Proyecciones

Vol. 18, N° 2, pp. 155-164, December 1999

Universidad Católica del Norte

Antofagasta - Chile

SOLUTIONS TO THE PRESCRIBED MEAN CURVATURE EQUATION

PABLO AMSTER

P. DE NÁPOLI

and

M. C. MARIANI

Universidad de Buenos Aires - Argentina

Abstract

We apply variational methods in order to prove that the nonlinear system (1) admits at least one regular solution.

1. Introduction

We consider the problem of finding $u \in H^1(\Omega, \mathbf{R}^3)$ such that

$$(1.1) \quad \begin{cases} \Delta u = 2H(u) u_x \wedge u_y & \text{in } \Omega \\ u = \gamma & \text{in } \partial\Omega \end{cases}$$

where H is a given continuously differentiable function and Ω is an open subset of $\mathbf{R}^2$ with C^1 boundary.

The system of differential equations above is the Dirichlet problem for the prescribed mean curvature equation in isothermal coordinates, also called H-system, which arises in the generalized Plateau's problem (see [O]).

Hildebrandt, Wente (see [W1, W2]) and others have studied the problem for constant H . For this case, Brezis-Coron and Struwe have shown by variational methods (see [BC], [S]) that if the boundary data is small and non constant, then there are at least two weak solutions. The problem has been also studied by fixed point methods (see [AMR]).

The goal of this work is to show that also for non constant H , the problem may be solved by variational methods, under rather general conditions on H and the boundary data.

Our main result is the following:

Theorem 1 : Let $\gamma \in H^{1/2}(\partial\Omega, \mathbf{R}^3) \cap L^\infty(\partial\Omega, \mathbf{R}^3)$, $R = \|\gamma - c\|_{L^\infty(\partial\Omega, \mathbf{R}^3)}$ for some constant c , and $B \subset \mathbf{R}^3$ the closed ball of radius R centered at c . Then, if $H \in C^1(B)$ satisfies

$$\|H\|_{L^\infty(B)} < \frac{1}{R}$$

then problem (1) admits a weak solution $u \in H^1(\Omega, \mathbf{R}^3) \cap L^\infty(\Omega, \mathbf{R}^3)$, with $\|u - c\|_{L^\infty(B)} \leq R$.

Remark : By a result of Bethuel ([B]) weak solutions of the mean curvature equation are $C^{2,\alpha}(\Omega)$ for H bounded and Lipschitz continuous. Moreover, by a result of Chanillo and Li ([CL]) the solutions are continuous up to the boundary.

2. Variational formulation of the problem

For simplicity, we may assume that $c = 0$ and extend H to a function of class $C^1(\mathbf{R}^3)$ such that $\|H\|_{L^\infty(\mathbf{R}^3)} < \frac{1}{R}$.

We define the vector field $q : \mathbf{R}^3 \rightarrow \mathbf{R}^3$:

$$q(x, y, z) = \frac{4}{3} \left(\int_0^x H(t, y, z) dt, \int_0^y H(x, t, z) dt, \int_0^z H(x, y, t) dt \right)$$

It is immediate that $\operatorname{div} q = 4H$ and $|q(x, y, z)| \leq \frac{4}{3} \|H\|_{L^\infty} |(x, y, z)|$.

We'll work in the Banach space $H^1(\Omega, \mathbf{R}^3) \cap L^\infty(\Omega, \mathbf{R}^3)$, with the norm:

$$\|u\| = \|u\|_{H^1} + \|u\|_{L^\infty}$$

and consider energy functional

$$E(u) = D(u) + Q(u)$$

where D is the Dirichlet's integral and Q is the functional

$$Q(u) = \int_{\Omega} q(u) \cdot (u_x \wedge u_y)$$

We'll use the estimatives

$$(2.1) \quad |Q(u)| \leq \int_{\Omega} |q(u)| \cdot |u_x \wedge u_y| \leq \frac{2}{3} \|H\|_{L^\infty} \|u\|_{L^\infty} \int_{\Omega} |\nabla u|^2$$

$$(2.2) \quad E(u) \geq \left(1 - \frac{2}{3} \|H\|_{L^\infty} \|u\|_{L^\infty} \right) \int_{\Omega} |\nabla u|^2$$

3. Technical lemmas

In this section we present some technical facts.

