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ON THE PARABOLIC DIRICHLET TO NEUMANN FUNCTIONAL

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Abstract

We prove the injectivity of the linearization of the Parabolic Dirichlet to Neumann Functional in a bounded domain of $\mathbf{R}^n$ under suitable transversality condition.

1. Introduction and Statement of the Result

Let $\overline{\Omega}$ be a compact manifold of dimension n with smooth boundary $\partial\Omega$ and $\text{Met}(\overline{\Omega})$ denote the set of all Riemannian metrics g on $\overline{\Omega}$. We consider the mixed problem for the heat equation:

$$(1.1) \quad \begin{aligned} L_g u &:= \frac{\partial u}{\partial t} - \Delta_g u = 0, & \text{in } \Omega \times (0, T), \\ u &= f & \text{on } \Gamma = \partial\Omega \times (0, T), \quad f \in C_0^\infty(\Gamma), \\ u(x, 0) &= 0 & \text{in } \Omega. \end{aligned}$$

There is a unique solution to (1.1); hence we may define the parabolic Dirichlet to Neumann map as the linear operator $\Lambda_g : C_0^\infty(\Gamma) \rightarrow C^\infty(\Gamma)$ given by

$$(1.2) \quad \Lambda_g f = du \cdot \nu_g \Big|_{\Gamma} = \frac{\partial u}{\partial \nu_g} \Big|_{\Gamma},$$

where u is the unique solution to (1.1) and ν_g is the g -interior unit normal to $\partial\Omega$. The parabolic Dirichlet to Neumann Functional:

$$(1.3) \quad \begin{aligned} \Lambda : \text{Met}(\overline{\Omega}) &\longrightarrow O_p(\Gamma), \\ g &\longmapsto \Lambda_g, \end{aligned}$$

where $O_p(\Gamma)$ denotes the space of all linear operators from $C_0^\infty(\Gamma)$ into $C^\infty(\Gamma)$, is known to be invariantly defined on the orbit obtained by the action over $\text{Met}(\overline{\Omega})$ of the group $\mathcal{D}$ of all diffeomorphisms ψ of $\overline{\Omega}$, each of which restricts to the identity on $\partial\Omega$. In fact, any such ψ can be used to construct a new metric, the pull-back metric, ψ^*g , such that $\Lambda_{\psi^*g} = \Lambda_g$. A natural conjecture is that this is the only obstruction to the injectivity of Λ . This article may be considered as a first step towards this claim.

The elliptic Dirichlet to Neumann map $h \mapsto \frac{\partial v}{\partial \nu_g}|_{\partial\Omega}$, where v solves the Dirichlet problem $\Delta_g v = 0$ in Ω , $v|_{\partial\Omega} = h$, was treated by several authors and is closely related to the physical problem referred as Electrical Impedance Tomography, of determining the conductivity of a body from measurement of voltage potentials and corresponding current fluxes at the boundary (see Uhlmann, 1992). The hyperbolic Dirichlet to Neumann map was considered in the articles (Cardoso & Mendoza, 1996), (Lee, Mendoza, Sylvester & Uhlmann) and (Uhlmann, 1992).

For fixed g , we consider the following map:

$$(1.4) \quad \mathcal{D} \ni \psi \xrightarrow{\Lambda_g} \psi^*g \in \text{Met}(\overline{\Omega}).$$

It is easy to see that the tangent space $T_I \mathcal{D}$ of $\mathcal{D}$ at the identity mapping I is the vector space $\Gamma_0(T\overline{\Omega})$ of all smooth vector fields on $\overline{\Omega}$ which vanish on $\partial\Omega$. On the other hand, the tangent space $T_g \text{Met}(\overline{\Omega})$ of $\text{Met}(\overline{\Omega})$ at g is the vector space $\Gamma(S^2\overline{\Omega})$ of all smooth sections of symmetric 2-tensor on $\overline{\Omega}$. We introduce respectively on $\Gamma_0(T\overline{\Omega})$ and on $\Gamma(S^2\overline{\Omega})$, the inner products

$$(1.5) \quad \langle X, Y \rangle = \int_{\overline{\Omega}} g(X, Y) v_g, \quad X, Y \in \Gamma_0(T\overline{\Omega}),$$

$$(1.6) \quad \langle \langle m, l \rangle \rangle = \frac{1}{n} \int_{\overline{\Omega}} \text{tr}(\hat{m} \circ \hat{l}) v_g, \quad m, l \in \Gamma(S^2\overline{\Omega}),$$

where v_g (resp., tr) denotes the volume element (resp., the trace) associated to g and $\hat{m}$ is the unique linear map (in fact a section of $\text{End}(T\overline{\Omega})$) defined by

$$(1.7) \quad g(\hat{m}u, v) = m(u, v), \quad \text{for all } u, v \in \Gamma(T\overline{\Omega}).$$

