ON THE RESOLUTION OF FREDHOLM INTEGRAL EQUATIONS BY MULTIGRID DOMAIN SEPARATION TECHNIQUES *

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Abstract

We introduce a technique for split discretizations of Fredholm integral equations of the second kind by means of projections on some subsets of quadrature nodes. The technique has its roots in the traditional framework of integral operators of the second kind. We combine this approach with two level methods and generate a numerical parallel scheme.

Key Words : *Nyström methods, Multigrid methods, Domain Reduction methods, Frequency Decomposition methods, Parallel computing.*

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1. Introduction

It is common in the mathematical modelling of a physical problem or in the reformulation of some differential problem, to arrive to a Fredholm integral equation. For these equations, there is in general a strong theoretical framework which guarantees both existence and unicity of solutions. For instance, the inner hypothesis of a particular problem leads usually to sufficient conditions for the use of the Fredholm alternative theoreme.

It is well knows also in the case of equations with weakly singular kernels, for which with a little bit more of care, it is possible again to get existence and uniqueness (see [16]).

In any case, the integral equations that we are looking to solve (specially, those of the second kind) have two common characteristics. First, the a *priori* error bounds are overdimensioned. As a consequence, to guarantee a given precision the size of the discrete problem must be very large. Secondly, the discretization process generates a *full matrix*. Because of this, techniques or ideas searching to separate the work in independent-simultaneous parts are always welcome.

In this article, we are interested in taking up the numerical resolution of Fredholm integral equations of the second kind through *Domain Separation* techniques. The methods can be applied to the continuous problem or to the discretized one (see [5], [7]) The main idea will be to project the discretized equation into two subspaces, performing in them a coarse grid correction-type proces, defining in this way an iterative scheme.

The rest of this article is a follows. In section 2 we resume the Nyström discretization process and some further comments about it. Section 3 presents the basic two-grid analysis for integral equations of the second kind with properties required for the following section. Section 4 deals with multiple coarse correction schemes and their implementation in the domain separation techniques. The final section presents numerical results based in the computational implementation of this techniques, by comparing with both standard direct and iterative methods. These results were obtained in a *Silicon Graphics O₂*. (CPU time in a parallel *Origin 2000* machine, together with analysis of these techniques for the continuous problem, can be seen in [7]).

2. Discrete Framework

Let us consider the second kind equation in a domain $\Omega \subseteq \mathbf{R}^n$,

$$(2.1) \quad \varphi(x) - \int_{\Omega} k(x, y) \varphi(y) dy = f(x), \quad x \in \Omega,$$

or, equivalently,

$$(2.2) \quad (I - K) \varphi = f, \quad \text{in } \Omega,$$

where,

$$(2.3) \quad (K\varphi)(x) := \int_{\Omega} k(x, y) \varphi(y) dy.$$

In various engineering applications, it is sufficient to set the abstract framework in $C(\Omega)$ or $L^2(\Omega)$, and $\Omega \subset \mathbf{R}^n$ compact. We suppose that there are sufficient conditions to get the Fredholm alternative theorem.

A discretization process leads to state the subproblems :

$$(2.4) \quad (I - K_n) \varphi_n = f_n, \quad \text{in } \Omega,$$

in which the distinct discretization methods are defined according to a kind of discrete operator considered. In this section, as well as in the rest of this article, we assume quadrature (Nyström) discretization :

$$(2.5) \quad (K_n \varphi)(x) := \sum_{k=1}^n \alpha_k^n k(x, x_k^n) \varphi(x_k^n), \quad \forall \varphi \in C(\Omega), \quad x \in \Omega,$$

where the values α_k^n are the weights which define the current quadrature. In general α_k^n is constant, although in the case of weakly singular kernels, i. e., those for which

$$(2.6) \quad |k(x, y)| \leq M \|x - y\|^{\alpha-n}, \quad \forall x \neq y.$$

for some constant $M > 0$ and $\alpha \in [0, n[$, the weights must be chosen as $\alpha_k^n = \alpha_k^n(x_i)$. It can be proved that in both cases the discrete operators (2.5) determine a collectively compact sequence which is pointwise convergent to the solution φ of (2.1) (see [16]).

