1$.

$B = \left\{f^*(e_{ij}^k) : f^*(e_{ij}^k) = ((k-1)(n-1) + \beta + 2\alpha i + (j+3))\,(mod\ q)\right\}$,
where $e_{ij}^k = v_{ij}^k v_{ij+1}^k,\ 0 \leq i \leq m-1;\ 1 \leq j \leq n-2;\ 1 \leq k \leq l,\ k$ odd.

$C = \left\{f^*(e_{ij}^k) : f^*(e_{ij}^k) = (2\alpha i + \beta + k(n-1) + 3 - j)\,(mod\ q)\right\}$,
where $e_{ij}^k = v_{ij}^k v_{ij+1}^k,\ 0 \leq i \leq m-1;\ 1 \leq j \leq n-2;\ 1 \leq k \leq l, k$ even.

$D = \left\{f^*(e_{i0}^k) : f^*(e_{i0}^k) = \left(2\alpha i + \beta + 3 + \frac{(n-1)(k-1)}{2}\right)(mod\ q)\right\}$,
where $e_{i0}^k = v_i^k v_{i1}^k,\ 0 \leq i \leq m-1;\ 1 \leq k \leq l, k$ odd.

$E = \left\{f^*(e_{i0}^k) : f^*(e_{i0}^k) = \left(2\alpha i + \frac{k(n-1)}{2} + 2\right)(mod\ q)\right\}$
where $e_{i0}^k = v_i v_{i1}^k,\ 0 \leq i \leq m-1;\ 1 \leq k \leq l,\ \ k$ even.

It is clear that the vertex labels are distinct. Also it is easily verified that the edge label sets A, B, C, D and E are mutually disjoint and $A \cup B \cup C \cup D \cup E = \{0, 1, 2, \cdots, q-1\}$. Therefore, it follows that the induced mapping $f^* : E \to \{0, 1, 2, \cdots, q-1\}$ is a bijection. Hence the graph G is vertex-graceful when n is even.

Case 2. n odd.

Define $f(v_i) = \alpha i + 1, 0 \leq i \leq m-1$. Let $\beta = \frac{l(n-1)}{2}$

f(v_{ij}^k) = { $\alpha\, i + \frac{(k-1)}{2}\frac{(k+1)}{2}\frac{(n-1)}{2} + \frac{j+2}{2}$,

$0 \leq i \leq m-1; 1 \leq j \leq n-1,\ j$ even; $1 \leq k \leq l,\ k$ odd.

$\alpha i + \beta + \frac{(k-1)}{2}\frac{(k+1)}{2}\frac{(n-1)}{2} + \frac{j+3}{2}$,

$0 \leq i \leq m-1; 1 \leq j \leq n-1,\ j$ odd; $1 \leq k \leq l,\ k$ odd.

$\alpha i + \frac{k(n-1)}{2} - \frac{j-3}{2}$,

$0 \leq i \leq m-1; 1 \leq j \leq n-1,\ j$ odd; $1 \leq k \leq l,\ k$ even.

$\alpha i + \beta + \frac{k(n-1)}{2} + 2 - \frac{j}{2}$,

$0 \leq i \leq m-1; 1 \leq j \leq n-1,\ j$ even; $1 \leq k \leq l,\ k$ even.

The corresponding edge labeling is defined as follows:

Let $A = \{f^*(e_i) : f^*(e_i) = (\alpha(2i+1)+2)\,(mod\ q)\}$,
where $e_i = v_i v_{(i+1) mod\ m},\ 0 \leq i \leq m-1$.

$B = \left\{f^*(e_{ij}^k) : f^*(e_{ij}^k) = \left((k-1)\left(\frac{n-1}{2}\right)\left(\frac{k+1}{2}\right) + \beta + 2\alpha i + j + 3\right)(mod\ q)\right\}$,
where $e_{ij}^k = v_{ij}^k v_{ij+1}^k,\ 0 \leq i \leq m-1;\ 1 \leq j \leq n-2;\ 1 \leq k \leq l,\ k$ odd.