Lemma 2 Q is continuous in $H^1 \cap L^\infty$.

Lemma 3 We consider the functional

$$I(u, h) = \int_{\Omega} H(u) \cdot (u_x \wedge u_y) \cdot h$$

where $u \in H^1 \cap L^\infty$ and $h \in L^\infty$.

Then I is continuous.

Proof :

Lemma 2 and Lemma 3 follow from simple computations.

Lemma 4

The functional E is Gateaux differentiable in $H^1 \cap L^\infty$ for variations in $H_0^1 \cap L^\infty$, and

$$dE(u)(h) = 2 \int_{\Omega} \nabla u \cdot \nabla h + \int_{\Omega} 4H(u) \cdot (u_x \wedge u_y) \cdot h$$

for every $u \in H^1 \cap L^\infty, h \in H_0^1 \cap L^\infty$.

Proof :

Let us first take $u, h \in C^1$ and define a function $U : \Omega \times \mathbf{R} \rightarrow \mathbf{R}^3$ by:

$$U(x, y, t) = u(x, y) + t \cdot h(x, y)$$

We'll write $u^t(x, y) = U(x, y, t)$.

We recall that the field q in $\mathbf{R}^3$ is associated to the 2-form ω given by:

$$\omega(X, Y) = (X \wedge Y) \cdot q(u)$$

with

$$d\omega = \operatorname{div}(q) du^1 \wedge du^2 \wedge du^3$$

Applying Stokes theorem to the cylinder $C = \Omega \times [0, \varepsilon]$, we obtain:

$$\int_C d(U^* \omega) = \int_{\partial C} U^* \omega$$

Moreover,

$$d(U^* \omega) = U^*(d\omega) = U^*(\operatorname{div} q(u) du^1 \wedge du^2 \wedge du^3)$$

$$= 4H(U) \cdot \frac{\partial(U^1, U^2, U^3)}{\partial(x, y, t)} dx \wedge dy \wedge dt = 4H(u^t) \cdot (u_x^t \wedge u_y^t) \cdot h$$

We have:

$$U^*\omega = \omega\left(\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}\right)dx \wedge dy = q(u^t) \cdot (u_x^t \wedge u_y^t)$$

on $\Omega \times \{\varepsilon\}$ and $\Omega \times \{0\}$ and

$$U^*\omega = \omega\left(\frac{\partial U}{\partial \sigma}, \frac{\partial U}{\partial t}\right)d\sigma \wedge dt = q(u^t) \cdot \left(\frac{\partial U}{\partial \sigma} \wedge h\right)d\sigma \wedge dt$$

on the lateral surface $\partial\Omega \times [0, \varepsilon]$, where σ is the unit tangent vector on $\partial\sigma$.

Thus,

$$\begin{aligned} & \int_0^\varepsilon \int_\Omega 4H(u^t) \cdot (u_x^t \wedge u_y^t) \cdot h dx \wedge dy \wedge dt = \\ & \int_{\Omega \times \{\varepsilon\}} q(u^\varepsilon) \cdot (u_x^\varepsilon \wedge u_y^\varepsilon) dx \wedge dy \wedge dt - \int_{\Omega \times \{0\}} q(u) \wedge (u_x \wedge u_y) \cdot h \\ & + \int_{\partial\Omega \times [0, \varepsilon]} q(u) \cdot \left(\frac{\partial U}{\partial \sigma} \wedge h\right) d\sigma \wedge dt \end{aligned}$$

Assuming that $h = 0$ on the boundary of Ω , we get:

$$\int_0^\varepsilon \int_\Omega 4H(u^t) \cdot (u_x^t \wedge u_y^t) \cdot h = Q(u^\varepsilon) - Q(u).$$

By density, this formula holds for $u \in H^1 \cap L^\infty$ and $h \in H_0^1 \cap L^\infty$.

By dominated convergence we conclude that

$$dQ(u)(h) = \int_\Omega 4H(u) \cdot (u_x \wedge u_y) \cdot h$$

which completes the proof.