Of course, $\hat{g}$ is the identity on $\Gamma(T\overline{\Omega})$ and the factor $1/n$ in (1.6) is taken so as to have $\langle \langle g, g \rangle \rangle = \text{vol}_g(\overline{\Omega})$. By definition, the formal linearization $A'_g := A'_g[I]$, of A_g at I is the linear map $A'_g : \Gamma_0(T\overline{\Omega}) \rightarrow \Gamma(S^2\overline{\Omega})$ given by

$$(1.8) \quad A'_g(X) = \frac{d}{ds} A_g(X^s)|_{s=0},$$

where X^s is the flow associated to the vector fields X (i.e. a mapping $(-\varepsilon, \varepsilon) \ni s \mapsto X^s \in \mathcal{D}$, $\varepsilon > 0$, such that, $X^0 = I$ and $\frac{d}{ds} X^s|_{s=0} = X$).

Similarly we consider, the formal linearization of Λ at g , as the linear map $\Lambda'_g : \Gamma(S^2\overline{\Omega}) \rightarrow O_p(\Gamma)$, defined by

$$(1.9) \quad \Lambda'_g(m) = \frac{d}{ds} (\Lambda_{g_s})|_{s=0},$$

where g_s is a smooth curve on $\text{Met}(\overline{\Omega})$, $s \in (-\varepsilon, \varepsilon)$, such that, $g_0 = g$ and $\frac{d}{ds} g_s|_{s=0} = m$.

Let $(A'_g)^*$ denote the formal adjoint of A'_g with respect to the inner products (1.5) and (1.6). We state the following:

Conjecture 1.1. *Let $m \in \Gamma(S^2\overline{\Omega})$ and assume that (a) $\Lambda'_g(m) = 0$, (b) $(A'_g)^*(m) = 0$. Then m is identically zero.*

Remark 1.1. *The Condition (b) in Conjecture 1.1 is obviously necessary. In fact, the range of A'_g is contained in the kernel of Λ'_g . Therefore, we should expect Λ'_g be injective on a “Transversal” subspace of the range of A'_g ; hence we shall refer to Condition (b) as the Transversality Condition.*

The main result of this paper is:

Theorem 1.1. *Conjecture 1.1 holds if $\overline{\Omega}$ is a bounded domain of $\mathbf{R}^n$, $n \geq 2$, with smooth boundary $\partial\Omega$, g is the euclidean metric $\mathbf{e}$ and $m \in \Gamma_0(S^2\overline{\Omega})$ is a symmetric 2-tensor on $\mathbf{R}^n$ which is supported on $\overline{\Omega}$.*

The article is organized as follows: In Section 2 we prove some estimates relative to the heat equation. In Section 3 we exhibit invariant formulas for A'_g , $(A'_g)^*$ and Λ'_g . In Section 4 we present the proof of Theorem 1.1. It is worth while mentioning that in the unidimensional case, i.e. $\overline{\Omega} = [0, 1]$, the unit interval, we obtain that $\Lambda_g = \Lambda_h$ if and only if there exists a diffeomorphism $\psi : [0, 1] \rightarrow [0, 1]$, which preserves orientation and consequently, fixes the boundary, such that $\psi^*g = h$.