$C = \left\{f^*(e_{ij}^k) : f^*(e_{ij}^k) = (2\alpha i + \beta + k(n-1) + 3 - j)\,(mod\ q)\right\}$,
where $e_{ij}^k = v_{ij}^k v_{ij+1}^k, 0 \leq i \leq m-1;\ 1 \leq j \leq n-2;\ 1 \leq k \leq l,\ k$ even.

$D = \left\{f^*(e_{i0}^k) : f^*(e_{i0}^k) = \left(2\alpha i + \beta + 3 + \frac{(k-1)}{2}\frac{(k+1)}{2}\frac{(n-1)}{2}\right)(mod\ q)\right\}$,
where $e_{i0}^k = v_i v_{i1}^k, 0 \leq i \leq m-1; 1 \leq k \leq l,\ k$ odd.

$E = \left\{f^*(e_{i0}^k) : f^*(e_{i0}^k) = \left(2\alpha i + \frac{k(n-1)}{2} + 2\right)(mod\ q)\right\}$,
where $e_{i0}^k = v_i v_{i1}^k, 0 \leq i \leq m-1; 1 \leq k \leq l,\ k$ even .

It is clear that the vertex labels are distinct. Also it is easily verified that the edge label sets A, B, C, D and E are mutually disjoint and $A \cup B \cup C \cup D \cup E = \{0, 1, 2, \cdots, q-1\}$. Therefore, it follows that the induced mapping $f^* : E \to \{0, 1, 2, \cdots, q-1\}$ is a bijection. Hence the graph G is a vertex-graceful when n is odd. $\square$

Illustrative example of labeling of Theorem 3.1 is given in Figures 3.3

and 3.4.

We leave the following problem as an open question.

Problem 3.5. For any n, is it true that the graph $C_nP_n^{(l)}$ is vertex-graceful when

(i) m odd and l odd ?
(ii) m even and l odd ?
(iii) m even and l even ?