Remark: By lemmas 2 and 3 dE is continuous, i.e. $E \in C^1(H_0^1 \cap L^\infty)$.

Corollary 5 :

If u is a critical point of functional E for variations in $H_0^1 \cap L^\infty$ then u is a weak solution of (1).

4. Proof of Theorem 1

We fix $R' > R$ such that $\|H\|_{L^\infty} R' < 1$ and consider the nonempty set:

$$K = \{u \in H^1(\Omega, \mathbf{R}^3) : u = \gamma \text{ in } \partial\Omega, \|u\| \leq R'\}$$

Then we obtain

Lemma 6 :

E achieves a minimum in K .

Proof :

By (3), the functional E is bounded from below in K . Let $E_0 = \inf_{v \in K} E(v)$ and consider a minimizing sequence (u^n)

By (3) and Poincaré's inequality u^n is bounded in H^1 , and then we may suppose that

$$u^n \rightarrow \underline{u} \text{ in } H^1 \text{ weakly}$$

and

$$u^n \rightarrow \underline{u} \text{ a.e.}$$

By the continuity of q we conclude that:

$$q(u^n) \rightarrow q(\underline{u}) \text{ a.e.}$$

and since $|q(u^n)| \leq \frac{4}{3}\|H\|_{L^\infty}$, we may also assume that:

$$q(u^n) \rightarrow q(\underline{u}) \text{ in } L^\infty \text{ weak}^*$$

Set $\theta^n = u^n - u \in H_0^1$. Then $\|\theta^n\|_{L^\infty} \leq 2R'$, and

$$\begin{aligned} E(u^n) &= \int_{\Omega} |\nabla \underline{u}|^2 + 2 \int_{\Omega} \nabla u \cdot \nabla \theta^n \\ &+ \int_{\Omega} |\nabla \theta^n|^2 + \int_{\Omega} q(u^n) \cdot (\underline{u}_x + \theta_x^n) \wedge (\underline{u}_y + \theta_y^n) \end{aligned}$$

Thus,

$$\int_{\Omega} \nabla u \cdot \nabla \theta^n \rightarrow 0$$

Moreover, being

$$\int_{\Omega} q(u^n) \cdot (\underline{u}_x \wedge \theta_y^n) = - \int_{\Omega} \theta_y^n \cdot (q(u^n) \wedge \underline{u}_x)$$

then

$$\int_{\Omega} q(u^n) \cdot (\underline{u}_x \wedge \theta_y^n) \rightarrow 0,$$

since $\theta_y^n \rightarrow 0$ weakly in L^2 , and

$$|q(u^n) \wedge \underline{u}_x|^2 \leq \left(\frac{4}{3} |u^n| \right)^2 \cdot |\underline{u}_x|^2 \leq \left(\frac{4}{3} R' \right) |\underline{u}_x|^2.$$

In the same way, we conclude that

$$\int_{\Omega} q(u^n) \cdot (\theta_x^n \wedge \underline{u}_y) \rightarrow 0$$

We also have the estimative:

$$\begin{aligned} \left| \int_{\Omega} q(u^n) \cdot \theta_x^n \wedge \theta_y^n \right| &\leq \int_{\Omega} 4 \|H\|_{L^\infty} \|u^n\|_{L^\infty} |\theta_x^n \wedge \theta_y^n| \\ &\leq \frac{2}{3} \|H\|_{L^\infty} R' \int_{\Omega} |\nabla \theta^n|^2 \leq \frac{2}{3} \int_{\Omega} |\nabla \theta^n|^2 \end{aligned}$$

Finally as $q(u^n) \rightarrow q(\underline{u})$ weak* in L^∞ :

$$\int_{\Omega} q(u^n) \cdot (\underline{u}_x \wedge \underline{u}_y) = \int_{\Omega} q(\underline{u}) \cdot (\underline{u}_x \wedge \underline{u}_y) + o(1)$$

Then,

$$E(u^n) \geq E(u) + o(1) + \frac{1}{3} \int_{\Omega} |\nabla \theta^n|^2$$

and we can conclude that $E(u) = E_0$ and $\int_{\Omega} |\nabla \theta^n|^2 \rightarrow 0$.