2. Estimates

Let Ω be a bounded domain of $\mathbf{R}^n$, with smooth boundary $\partial\Omega$ and let $P(x, D)$ be a second order differential operator of the form

$$(2.1) \quad P(x, D) = - \sum_{i,j} \frac{\partial}{\partial x_i} \left(a^{ij} \frac{\partial}{\partial x_j} \right) + \sum_j b^j \frac{\partial}{\partial x_j} + c,$$

with coefficients in $C^\infty(\overline{\Omega})$. We assume that (a^{ij}) is real and positive definite in $\overline{\Omega}$ and that P is symmetric with respect to a density γdx with $0 < \gamma \in C^\infty(\overline{\Omega})$:

$$(2.2) \quad (P(x, D)u, v)_\gamma = (u, P(x, D)v)_\gamma, \quad u, v \in C^\infty(\overline{\Omega}),$$

where

$$(2.3) \quad (u, v)_\gamma = \int u \bar{v} \gamma dx.$$

(The density simplifies coordinates changes but should be disregarded as otherwise irrelevant). Let $\mathcal{P}$ be the operator defined by $P(x, D)$ in $L^2_\gamma(\Omega)$ with Dirichlet boundary conditions. We have $u \in \mathcal{D}_\mathcal{P}$ if and only if

$P(x, D)u = f \in L^2$ and u is in the closure $\overset{\circ}{H}$ of $C_0^\infty(\overline{\Omega})$ in $H_1(\overline{\Omega})$. Then we have (see Section 17.5 of Hörmander, 1985),

$$(2.4) \quad \|u\|_{(1)}^2 \leq C_1(\mathcal{P}u, u), \quad u \in \mathcal{D}_{\mathcal{P}},$$

$$(2.5) \quad \|u\|_{(k+2)} \leq C_k \|\mathcal{P}u\|_{(k)}, \quad u \in \mathcal{D}_{\mathcal{P}},$$

where we have used the notation

$$\|u\|_{(k)}^2 = \sum_{|\alpha| \leq k} \|D^\alpha u\|^2.$$

We shall also write $H_{(k)}(\overline{\Omega})$ for the set of all u with $D^\alpha u \in L^2(\Omega)$, $|\alpha| \leq k$, with the norm just defined.

We shall denote by $P_z(x, D)$ an elliptic operator of the same form as before, except that it now depends on a parameter $z \in K$, where K is a compact convex body in $\mathbf{R}^N$, $N \geq 1$. We assume that the coefficients of P_z are in $C^\infty(\overline{\Omega} \times K)$ and that (2.4) is valid uniformly for all P_z . We have the following:

Proposition 2.1. *Let $v \in C^\infty([0, T] \times \overline{\Omega} \times K)$ be a solution of the mixed problem*

$$(2.6) \quad \begin{aligned} \left(P_z + \frac{\partial}{\partial t}\right) v_z &= h_z & \text{in } [0, T] \times \overline{\Omega} \times K, \\ [0.5cm] v_z &= 0 & \text{in } [0, T] \times \partial\Omega \times K, \\ [0.5cm] v_z(0, x) &= 0 & \text{in } \Omega. \end{aligned}$$

Assume that $\partial^j h_z / \partial t^j = 0$ when $t = 0$, if $j \leq k$. Then it follows that (with the supremum taken over $[0, T] \times \overline{\Omega} \times K$)

$$(2.7) \quad \sum_{|\alpha| + \frac{1}{2}(n+2) < k} \sup |D_{t,x,z}^\alpha v_z(t, x)| \leq C \sum_{|\alpha| \leq k} \sup |D_{t,x,z}^\alpha h_z(t, x)|.$$

To prove Proposition 2.1, we shall use the following:

Lemma 2.1. *Let $v \in C^\infty([0, T] \times \overline{\Omega})$ be a solution of the mixed problem*

$$(2.8) \quad \begin{aligned} \left(P + \frac{\partial}{\partial t}\right) v &= h & \text{in } [0, T] \times \overline{\Omega}, \\ v &= 0 & \text{in } [0, T] \times \partial\Omega, \\ v(0, x) &= 0 & \text{in } \Omega. \end{aligned}$$

Assume that

$$(2.9) \quad \frac{\partial^j h}{\partial t^j} \Big|_{t=0} = 0, \quad \text{for all } j \leq k.$$

Then it follows that

$$(2.10) \quad \sum_{j=0}^k \|D_t^{k-j} v(t, \bullet)\|_{(j)} \leq C_k \left(\int_0^t \|D_s^k h(s, \bullet)\| ds + \sum_{j=0}^{k-1} \|D_t^{k-1-j} h(t, \bullet)\|_{(j)} \right).$$

