Proyecciones

Vol. 18, N° 2, pp. 195-205, December 1999

Universidad Católica del Norte

Antofagasta - Chile

ON (*)-IDEALS AND POSITIVE IMPLICATIVE IDEALS IN BCI-ALGEBRAS

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Abstract

In the present paper, we given some equivalent conditions of ()-ideals and positive implicative ideals in BCI-algebras and we clarify the relation of the two classes of ideals. Also, we obtain further properties of these ideals. Finally, we discuss a quotient algebra of a BCI-algebra by a closed (*)-ideal and a quotient algebra by positive implicative ideal.*

KEY WORDS AND PHRASES : *BCI-algebra, closed ideal, (*)-ideal, positive implicative ideal, quotient algebra.*

1991 AMS Subject classification : *03G25, 06F35.*

The study of ideals forms an important part of the theory of BCI-algebra. Since K. Iséki [6] generalized the notion of ideals in BCK-algebras to BCI-algebras, several classes of ideals in BCI-algebras have occurred, for instance, closed ideals [2], strong ideals [1], p -ideals [16], positive implicative ideals [3], and so on. In [4] and [9] closed ideals, strong ideals, and p -ideals were further investigated. In particular, it is shown that in a BCI-algebra the notion of strong ideals and closed $(*)$ -ideals coincide. As a continuation of [4] and [9], we now, will deeply study further properties of $(*)$ -ideals and positive implicative ideals and clarify the relation of the two classes of ideals.

Let X be a nonempty set. Let $*$ be a binary operation on X and 0 is a constant of X . An algebra $\langle X; *, 0 \rangle$ of type $(2, 0)$ is said to be a BCI-algebra if for all $x, y, z \in X$,

- (I) $((x * y) * (x * z)) * (z * y) = 0$,
- (II) $(x * (x * y)) * y = 0$,
- (III) $x * x = 0$,
- (IV) $x * y = 0$ and $y * x = 0$ imply $x = y$

A binary relation $\leq$ on X can be defined by putting $x \leq y$ if and only if $x * y = 0$. Then $\langle X; \leq \rangle$ is a partially ordered set with a minimal element 0 .

In any BCI-algebra X the following properties hold :

- (1) $(x * y) * z = (x * z) * y$,
- (2) $x * 0 = x$,
- (3) $(x * z) * (y * z) \leq x * y$
- (4) $x \leq y$ implies $x * z \leq y * z$ and $z * y \leq z * x$.

A BCI-algebra X with the condition $0 \leq x$ for all $x \in X$ is called a BCK-algebra.

A nonempty subset I of a BCI-algebra X is called an ideal of X if

- (i) $0 \in I$,
- (ii) $x \in I$ whenever $x * y \in I$ and $y \in I$.

Every ideal I of X satisfies

- (iii) $x \leq y$ and $y \in I$ imply $x \in I$.

An ideal I of a BCI-algebra X is said to be closed if $0 * x \in I$ for all $x \in I$. An ideal I of a BCI-algebra X is closed if and only if I is a subalgebra of X . Every ideal of a BCK-algebra is always closed. An element a in a BCI-algebra X is said to be an atom if for all $x \in X$, $x * a = 0$ implies $x = a$. Let $L(X)$ be the set of all atoms of a BCI-algebra X . For any $a \in L(X)$, the set $\{x \in X \mid a \leq x\}$ is called the branch of a BCI-algebra X and denoted by $V(a)$. The branch $V(0)$ is the BCK-part of X , which

is denoted by $B(X) = \{x \in X \mid 0 \leq x\}$. For all x in a BCI-algebra X , $a_x = 0 * (0 * x) \in L(X)$. In the sequel, we will use the following properties:

(5) $L(X)$ is a subalgebra of X ,

(6) $L(X) = \{0 * (0 * x) \mid x \in X\} = \{0 * x \mid x \in X\}$,

(7) If $x, y \in X$, then x and y belong to the same branch if and only if $x * y \in B(X)$ and so $x * a_x \in B(X)$.

A BCI-algebra X is said to be p -semisimple if $B(X) = \{0\}$ or equivalently, $L(X) = X$.

The above concepts and results can be found in [1], [7] and [12]. Throughout this paper, X will mean a BCI-algebra unless mentioned otherwise.

Before starting to discuss $(*)$ -ideals and positive implicative ideals, we give an elementary property of BCI-algebras.

Theorem 1 : Let X be a BCI-algebra. Then for all $x \in X$ and $y \in B(X)$, $x * y \leq x$.

Proof : Since $y \in B(X)$ implies $0 * y = 0$, we have $(x * y) * x = (x * x) * y = 0 * x = 0$, that is $x * y \leq x$. $\diamond$

E. H. Roh, Y. B. Jun and S. M. Wei [13] introduced the notion of $(*)$ -ideals in BCI-algebras and obtain some of its properties.