The discrete solution of (2.4) is obtained by solving the following linear system :

$$(2.7) \quad \varphi_j^{(n)} - \sum_{k=1}^n \alpha_k^n k(x_j^n, x_k^n) \varphi_k^{(n)} = f_j^{(n)}$$

where we denote

$$\varphi_j^{(n)} := \varphi_n(x_j^n), f_j^{(n)} := f(x_j^n)$$

The coefficients $\varphi_j^{(n)}$ allow to define as solution for problem (2.4) the Nyström Interpolate :

$$(2.8) \quad \varphi_n(x) := f(x) + \sum_{k=1}^n \alpha_k^n k(x, x_k^n) \varphi_k^{(n)}, \quad x \in \Omega$$

This interpolant is, in general, of very high accuracy. Now, in order to solve the system (2.7) we have several possibilities. The first is to take out directly the resolution of this system (of order n and non-sparse) by direct or iterative methods. The second possibility is to make use of Coarse Grid Correction methods or Multigrid techniques. Another alternatives of resolution can be found in [1], [11], [15]. In general, there is no much innovation in this subject for solving (2.4).

Recently, the use of *Domain Decomposition* techniques to seek inherent parallelism of the problem have been reconsidered. Hackbusch presents as a variant his *Frecuency Decomposition* techniques (see [13]), originally designed for dump as long as possible, the oscillatory part of the error.

On the other hand, Chan and Tuminaro (see [8]) state a multiple coarse correction scheme, based on simultaneous coarse correction steps. For our case, we take relevant aspects of each of them to define our Domain Separation techniques.

3. Multigrid Analysis

The classical scheme for Multigrid techniques of the second is based on coarse grid correction steps, which consider as smoother the Picard iteration.

This coarse grid correction consider two levels of discretization (fine and coarse), and begins with a given approximation $\varphi_l^{(i)}$ associate to the level l . Then, it performs the following two procedures :

a) Smoothing step :

$$(3.1) \quad \varphi_l^{(i)} \longmapsto \varphi_l^{(i+\frac{1}{2})} := \mathbf{K}_{l\varphi_l^{(i)}} + f_l$$

b) Coarse Grid Correction step :

$$\varphi_l^{(i+1)} := \varphi_l^{(i+\frac{1}{2})} - p(I - \mathbf{K}_{l-1})^{-1} r \left(\varphi_l^{(i+\frac{1}{2})} - \mathbf{K}_{l\varphi_l}^{(i+\frac{1}{2})} - f_l \right), \quad (3.2)$$

where p and r are prolongation and restriction operators, respectively, which allow to relate the distinct levels, l , $l-1$, and so on. The recursivity of this scheme towards inferior levels defines the Multigrid iterations. These algorithms are *serial*, that is, it is required to have completed the tasks in the preceding level to be able to access the next level. Some further manners to avoid this problem can be found in [18].

The convergence analysis of these methods can be based on the same kind of properties that those required for the Collectively Compact convergence, as it is shown in [12].

For the class of operators defined in (2.4), let us consider the following properties :

Discrete Regularity :

$$(3.3) \quad \|\mathbf{K}_l\|_{X_l \rightarrow Y_l} \leq C_R, \quad \forall l = 1, 2, \dots$$

Stability :

$$(3.4) \quad \|(I - \mathbf{K}_l)^{-1}\| \leq C_S, \quad \forall l = 1, 2, \dots$$

Uniform boundedness of inter-grid operators :

$$(3.5) \quad \|p\| \leq C_p, \quad \|r\| \leq C_r, \quad \forall l = 1, 2, \dots$$

Interpolation error estimate :

$$(3.6) \quad \|I - pr\| \leq C_I h_l^\beta, \quad \forall l = 1, 2, \dots$$

Relative consistency of order β :

$$(3.7) \quad \|r\mathbf{K}_l - \mathbf{K}_{l-1}r\| \leq C_C h_l^\beta, \quad \forall l = 1, 2, \dots$$

These properties permit to find sufficient conditions for the convergence of Two-grid and Multigrid schemes. The following result can be found in [12].