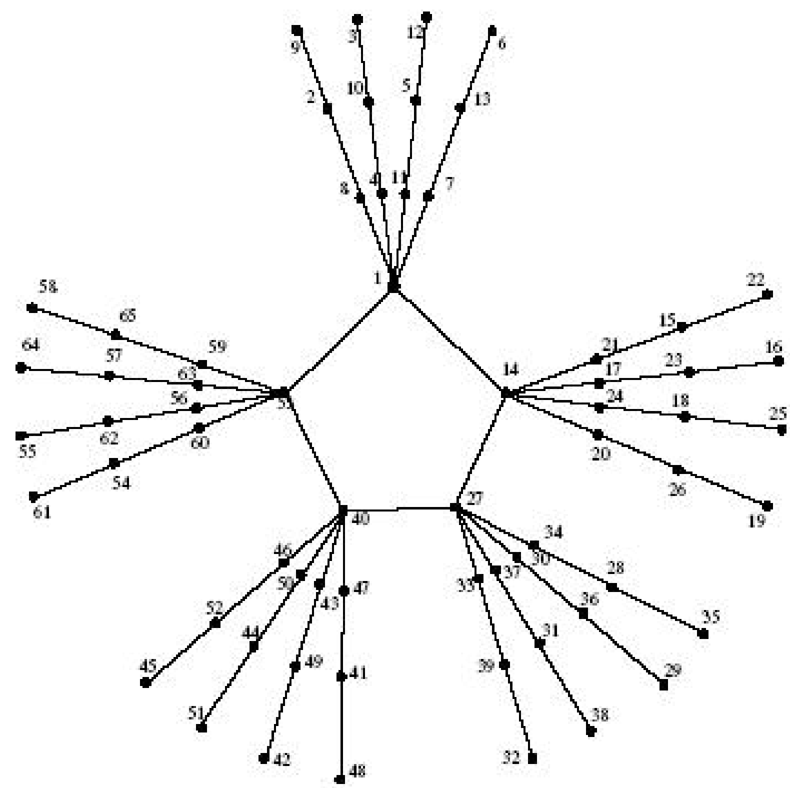

Figure 3.3: Vertex graceful labeled graph of $C_5 \cdot P_4^{(4)}$

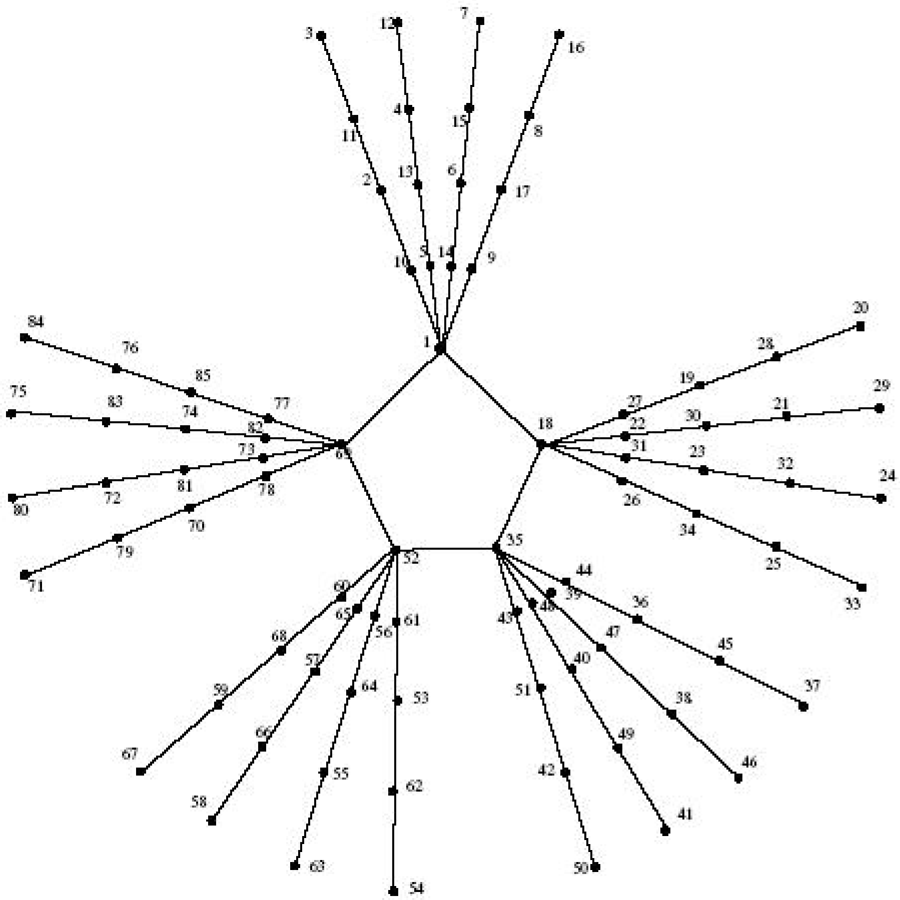

Figure 3.4: Vertex graceful labeled graph of $C_5 \blacksquare P_5^{(4)}$

4. Vertex-graceful Labeling of l copies of Actinia Graphs

In this section, it is proved that l copies of the actinia graphs $A(m_j, n)$ is vertex-graceful. Let $v_0^k, v_1^k, v_2^k, \cdots, v_{n-1}^k$ be the vertices of k^{th} copy of the cycle in $lA(m_j, n)$ for $0 \le j \le n-1$. Let v_{ij}^k denote the k^{th} pendant vertices of the k^{th} copy of $lA(m_j, n)$ such that v_{ij}^k are adjacent to vertices v_i^k for $0 \le i \le n-1, 1 \le j \le m_i, 1 \le k \le l$. The graph $lA(m_j, n)$ is shown in Figure 4.1.