Definition 1 ^[13] : If an ideal I of a BCI-algebra X satisfies the condition

(iv) $x \in I$ and $a \in X - I$ imply $x * a \in I$, then I is called a $(*)$ -ideal of X .

Obviously, each ideal of a BCK-algebra is always a $(*)$ -ideal, but for a proper BCI-algebra, $\{0\}$ is not a $(*)$ -ideal.

Theorem 2 : If I is an ideal of a BCI-algebra X , and $L(X) \subseteq I$, then I is a $(*)$ -ideal of X .

Proof : Assume that $x \in I$ and $y \in X - I$. Since $y * a_y \in B(X)$, by Theorem 1, we have $x * (y * a_y) \leq x$, and so $x * (y * a_y) \in I$. Since

$$\begin{aligned} ((x * y) * a_y) * (x * (y * a_y)) &= ((x * (x * (y * a_y))) * y) * a_y \\ &\leq ((y * a_y) * y) * a_y \\ &= (0 * a_y) * a_y \in L(X), \end{aligned}$$

and $L(X) \subseteq I$, we have $((x * y) * a_y) * (x * (x * a_y)) \in I$. Combining $x * (y * a_y) \in I$ and using (ii), we have $(x * y) * a_y \in I$. Since $a_y \in L(X) \subseteq I$, it follows that $x * a \in I$. Therefore, I is a $(*)$ -ideal of X . $\diamond$

The converse of Theorem 2 need not hold, as is shown in the following example.

Example 1 ^[9] : Let $X = \{2^n \mid n = \pm 1, \pm 2, \dots\}$, and let $\div$ be the usual division. Then $\langle X; \div, 1 \rangle$ is a p -semisimple BCI-algebra and $I = \{1, 2, 2^2, \dots\}$ is an ideal of X . Observe that for all natural numbers m and n , we have $2^{-n} \in X - I$, $1 \div 2^{-n} \in I$ and $2^m \div 2^{-n} = 2^{m+n} \in I$. Hence I is a $(*)$ -ideal of X . But $L(X) \not\subseteq I$ as $L(X) = X$.

One easily sees that I is not closed, because $1 \div 2 = \frac{1}{2} \notin I$.

It is natural to ask whether or not for closed ideals the converse of Theorem 2 holds. The answer is positive.

Theorem 3 : Let I be a closed $(*)$ -ideal of a BCI-algebra. Then $L(X) \subseteq I$.

Proof : If $L(X) \not\subseteq I$, then there is $a \in L(X) - I$, that is, $a \in L(X)$ and $a \notin I$. But $a \in L(X)$ implies $a = 0 * (0 * a)$. By (iv), $a \notin I$ implies $0 * a \in I$. Furthermore, $a = 0 * (0 * a) \in I$, because I is closed. Therefore, we have a contradiction. Hence $L(X) \subseteq I$. $\diamond$

By Theorem 2 and Theorem 3, we have

Corollary 4 : If I is a closed ideal of a BCI-algebra X , then I is a $(*)$ -ideal if and only if $L(X) \subseteq I$.

Definition 2 ^[11] : Let $\mathbf{N}$ be the set of all natural numbers. Let X be a BCI-algebra. For $x \in X$, we define x^n by $x^1 = x$, $x^{n+1} = x * (0 * x^n)$. If there is $n \in \mathbf{N}$ such that $x^n \in B(X)$, then x is called finite periodic and $|x| = \min \{n \in \mathbf{N} \mid x^n \in B(X)\}$ is the period of x . The set $P(X) = \{x \in X \mid |x| < \infty\}$ is called the periodic part of X . If $X = P(X)$, then X is said to be periodic.

Proposition 5 [11, Theorem 11] : Let X be a periodic BCI-algebra. Then each ideal of X is closed.

Combing Corollary 4 and Proposition 5, we get

Theorem 6 : If X is a periodic BCI-algebra, and I is an ideal of X , then I is a $(*)$ -ideal of X if and only if $L(X) \subseteq I$.

Since a finite BCI-algebra is periodic (see [11, Theorem 8], we obtain

Corollary 7 : An ideal I of a finite BCI-algebra is a $(*)$ -ideal if and only if $l(X) \subseteq I$.

Now, we give simpler characterizations of $(*)$ -ideals and closed $(*)$ -ideals.

Theorem 8 : Suppose I is an ideal of a BCI-algebra X . Then I is a $(*)$ -ideal of X if and only if $a \in X - I$ implies $0 * a \in I$.

Proof ($\implies$) : Trivial.