Theorem 3.1. *Assume the properties of Stability (3.4), regularity (3.3), relative consistency condition (3.7), stability of prolongation operators (3.5) and the interpolation error (3.6). Then the Two-grid iteration with respect to the associate space norm satisfies :*

$$M_l^{tgm} = \left[I - p(I - \mathbf{K}_{l-1})^{-1} r(I - \mathbf{K}_l) \right] \mathbf{K}_l$$

and

$$\|M_l^{tgm}\| \leq C_{tgm} h_l^\beta,$$

where

$$C_{tgm} \leq (C_I + C_p C_S C_C) C_R.$$

Proof : It follows by noting that

$$M_l^{tgm} = \left[(I - pr) + p(I - \mathbf{K}_{l-1})^{-1} [(I - \mathbf{K}_{l-1}) r - r(I - \mathbf{K}_l)] \right] \mathbf{K}_l.$$

4. Multiple Corrections. Domain Separation.

Let us consider a decomposition of the discrete space Ω_h in d subspaces,

$$\Omega = \bigcup_{i=1}^d \Omega^{p_i}.$$

The way of relating each subspace Ω^{p_i} with Ω_h will be defined by operators R_i, P_i , where,

$$(4.1) \quad P_i : \Omega^{p_i} \longmapsto \Omega_h,$$

and usually $R_i = P_i^*$. In this form, the solution that we are looking for will be expressed in the form,

$$\varphi_n = \sum_{i=1}^d P_i \varphi^i, \quad \varphi^i \in \Omega^{p_i}.$$

This decomposition need not be unique. It just requires as necessary condition that $\sum_{i=1}^d P_i(\Omega^{p_i}) = \Omega_h$. Usually, the operators of those grids are defined by the Galerkin condition :

$$(4.2) \quad \mathbf{A}_{p_i} := R_i \mathbf{A}_h P_i, \quad \forall i = 1, 2, \dots, d$$

Remark : This definition for the associate operators to integral equations is utilized in both discrete Collocation and discrete Galerkin methods (see [3]). In our case, the simultaneous implementation of coarse-grid corrections following the multiple grid correction defined in [10] leads to define the correction operator of our parallel scheme, as

$$(4.3) \quad \left(I - \left(\sum_{j=1}^d P_j (R_j \mathbf{A}_h P_j)^{-1} R_j \right) \mathbf{A}_h \right).$$

In the traditional sense, the operators $\mathbf{K}_l$, $\mathbf{K}_{l-1}$ are those defined by quadrature at the respective level. Nevertheless, under this approach, it must be defined by

$$(4.4) \quad \mathbf{A}_{l-1}^i := R_i (I - \mathbf{K}_l) P_i,$$

which leads to the multiple correction operators,

$$(4.5) \quad \left[I - \sum_{j=1}^d P_j (I - R_j \mathbf{K}_l P_j)^{-1} R_j (I - \mathbf{K}_l) \right] \mathbf{K}_l.$$

In the particular case for which the operators R_j , P_j satisfy $R_j P_j = I$, $\forall j = 1, \dots, d$; the above expression becomes

$$(4.6) \quad M'_{mcg} = \left[I - \sum_{j=1}^d P_j (I - R_j \mathbf{K}_l P_j)^{-1} R_j (I - \mathbf{K}_l) \right] \mathbf{K}_l.$$

We denote

$$(4.7) \quad (I - R_j \mathbf{K}_l P_j) =: (I - \mathbf{K}_{l-1}^j),$$

indicating that it corresponds to the operator in the j^{th} subspace, and in the particular case $R_j P_j = I$. In this way, we rewrite the relative consistency property (3.7) :

$$r \mathbf{K}_l - \mathbf{K}_{l-1} r = r \mathbf{K}_l (I - pr)$$

Moreover, if we have $\|r\| \leq 1$, $\|p\| \leq 1$, then the discrete regularity (3.3) inherits to all the coarser operators.