Proof: Let $k = 0$; since P is symmetric, we have

$$(2.11) \quad \frac{\partial}{\partial t} \|v\|_\gamma^2 \leq 2\text{Re}(h, v)_\gamma.$$

Integrating from 0 to t and letting $M^2 = \sup_{0 \leq s \leq t} \|v(s, \bullet)\|^2$, we obtain

$$(2.12) \quad \|v(t, \bullet)\|_\gamma^2 \leq 2M \int_0^t \|h(s, \bullet)\|_\gamma ds.$$

The same estimate holds for smaller values of t , so

$$(2.13) \quad M^2 \leq 2M \int_0^t \|h(s, \bullet)\|_\gamma ds,$$

which proves (2.10) when $k = 0$. If $k > 0$, taking into account (2.9), assume that (2.10) is proved for k ; we can therefore apply (2.10) to $\frac{\partial v}{\partial t}$ and $\frac{\partial h}{\partial t}$. We get (2.10) with the term for $j = k + 1$ missing in the left-hand side. Now it follows from (2.5) that

$$\|v(t, \bullet)\|_{(k+1)} \leq \|v(t, \bullet)\|_{(k+2)} \leq C_k \left(\|h(t, \bullet)\|_{(k)} + \left\| \frac{\partial v}{\partial t}(t, \bullet) \right\|_{(k)} \right).$$

This concludes the proof of Lemma 2.1. ■

Proof of Proposition 2.1. Let the sum in the right-hand side of (2.7) be equal to 1. Then it follows from (2.10) that we have an uniform bound for $\|D_t^{k-j} v_z(t, \bullet)\|_{(j)}$, if $j \leq k$. Differentiating the differential equation $(\frac{\partial}{\partial t} + P_z)v_z = h_z$ with respect to z_ν , we obtain

$$\left(\frac{\partial}{\partial t} + P_z \right) \frac{\partial v_z}{\partial z_\nu} = \frac{\partial h_z}{\partial z_\nu} - \left(\frac{\partial P_z}{\partial z_\nu} \right) v_z := H.$$

We have a bound for $\|D_t^{k-2-j} H(t, \bullet)\|_{(j)}$, if $j \leq k - 2$, and then we may conclude, using (2.10) again, that there is an uniform bound for

$\|D_t^{k-2-j} \frac{\partial v_z}{\partial z_\nu}(t, \bullet)\|_{(j)}$ if $j \leq k-2$. Continuing in this way we obtain uniform bounds for $\|D_t^{k-2|\beta|-j} \partial_z^\beta v_z(t, \bullet)\|_{(j)}$ when $j \leq k-2|\beta|$, $|\beta| \leq \frac{k}{2}$, and the proposition is therefore a consequence of Lemma 17.5.2 in (Hörmander, 1985). The proof is complete. ■

3. Invariant Formulas

Let Ω be a bounded domain of $\mathbf{R}^n$ with smooth boundary $\partial\Omega$ and $g \in \text{Met}(\mathbf{R}^n)$.

(Cardoso & Mendoza, 1996), proved that if $X \in \Gamma_0(T\bar{\Omega})$ and $m \in \Gamma(S^2\bar{\Omega})$, then it follows that

$$(3.1) \quad A'_g(X)(\bullet, \bullet) = g(\nabla_\bullet^g X, \bullet) + g(\bullet, \nabla_\bullet^g X),$$

$$(3.2) \quad (A'_g)^*(m)(\bullet) = -\frac{2}{n} \sum_{k=1}^n \nabla_{\mathbf{e}_k} m(\bullet, \mathbf{e}_k).$$

In (3.1) ∇^g denotes the g Levi-Civita connection on $\Gamma_0(T, \bar{\Omega})$ and in (3.2) ∇ is the g Levi-Civita connection on $\Gamma(S^2\bar{\Omega})$, $(e_i)_{i=1, \dots, n}$ in $\Gamma(T\bar{\Omega})$ is a g orthonormal frame. We also observe that the right-hand side of (3.2) is independent of the chosen orthonormal frame.

We denote by $\tilde{m} \in \tilde{\Gamma}(S^2\bar{\Omega})$ the symmetric 2-tensor on $\Gamma(T^*\bar{\Omega})$ corresponding to m via g , i.e.

$$\tilde{m}(U^\#, V^\#) = m(U, V) \quad \text{for all } U, V \in \Gamma(T\bar{\Omega}),$$

where $U^\#(\bullet) = g(U, \bullet)$.