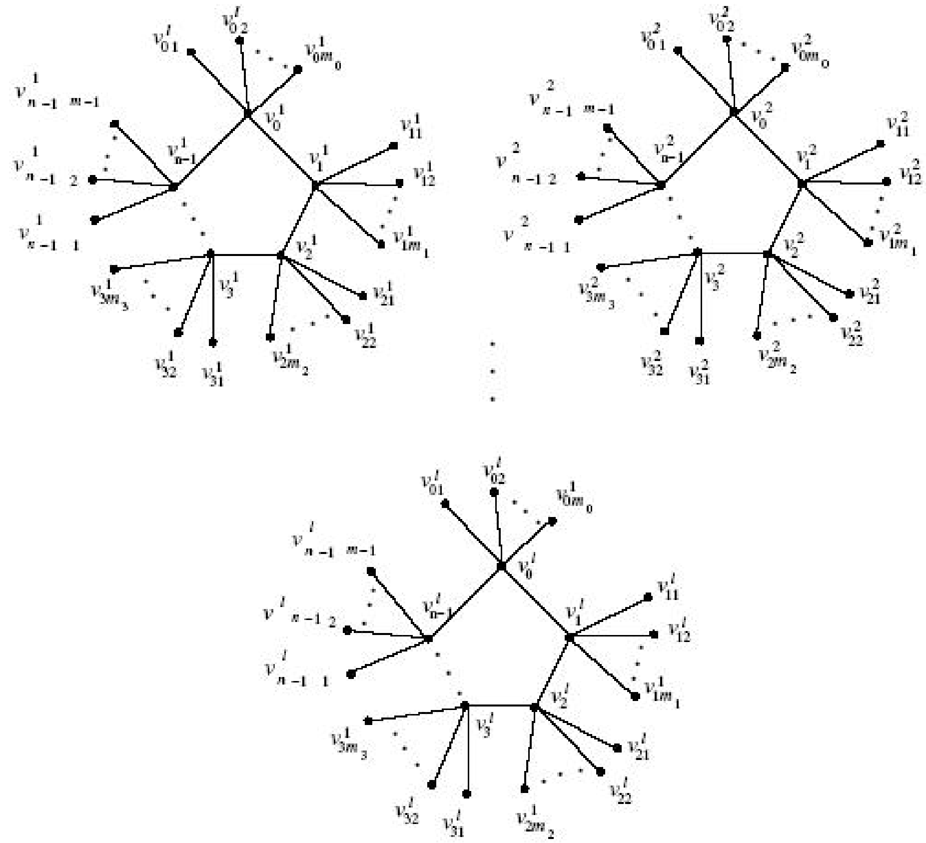

Figure 4.1: The graph $lA(m_j, n)$

Theorem 4.1. The graph $lA(m_j, n)$ is vertex-graceful for both n and l odd, $0 \leq i \leq n-1, 1 \leq j \leq m_i$.

Proof. Consider the graph $G = lA(m_j, n)$ for both n, l odd, $0 \leq i \leq n-1, 1 \leq j \leq m_i$. Let $v_0^k, v_1^k, v_2^k, \cdots, v_{n-1}^k$ be the vertices of the k^{th} copy of the cycle in G. Let v_{ij}^k denote the pendant vertices of the k^{th} copy such that v_{ij}^k are adjacent to the vertices v_i^k for $0 \leq i \leq n-1, 1 \leq j \leq m_i, 1 \leq k \leq l$. It is easily verified that G has $p = l\left(n + \sum_{i=1}^{n} m_i\right)$ vertices and $q = p$ edges. The required vertex-graceful labeling for vertices is defined as follows:

Let $f : V \to \{1, 2, \cdots, p\}$ be such that

$$f(v_i^l) = \begin{cases} p(k-1)l + \frac{i}{2} + \sum_{a=1}^{\frac{i}{2}} m_{2a-1} + 1, & 0 \le i \le n-1,\ i \text{ even};\ 1 \le k \le l. \\ \frac{p(k-1)}{l} + \frac{n-1}{2} + \sum_{a=1}^{\frac{i-1}{2}} m_{2a-1} + \sum_{a=0}^{\frac{i-1}{2}} m_{2a} + \frac{i+3}{2}, & 1 \le i \le n-1,\ i \text{ odd};\ 1 \le k \le l. \end{cases}$$

$$f(v_{ij}^l) = \begin{cases} p(k-1)l + \frac{n-1}{2} + \sum_{a=1}^{\frac{n-1}{2}} m_{2a-1} + \sum_{a=0}^{\frac{i-2}{2}} m_{2a} + \frac{i}{2} + j + 1, & 0 \le i \le n-1,\ i \text{ even};\ 1 \le j \le m_i;\ 1 \le k \le l. \\ \frac{p(k-1)}{l} + \frac{i+1}{2} + \sum_{a=1}^{\frac{i-1}{2}} m_{2a-1} + j, & 1 \le i \le n-1,\ i \text{ odd};\ 1 \le j \le m_i;\ 1 \le k \le l. \end{cases}$$

The corresponding edge labeling is defined as follows:

Let $A = \left\{ f^*(e_i^k) : f^*(e_i^k) = \left(\frac{2p(k-1)}{l} + \sum_{a=1}^{\frac{n-1}{2}} m_{2a-1} \frac{n-1}{2} + 2 \sum_{a=1}^{(i+1) mod\ n} m_{a-1} + (i+1) mod\ n \right) (mod\ q) \right\}$.

$B = \left\{ f^*(e_{ij}^k) : f^*(e_{ij}^k) = \left(\frac{2p(k-1)}{l} + \sum_{a=1}^{\frac{n-1}{2}} m_{2a-1} \frac{n-1}{2} + \sum_{a=0}^{i-1} m_a + i + j + 2 \right) (mod\ q) \right\}$.

It is clear that the vertex labels are distinct. Also it is easily verified that the edge labels sets are mutually disjoint and $A \cup B = \{0, 1, 2, \cdots, q-1\}$. Therefore it follows that the induced mapping $f^* : E \to \{0, 1, 2, \cdots, q-1\}$ is a bijection. Hence the graph $lA(m_j, n), 0 \le i \le n-1, 1 \le j \le m_i$ is vertex-graceful for both n and l odd. □

We leave the following problem as an open question.

Problem 4.2. Is it true that the graph $lA(m_j, n)$ is vertex-graceful when

(i) n odd and l even ?
(ii) n even and l even ?
(iii) n even and l odd ?

Illustrative example of the labeling of G of Theorem 4.1 is given in Figure 4.2.

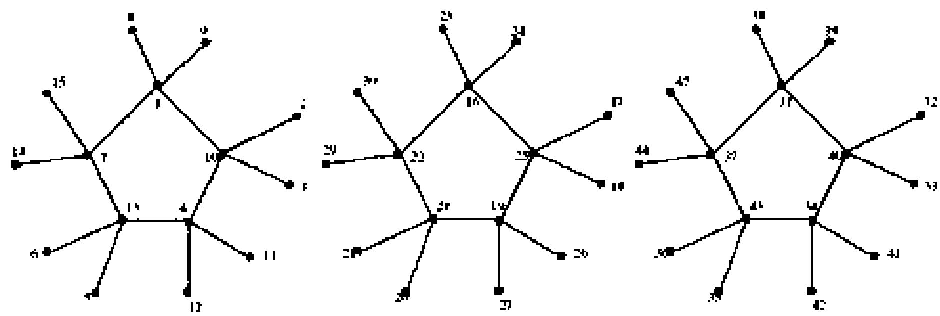

Figure 4.2: A vertex graceful labeled graph of 3A(2,5), $m_1 = m_2 = m_3 = m_4 = m_5 = 2$.

Remark 4.3. If $l = 1$ in Figure 4.1, then the graph G represents a actinia. Hence it follows from Theorem 4.1 that the graph $A(m_j, n)$ is vertex-graceful for any $m_j, 0 \leq j \leq n-1, n \geq 3$ with n odd.

Theorem 4.4. The graph $A(m_j, n)$ is strong vertex-graceful for n odd, $0 \leq i \leq n-1,\ 1 \leq j \leq m_i$.