Lemma 4.1 : Define in each Ω^{p_h} the coarse operators $\mathbf{K}_{l-1}^j$ as in (4.7), and assume operators R_j , P_j satisfying $R_j P_j = I$, $\forall j$, and $\|R_j\| \leq 1$, $\|P_j\| \leq 1$. Assume also the properties of discrete regularity

(3.3) and interpolation error (3.6). Then, the multiple correction operator defined by

$$(4.8) \quad \left[I - \sum_{j=1}^d P_j (I - \mathbf{K}_{l-1}^j)^{-1} R_j (I - \mathbf{K}_l) \right] \mathbf{K}_l,$$

generates an iterative scheme with the same convergence characteristics that the required for the case of standard Coarse Grid Correction (see [18]).

5. Discrete Domain Separation. Case $d=2$. Numerical results.

By considering averages of operators used by Hackbusch in his Frequency Decomposition techniques (see [13]), we are able to define pairs of prolongation-restriction operators which defines a convergent multiple coarse correction scheme.

The operators utilized by Hackbusch for one-dimensional problems are given by,

$$(5.1) \quad y_i = \begin{cases} \frac{1}{2} [x_{i-1} + 2x_i + x_{i+1}], & i \text{ odd} \\ \frac{1}{2} [-x_{i-1} + 2x_i - x_{i+1}], & i \text{ even} \end{cases}$$

By analyzing the distribution of the coefficients in the matrices $(I - \mathbf{K}_l)$, we have found, for instance, the following operators,

$$(5.2) \quad P_1 := \frac{1}{4} \begin{bmatrix} 1 & & & & & \\ 2 & & & & & \\ 1 & 1 & & & & \\ & 2 & & & & \\ & 1 & & & & \\ & & \ddots & & & \\ & & & 1 & & \\ & & & 2 & & \end{bmatrix}_{N \times \frac{N}{2}}$$

$$(5.3) \quad P_2 := \frac{1}{4} \begin{bmatrix} 2 & & & & & \\ -1 & -1 & & & & \\ & 2 & & & & \\ & -1 & & & & \\ & & \ddots & & & \\ & & & -1 & & \\ & & & 2 & & \\ & & & -1 & & \end{bmatrix}_{N \times \frac{N}{2}}$$

With these operators, it is possible to separate a given one-dimensional problem in two lower-size subproblems by projecting over the set of even and odd quadrature nodes.

As numerical examples, we present some equations of the second kind treated by several ways. First, with the traditional Nyström method, discretized with both Simpson's quadrature (N-S) and Gauss-Legendre quadrature (N-G. L.). The solution of all the systems is obtained with Gaussian elimination procedure with partial pivoting.

Then, the implementation of iterative Coarse Grid Correction methods considers Simpson's (CG-S) and Gauss-Legendre (CG-G. L.). The resolution of all the coarser problems are again with Gaussian elimination. We indicate with parenthesis the number of iterations needed to stabilize the absolute error.

Finally, we present the results on the implementation of the multiple-coarse scheme, with operators P_1 and P_2 defined in (5.2) and (5.3), and again with Simpson's quadrature (MC-S) and Gauss Legendre quadrature (MC-G. L.). As before, we indicate with parenthesis the number iterations needed to stabilize the absolute error.

The iterations of the last method are counted at pairs (one iteration means two correction steps : on even-nodes set and odd-nodes set). This is because in the parallel implementation, both corrections -even and odd nodes- can be realized simultaneously.

CPU times results considering both serial and parallel implementations, can be found in [7].

The following results were obtained in a *Silicon Graphics O₂* workstation, of 180 *MHz* and 64 *Mb* RAM.