($\impliedby$) : Suppose an ideal I satisfies that $a \in X - I$ implies $0 * a \in I$. If $x \in I$ and $a \in X - I$, then $(x * a) * x = (x * x) * a = 0 * a \in I$. By (ii), $x * a \in I$. Therefore, I is a $(*)$ -ideal. $\diamond$

Theorem 9 : An ideal I of a BCI-algebra X is a closed $(*)$ -ideal if and only if $0 * x \in I$ for all $x \in X$.

Proof : Let I be a closed $(*)$ -ideal and $x \in X$. If $x \in X - I$, then $0 * x \in I$, because I is a $(*)$ -ideal. If $x \in I$, then $0 * x \in I$, because I is a closed ideal. Hence $0 * x \in I$, for all $x \in X$.

Conversely, suppose $0 * x \in I$ for all $x \in X$. In other words, $L(X) \subseteq I$ by (6). It follows from the definition of closed ideals that I is closed. Then, applying Corollary 4, we get that I is a $(*)$ -ideal. $\diamond$

Theorem 10 : A nonempty subset I of a BCI-algebra X is a closed $(*)$ -ideal of X if and only if (i) $0 \in I$, and (v) for all $x, y, z \in X$, $x * y \in I$ and $y \in I$ imply $x * z \in I$.

Proof : Suppose that I satisfies (i) and (v). Assume that $z = 0$ in (v). Then I satisfies $x * y \in I$ and $y \in I$ imply $x \in I$. Hence I is an ideal of X . Let $x = y = 0$ in (v). Then $0 * z \in I$ for all $z \in X$. By Theorem 9, I is a closed $(*)$ -ideal of X .

Conversely, let I be a closed $(*)$ -ideal of X . If $x * y \in I$ and $y \in I$, then by closeness of I , we have $x * z \in I$ whenever $z \in I$. Thus, for all $x, y, z \in X$, $x * y \in I$ and $y \in I$ imply $x * z \in I$. $\diamond$

For a subset A of X , let $[A]$ resp. $(A]_{*}$ denotes the least ideal (resp. least closed $(*)$ -ideal) containing A in X .

Theorem 11 : Let A be a subset of a BCI-algebra X . Then $(A]_{*} = (A \cup L(X)]$.

Proof : It follows directly from Theorem 9 and (6). $\diamond$

Next, we discuss positive implicative ideals in BCI-algebras and their relation with $(*)$ -ideals. The notion of positive implicative ideals in BCK-algebras was introduced by K. Iséki in [5] and generalized to BCI-algebras by C. S. Hoo in [3].

Definition 3 ^[5] : A nonempty subset I of a BCI-algebra X is called a **positive implicative ideal** of X if it satisfies (i) $0 \in I$ and (vi) $(x * y) * z \in I$ imply $x * z \in I$.

Any positive implicative ideal must be an ideal, but the converse need not be hold.

Definition 4 ^[14] : A BCI-algebra X is **quasi-associative** if for $x, y, z \in X$, $(x * y) * z \leq x * (y * z)$.

Proposition 12 ^[14] : A BCI-algebra X is **quasi-associative** if and only if for all $x \in X$, $0 * x \leq x$, or equivalently $(0 * x) * x = 0$.

For a nonempty subset A of a BCI-algebra X , and a fixed element a of X denote $A_a = \{x \in X \mid x * a \in A\}$.

Theorem 13 : Let X be a quasi-associative BCI-algebra and let A be a positive implicative ideal of X . Then for any fixed $a \in X$, A_a is the least ideal containing A and a .

Proof : Since X is quasi-associative, $(0 * a) * a = 0 \in A$. Combining $a * a = 0 \in A$ any by (vi), we have $0 * a \in A$, that is, $0 \in A_a$. Also, let $x * y \in A_a$ and $y \in A_a$. Then $(x * y) * a \in A$ and $y * a \in A$. By (vi), $x * a \in A$ and so $x \in A_a$. Therefore, A_a is an ideal of X .

Since X is quasi-associative, for all $x \in X$, we have $(x * a) * a \leq x * (a * a) = x * 0 = x \in A$ and so $(x * a) * a \in A$ by (iii). Observe that $a * a = 0 \in A$. Hence $x * a \in A$. Namely, $x \in A_a$. Thus $A \subseteq A_a$. Clearly, $a \in A_a$.

Suppose that I is any ideal containing A and a . If $x \in A_a$, then $x * a \in A \subseteq I$, and so $x * a \in I$. It follows from $a \in I$ that $x \in I$. Hence $A_a \subseteq I$. This means that A_a is the least ideal containing A and a . $\diamond$

Theorem 14 : Let A be an ideal of a BCI-algebra X . If for all $a \in X$, A_a is an ideal of X , then A is a positive implicative ideal of X .