Proof. Consider the graph $G = A(m_j, n)$ for n odd, $0 \leq i \leq n-1,\ 1 \leq j \leq m_i$. Let $v_0, v_1, v_2, \cdots, v_{n-1}$ be the vertices of the cycle in G. Let v_{ij} denote the pendant vertices of G such that v_{ij} are adjacent to the vertices v_i for $0 \leq i \leq n-1,\ 1 \leq j \leq m_i$. It is easily verified that G has $p = n + \sum_{i=0}^{n-1} m_i$ vertices and $q = p$ edges. The required strong vertex-graceful labeling for vertices is defined as follows:
Let $f : V \to \{1, 2, \cdots, p\}$ be such that

$$\mathrm{f}(\mathrm{v}_i) = \Big\{\ i2 + \sum_{a=1}^{\frac{i}{2}} m_{2a-1} + 1,\ \ 0 \leq i \leq n-1,\ i \text{ even}.$$

$$\tfrac{n-1}{2} + \sum_{a=1}^{\frac{n-1}{2}} m_{2a-1} + \sum_{a=0}^{\frac{i-1}{2}} m_{2a} + \frac{i+3}{2},\ \ 1 \leq i \leq n-1,\ i \text{ odd}.$$

$$\mathrm{f}(\mathrm{v}_{ij}) = \{\ \textstyle\sum_{a=1}^{\frac{n-1}{2}} m_{2a-1} + \frac{n-1}{2} + \displaystyle\sum_{a=0}^{\frac{i-2}{2}} m_{2a} + \frac{i}{2} + j + 1,$$
$$0 \leq i \leq n-1,\ i \text{ even};\ 1 \leq j \leq m_i.$$

$$\sum_{a=1}^{\frac{i-1}{2}} m_{2a-1} + \frac{i+1}{2} + j,$$
$$1 \leq i \leq n-1,\ i \text{ odd},\ 1 \leq j \leq m_i.$$

The corresponding edge labeling is defined as follows:

Let $\beta = \sum_{a=1}^{\frac{n-1}{2}} m_{2a-1} + \frac{n-1}{2}$

Let $A = \left\{ f_1(e_i) : f_1(e_i) = \beta + \sum_{a=1}^{(i+1)mod\ n} m_{a-1} + ((i+1)mod\ n) + 2 \right\},$
where $e_i = v_i v_{(i+1)mod\ n}, 0 \leq i \leq n-1.$

$$= \{\beta + m_0 + 3, \beta + m_0 + m_1 + 4, \beta + m_0 + m_1 + m_2 + 5, \cdots, \beta + 2\}.$$

$$B = \left\{ f_1(e_{ij}) : f_1(e_{ij}) = \beta + \sum_{a=0}^{i-1} m_a + i + j + 2 \right\},$$
where $e_{ij} = v_i v_{ij},\ 0 \leq i \leq n-1,\ 1 \leq j \leq m_i.$

$= \{\beta+3, \beta+4, \cdots, \beta+m_0+2; \beta+m_0+4; \beta+m_0+5, \cdots, \beta+m_0+m_1+3, \cdots; \beta+m_0+m_1+\cdots, m_{n-2}+n+2, \beta+m_0+m_1+\cdots, m_{n-2}+n+3, \cdots, \beta+m_0+m_1+\cdots, m_{n-2}+n+1+m_{n-1}\}.$

It is clear that the vertex labels are distinct. Also it is easily verified that the edge labels sets are mutually disjoint and $A \cup B = \{\beta + 2, \beta + 3, \cdots, \beta + m_0 + m_1 + \cdots, +m_{n-2} + m_{n-1} + n + 1\}$. Therefore the induced edge labels of G have q consecutive values. Hence the graph G is strong vertex-graceful. $\square$

Illustrative examples of the labeling of Theorem 4.4 is given in Figure 4.3.