We shall consider the “adjoint” problem

$$(3.3) \quad \begin{aligned} M_g v &:= \frac{\partial v}{\partial t} + \Delta_g v = 0 \quad \text{in } \Omega \times (0, T) \\ v|_\Gamma &= h \in C_0^\infty(\Gamma), \quad v(x, T) = 0 \quad \text{in } \Omega. \end{aligned}$$

Let $g_s, |s| < \varepsilon$, be any curve in $\text{Met}(\bar{\Omega})$ such that $g_0 = g$ and $\frac{dg_s}{ds} = m$. We need the following:

Proposition 3.1. *The linearization of Λ at $g \in \text{Met}(\bar{\Omega})$, satisfies*

$$(3.4) \quad \langle \Lambda'_g(m)f, h \rangle = \int_0^T \int_\Omega \left[\tilde{m}(du, dv) + \frac{1}{2} \text{tr} \tilde{m} \left(\tilde{g}(du, dv) + v \frac{\partial u}{\partial t} \right) \right] v_g dt$$

for all $f, h \in C_0^\infty(\Gamma)$, where u (respect. v) is a solution of (1.1) (respect. 3.3), and $\text{tr} \tilde{m}$ means the trace of $\tilde{m}$ associated to g .

Proof: By applying Stoke's theorem, we obtain

$$(3.5) \quad \begin{aligned} \langle \Lambda' g(m)f, h \rangle &= \frac{d}{ds}|_{s=0} \int_0^T \int_{\partial\Omega} h \frac{\partial u_s}{\partial \nu} v_{g_s}^* dt = \\ &= \frac{d}{ds}|_{s=0} \int_0^T \int_{\Omega} [v_s \Delta_{g_s} u_s + \tilde{g}_s(du_s, dv_s)] v_{g_s} dt, \end{aligned}$$

where u_s (respect. v_s) is the solution of (1.1) (respect. (3.3)) with the metric g replaced by g_s and $v_{g_s}^*$ is the element of volume on $\partial\Omega$, induced by g_s .

From Proposition 2.1, we obtain that u_s, v_s depend differentiably of s . Hence, we can write

$$(3.6) \quad \begin{aligned} u_s &= u + s\omega_1 + O(s^2), \\ v_s &= v + s\omega_2 + O(s^2), \end{aligned}$$

where of course $\omega_i|_{\Gamma} = \omega_i|_{t=0} = \omega_i|_{t=T} = 0$, $i = 1, 2$. If we substitute (3.6) into (3.5) and make use of the facts that $\frac{d}{ds}\tilde{g}_s|_{s=0} = \tilde{m}$ and $\frac{d}{ds}v_{g_s}|_{s=0} = \frac{1}{2}\text{tr}(\tilde{m})v_g$, we obtain (3.4), after noticing that (by Stoke's Theorem),

$$\int_0^T \int_{\Omega} [\tilde{g}(d\omega_1, dv) + \tilde{g}(du, d\omega_2)] v_g dt = \int_0^T \int_{\Omega} \left[\omega_1 \frac{\partial v}{\partial t} - \omega_2 \frac{\partial u}{\partial t} \right] v_g dt.$$

■

4. Proof of Theorem 1.1

We shall only prove Theorem 1.1 when $n \geq 3$; the case $n = 2$ requires small modifications. Let $\Gamma_0(S^2\overline{\Omega})$ denote the vector space of all smooth sections of symmetric 2-tensor on $\mathbf{R}^n$ which are supported on $\overline{\Omega}$ and $m \in \Gamma_0(S^2\overline{\Omega})$. In the case of the euclidean metric $\mathbf{e}$, taking into account (3.2), we may write the Transversality Condition as

$$(4.1) \quad \sum_{i=1}^n \frac{\partial}{\partial x_i} m_{ij}(x) = 0, \quad \text{for all } x \in \Omega, \quad j = 1, \dots, n,$$

where m_{ij} are functions in $C^\infty(\overline{\Omega})$ given by

$$(4.2) \quad m_{ij} = m \left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right),$$

and x_i , $i = 1, \dots, n$ are the standard euclidean coordinates in $\mathbf{R}^n$. We define

$$(4.3) \quad \overline{m}_{ij}(x) = \begin{cases} m_{ij}(x) & \text{if } x \in \Omega, \\ [0.5cm]0 & \text{if } x \in \mathbf{R}^n \setminus \Omega. \end{cases}$$