Example 1 :

$$\varphi(x) - \frac{1}{2} \int_0^1 e^{x-y} \varphi(y) dy = f(x),$$

where $f(x)$ is chosen so that the solution is $\varphi(x) = x \sin \frac{x}{4}$.

Table 3.1 : Example 1

n	N-S	N-G.L.	CG-S	CG-G.L.
21	0.1322E-6	0.1330E-14	0.1313E-6 (2)	0.1300E-14 (6)
43	0.6838E-8	0.1166E-14	0.6838E-8 (6)	0.1070E-14 (6)
87	0.3867E-9	0.1554E-14	0.3578E-9 (5)	0.1734E-14 (4)
255	0.5000E-11	0.2178E-14	0.5005E-11 (5)	0.1915E-14 (4)
511	0.3139E-12	0.9514E-14	0.3131E-12 (4)	0.9087E-14 (4)

Table 3.2 : Example 1 (Continuation)

n	MC-S	MC-G.L.
21	0.3777E-2 (3)	0.1328E-14 (5)
43	0.1878E-2 (3)	0.1100E-14 (5)
87	0.9290E-3 (2)	0.1723E-14 (3)
255	0.3211E-3 (2)	0.1880E-14 (3)
511	0.1600E-3 (2)	0.9099E-14 (4)

Example 2 :

$$\varphi(x) - \int_0^1 xy(y-x)\varphi(y)dy = f(x),$$

where $f(x)$ is chosen so that the solution is $\varphi(x) = x \sin \frac{x}{4\pi}$.

Table 3.3 : Example 2

Size(n)	N-S	N-G.L.	CG-S	CG-G.L.
21	0.9450E-10	0.7299E-12	0.9451E-10 (2)	0.7373E-12 (4)
43	0.5243E-11	0.7888E-12	0.4249E-11 (4)	0.7888E-12 (4)
87	0.9441E-12	0.7521E-12	0.9399E-012 (4)	0.7499E-12 (5)
255	0.9011E-12	0.9712E-12	0.9011E-12 (1)	0.9720E-12 (3)
511	0.1018E-11	0.8110E-12	0.1017E-11 (3)	0.8123E-12 (3)

Table 3.4 : Example 2 (Continuation)

Size(n)	MC-S	MC-G.L.
21	0.2829E-3 (2)	0.2187E-8 (3)
43	0.1467E-3 (2)	0.2190E-8 (2)
87	0.7521E-4 (1)	0.2190E-8 (2)
255	0.2608E-4 (1)	0.2188E-8 (2)
511	0.1300E-4 (1)	0.2188E-8 (2)

Example 3 :

$$\varphi(x) - \int_0^1 (x^2 \cos y + x \sin y) \varphi(y) dy = f(x),$$

where $f(x)$ is chosen so that the solution is $\varphi(x) = e^x \cos x$.

Table 3.5 : Example 3

Size(n)	N-S	N-G.L.	CG-S	CG-G.L.
21	0.9179E-7	0.4149E-13	0.9178E-7 (4)	0.4060E-13 (7)
43	0.4680E-8	0.2990E-13	0.4680E-8 (8)	0.2980E-13 (7)
87	0.2660E-9	0.4814E-13	0.2658E-9 (7)	0.4856E-13 (7)
255	0.3490E-11	0.5100E-13	0.3499E-11 (6)	0.5101E-13 (6)
511	0.2178E-12	0.2615E-12	0.2178E-12 (7)	0.2600E-12 (6)

Table 3.6 : Example 3 (Continuation)

Size(n)	MC-S	MC-G.L.
21	0.7367E-1 (2)	0.1258E-13 (5)
43	0.3656E-1 (2)	0.3064E-13 (5)
87	0.1819E-1 (2)	0.4700E-13 (4)
255	0.6245E-2 (2)	0.5235E-13 (4)
511	0.3125E-2 (2)	0.2600E-12 (3)

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