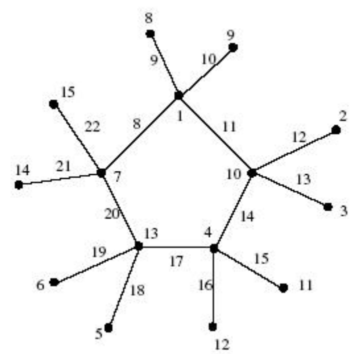

Figure 4.3: Strong vertex-graceful labeled graph of $A(2, 5)$.

References

[1] D. Acharya, S. Arumugam, and A. Rosa, *Labeling of Discrete Structures and Applications*, Narosa Publishing House, New Delhi, (2008).

[2] P. Bahl, S. Lake, and A. Wertheim, *Gracefulness of Families of Spiders*, Involve, 3, pp. 241-247, (2010).

[3] S. Bloom and D. F. Hsu, *On graceful digraphs and a problem in network addressing*, Congr. Numer., 35, pp. 91-103, (1982).

[4] S. Bloom and D. F. Hsu, *On graceful directed graphs that are computational models of some algebraic systems*, Graph Theory with Applications to Algorithms and Computers, Ed. Y. Alavi, Wiley, New York, (1985).

[5] G. Chartrand and P. Zhang,*An Introduction to Graph Theory*, Tata McGrawHill Edition, (2006).

[6] J. A. Gallian, *A Dynamic Survey of Graph Labeling*, The Electronic. J. Combin., DS17, pp. 1-384, (2014).

[7] S. W. Golomb, *How to number a graph*, Graph Theory and Computing, R. C.Read, ed., Academic Press, New York, pp. 23-37, (1972).

[8] F. Harary, Graph Theory, Addison-Wesley, (1969).

[9] F. Hsu and A. D. Keedwell, *Generalized complete mappings, neofields, sequenceable groups and block designs*, I, Pacific J. Math., 111, pp. 317-332, (1984).

[10] F. Hsu and A. D. Keedwell, *Generalized complete mappings, neofields, sequenceable groups and block designs*, II, Pacific J. Math., 117, pp. 291-312, (1985).

[11] J. Jeba Jesintha and G. Sethuraman, *A new class of graceful rooted trees*, J. Discrete Math. Sci. Cryptogr., 11, pp. 421-435, (2008).

[12] A. Rosa, *On certain valuations of the vertices of a graph*, Theory of Graphs, Gordon and Breach, N. Y. and Dunod Paris, pp. 349-355, (1967).

[13] P. Selvaraju, P. Balaganesan and J. Renuka, V. Balaji, *On Vertex-graceful Labeling*, Bulletin of Kerala Mathematics Association, 9, pp. 179-184, (2012).

[14] P. Selvaraju, P. Balaganesan and J. Renuka, *Vertex-graceful graph* $c_{2k} \cup C_{2k+1}$, European Journal of Scientific and Research, 97, pp. 192-196, (2013).

[15] P. Selvaraju, P. Balaganesan and J. Renuka, *Vertex-graceful Labeling of Some Path Related Graphs*, International J. Math. Combin., 3, pp. 44-49, (2013).

[16] P. Selvaraju, P. Balaganesan and J. Renuka and M. L. Suresh, *Vertex-graceful Labeling of* $C_i \cup C_j \cup C_l$, Applied Mathematical Sciences, 8 (82), pp. 4047-4052, (2014).

[17] P. Selvaraju, P. Balaganesan and J. Renuka and M. L. Suresh, *Harmonious and Vertex-graceful Labeling of Path and Star Related Graphs*, International Journal of Pure and Applied Mathematics, 93, pp. 501-509, (2014).

[18] Sin-min Lee, Y. C. Pan and Ming Chen Tsai, *On vertex-graceful graph* $(p, p+1)$ - *graphs*, Congressus Numerantium 172, pp. 65-78, (2009).

[19] W. C. Shiu, P. C. B. Lam, *Super-edge-graceful labelings of multi-level wheel graphs, fan graphs and actinia graphs*, Congressus Numerantium 174, pp. 49-63, (2005).

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