If we take the Fourier transform of (4.1), we obtain

$$(4.4) \quad \mathcal{F}(\overline{m})(k)k = 0, \quad \text{for all } k \in \mathbf{R}^n.$$

We consider the solutions u_τ^i , $i = 1, 2$, of $L_g u = 0$ and $M_g v = 0$ respectively, of the form

$$(4.5) \quad u_\tau^i(x, t) = e^{\langle x, \rho_i \rangle} \left(\frac{1}{\tau^2} + R_\tau^i(x, t) \right),$$

where

$$(4.6) \quad \begin{aligned} \rho_1 &= \frac{\eta}{2} + \frac{i(\tau w + k)}{2}, \\ [0.5cm]\rho_2 &= -\frac{\eta}{2} + \frac{i(-\tau w + k)}{2}, \end{aligned}$$

with $\eta, k, w \in \mathbf{R}^n$, $|w| = 1$, $\tau \in \mathbf{R}$ such that

$$(4.7) \quad \begin{aligned} \langle \eta, k \rangle &= \langle k, w \rangle = \langle \eta, w \rangle = 0, \\ [0.3cm]|\eta|^2 &= \tau^2 + |k|^2. \end{aligned}$$

This guarantees that $\langle \rho_i, \rho_i \rangle = 0$, $i = 1, 2$ and that therefore, the functions $e^{\langle x, \rho_i \rangle}$ are harmonic. As for $R_\tau^i(x, t)$, $i = 1, 2$, they satisfy

$$(4.8) \quad \begin{cases} (-1)^{i+1} \frac{\partial}{\partial t} R_\tau^i - \Delta R_\tau^i - 2\rho_i \cdot \nabla R_\tau^i = 0, \\ [0.5cm]R_\tau^i|_\Gamma = 0, \\ [0.5cm]R_\tau^i|_{t=t_i} = -\frac{1}{\tau^2}, t_1 = 0, t_2 = T. \end{cases}$$

We obtain from the standard estimate for the mixed parabolic problem (see Egorov & Shubin, 1991) that there exist a constant $C > 0$, such that

$$(4.9) \quad \|R_\tau^i\| + \|d R_\tau^i\| + \left\| \frac{\partial}{\partial t} R_\tau^i \right\| \leq \frac{C}{\tau^2}, \quad i = 1, 2$$

where $\|\bullet\|$ denotes the norm in $L^2(\Omega \times (0, T))$.

If we insert u_τ^i given by (4.5), into (3.4) and let τ goes to $+\infty$, we obtain (taking into account (4.3) and (4.9)) that

$$(4.10) \quad {}^t v \mathcal{F}(\overline{m})(k) v + \frac{|v|^2}{8} \text{tr } \mathcal{F}(\overline{m})(k) = 0,$$

for all $k, v \in \mathbf{R}^n$, with $\langle k, v \rangle = 0$.

Let us take $k = (1, 0, \dots, 0)$. From (4.4) we obtain $\mathcal{F}(\overline{m}_{1\alpha})(k) = 0$, $\alpha = 1, \dots, n$. On the other hand, if we take $v \in k^\perp, |v| = 1$, and use (4.10), we get $\text{tr } \mathcal{F}(\overline{m})(k) = 0$. In fact, if $v = e_i, i = 2, \dots, n$, e_i being the coordinator vectors, then (4.10) reads $\mathcal{F}(\overline{m}_{ii})(k) + \text{tr } \mathcal{F}(\overline{m})(k)/8 = 0$. If we add these equalities, reminding that $\mathcal{F}(\overline{m}_{11})(k) = 0$, we get $\text{tr } \mathcal{F}(\overline{m})(k) = 0$. Since $\overline{m}$ is symmetric we see by using (4.10) again that $\mathcal{F}(\overline{m})(k) = 0$. Rotating coordinates, it follows from the invariant formulas in Section 3 that $\mathcal{F}(\overline{m})(k) = 0$, for all $k \in \mathbf{R}^n$, and therefore $m = 0$.

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