Lemma 7 :

Let $u \in K$ satisfy $E(u) = \inf_{v \in K} E(v)$. Then u is a weak solution of (1).

Proof :

Let us fix a nonnegative test function $\eta \in \mathcal{D}(\Omega)$. Then $u(1 - \varepsilon\eta) \in K$ for $\varepsilon \geq 0$ small enough.

As the function $\varphi(\varepsilon) = E(u(1 - \varepsilon \cdot \eta))$ has a minimum in $\varepsilon = 0$, then $\varphi'(0) = dE(u)(-u \cdot \eta) \geq 0$, i.e.

$$2 \int_{\Omega} \nabla u \cdot \nabla(-\eta \cdot u) + \int_{\Omega} 4H(u)(u_x \wedge u_y) \cdot (-\eta u) \geq 0 \quad \forall \eta \in \mathcal{D}(\Omega)$$

By Green's formula,

$$\int_{\Omega} \Delta(|u|^2) \cdot \eta = - \int_{\Omega} \nabla(|u|^2) \nabla(\eta) = - \int_{\Omega} 2(\nabla u \cdot u) \cdot \nabla \eta$$

It follows that

$$-\frac{1}{2} \Delta|u|^2 + |\nabla u|^2 + 2H(u)u \cdot u_x \wedge u_y \leq 0 \text{ in } \mathcal{D}'(\Omega)$$

Then, as

$$|2H(u)u \cdot (u_x \wedge u_y)| \leq \|H\|_{L^\infty} R' |\nabla u|^2$$

it follows that

$$-\Delta|u|^2 \leq 0 \text{ in } \mathcal{D}'(\Omega)$$

and then by Stampacchia's maximum principle (see [GT]) we get:

$$\sup_{\Omega} |u| = \sup_{\partial\Omega} |u| = R$$

Since $R < R'$, u is interior in K . It follows that $dE(u)(v) = 0$ for any $v \in H_0^1 \cap L^\infty$, and by corollary 5 u is a weak solution of (1).

References

- [1] Amster P. Mariani, M. C., Rial, D. F. : Existence and uniqueness of H-System's solutions with Dirichlet conditions. To appear in *Nonlinear Analysis, Theory, Methods, and Applications*.
- [2] Bethuel, F. : Un résultat de régularité pour les solutions de l'équation des surfaces à courbure moyenne prescrite. *R. Acad. Sci. Paris Sér. I Math* 314, n° 13, pp. 1003 – 100, (1992).
- [3] Brezis, H., Coron, J., Multiple Solutions of H-Systems and Rellich's conjecture. *Comm. Pure Appl. Math* 37, pp. 149 – 187, (1984).
- [4] Chanillo S., Li Y. : Continuity of Solutions of Uniformly Elliptic Equations in $\mathbf{R}^2$. *Manuscripta math.* 77, pp. 415-433, (1992).
- [5] Gilbarg, D., Trudinger N. : *Elliptic Partial Differential Equations of Second Order*. Springer-Verlag (1977).
- [6] Osserman, R. : *A Survey of Minimal Surfaces*. Van Nostrand Reinhold Company, (1969).
- [7] Struwe, M.: *Variational Methods, Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems*, Springer-Verlag, pp. 180.
- [8] Wente, H. : An Existence Theorem for Surfaces of Constant Mean Curvature. *Journal of Mathematical Analysis and Applications* 26, pp. 318-344, (1969)
- [9] Wente, H. : The differential equation $\Delta X = 2H(X_u \wedge X_v)$ with vanishing boundary values. *Proceedings of the American Mathematical Society* 50, pp.131-7, (1975).

Received : December, 1998

P. Amster

P. De Nápoli

and

M. C. Mariani

Departamento de Matemática

Fac. de Cs. Exactas y Naturales

Universidad de Buenos Aires

Consejo Nacional de Investigaciones Científicas y Técnicas
(CONICET)

Pab. I, Ciudad Universitaria (1428)

Buenos Aires

Argentina