Proof : Let $x, y, z \in X$, $(x * y) * z \in A$ and $y * z \in A$. Then $x * y \in A_z$ and $y \in A_z$. Since A_z is an ideal of X , by (ii) $x \in A_z$ and so $x * z \in A$. Hence A is a positive implicative ideal of X . $\diamond$

By Theorem 13 and Theorem 15, we have

Corollary 15 : Let X be a quasi-associative BCI-algebra and A be an ideal of X . Then A is positive implicative if and only if for any $a \in X$, A_a is an ideal of X .

The following result is a generalization of [8, Theorem 4].

Theorem 16 : If I is an ideal of a BCI-algebra X , then the following are equivalent : (8) I is positive implicative,

(9) $(x * y) * y \in I$ implies $x * y \in I$, for all $x, y \in X$,

(10) $(x * y) * z \in I$ implies $(x * z) * (y * z) \in I$ for all $x, y, z \in I$.

Theorem 17 : Suppose X is a quasi-associative BCI-algebra and A is a positive implicative ideal of X . Then A is a closed $(*)$ -ideal of X .

Proof : For all $a \in X$, A_a is an ideal of X by Theorem 13. Since $0 \in A_a$, it follows that $0 * a \in A$. By Theorem 9, we know that A is a closed $(*)$ -ideal of X . $\diamond$

Example 2 : Let $X = \{0, 1, 2, 3, 4, 5\}$ and the $*$ table is given by

$*$	0	1	2	3	4	5
0	0	0	0	3	3	3
1	1	0	0	3	3	3
2	2	2	0	5	5	3
3	3	3	3	0	0	0
4	4	3	3	1	0	0
5	5	5	3	2	2	0

Then $\langle X : *, 0 \rangle$ is a quasi-associative BCI-algebra and $I = \{0, 3\}$ is closed $(+)$ -ideal of X , but I is not positive implicative as $(4 * 3) * 3 = 1 * 3 = 3 \in I$ and $4 * 3 = 1 \notin I$. Hence the converse of Theorem 17 does not hold.

Corollary 18 ^[15] : Let I be a positive implicative ideal of a quasi-associative BCI-algebra X . Then $L(X) \subseteq I$.

Proof : It follows from Theorem 3 and Theorem 17. $\diamond$

By quotient algebras, we can characterize $(*)$ -ideals. Let I be an ideal of a BCI-algebra X . Define a binary relation $\sim$ on x by $x \sim y$ if and only if $x * y \in I$. Then $\sim$ is a congruence relation on X . Let $C_x = \{y \in X \mid y \sim x\}$ and $X/I = \{C_x \mid x \in X\}$. Define $C_x * C_y = C_{x*y}$. Then C_0 is the greatest closed ideal contained in I , and $\langle X/I; *, C_0 \rangle$ is a BCI-algebra, called the quotient algebra of X by I (see [7]).

Theorem 19 : Let I be a closed ideal of a BCI-algebra X . Then I is a $(*)$ -ideal if and only if $\langle X/I; *, C_0 \rangle$ is a BCK-algebra.

Proof ($\implies$): Suppose I is a closed $(*)$ -ideal of X . By Theorem 9, $0 * x \in I$ for all $x \in X$, and so $C_0 * C_x = C_{0*x} = I = C_0$ for all $x \in X$. Hence $\langle X/I; *, C_0 \rangle$ is a BCK-algebra.

($\impliedby$): If $\langle X/I; *, C_0 \rangle$ is a BCK-algebra, then for all $x \in X$, $C_0 * C_x = C_0$. Thus $C_{0*x} = I$. Hence $0 * x \in I$. By Theorem 9, I is a closed $(*)$ -ideal of X . $\diamond$

Corollary 20 : Let X be a quasi-associative BCI-algebra, and let I be a positive implicative ideal of X . Then $\langle X/I; *, C_0 \rangle$ is a positive implicative BCK-algebra.

Proof : By Theorem 17 and Theorem 19, it suffices to prove that $\langle X/I; *, C_0 \rangle$ is positive implicative. So, we assume that $(C_x * C_y) * C_y \in \{C_0\}$. Hence $C_{(x*y)*y} = I$, and so $(x * y) * y \in I$. By (9), we have $x * y \in I$. By (9), we have $x * y \in I$. Thus $C_x * C_y = C_{x*y} = C_0 \in \{C_0\}$. This is to say that the zero ideal $\{C_0\}$ is positive implicative in BCK-algebra X/I . By [8, Corollary 7], $\langle X/I; *, C_0 \rangle$ is a positive implicative BCK-algebra. $\diamond$

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Received : March, 1998

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