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γ-HYPERELLIPTIC RIEMANN SURFACES *

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Abstract

We give some characterizations of γ–hyperelliptic Riemann surfaces of genus $g \geq 2$, that is, pairs (S, j) where S is a closed Riemann surface of genus g and $j : S \rightarrow S$ is a conformal involution with exactly $2g + 2 - 4\gamma$ fixed points. These characterizations are given by Schottky groups, special hyperbolic polygons and algebraic curves. These can be seen as generalizations of the works [5] and [11].

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Introduction

A γ–hyperelliptic Riemann surface of genus g is a pair (S, j), where S is a closed Riemann surface of genus g and j is a conformal involution of S with exactly $2g+2-4\gamma$ fixed points. Equivalently, a γ–hyperelliptic Riemann surface of genus g is a triple $(S, X, \pi : S \to X)$, where S and X are closed Riemann surfaces of genus g and γ, respectively, and $\pi : S \to X$ is a degree two holomorphic covering map. The equivalence is given by $X = S/ < j >$ and $\pi : S \to X$ the natural two fold covering induced by the action of j on S. A 0–hyperelliptic Riemann surface (S, j) consists of a hyperelliptic Riemann surface S and its hyperelliptic involution j.

If (S, j) is a γ–hyperelliptic Riemann surface of genus g, then $j : S \to S$ is called a γ–hyperelliptic Riemann surface and by abuse of language we call S a γ–hyperelliptic Riemann surface of genus g.

In general γ–hyperelliptic involutions are not unique. If the genus of S is large compared to γ, then it is possible to get uniqueness for γ-hyperelliptic involutions (see [4]).

In this paper, we discuss Schottky uniformizations of γ–hyperelliptic Riemann surfaces which reflect the γ-hyperellipticity property (see next section). These uniformizations are called γ–hyperelliptic Schottky uniformizations. The 0–hyperelliptic Schottky uniformizations are also called hyperelliptic Schottky uniformizations.

In [8], we treated γ–hyperelliptic Schottky uniformizations using different techniques. Here we use strongly the topology of the action of involutions on closed Riemann surfaces (Theorem A).

The particular case of hyperelliptic Riemann surfaces and hyperelliptic Schottky uniformizations is done in [11]. It will follow from our results as a particular case.

Let (S, j) be a γ–hyperelliptic Riemann surface of genus $g \geq 2$ such that j has fixed points. If F is a torsion–free Fuchsian group acting on the unit disc U, uniformizing S, then we can lift the involution j to a conformal involution J. In particular, J is in the normalizer of F in the group of conformal automorphisms of U. We show the existence of a strictly convex $4g$–sided hyperbolic fundamental polygon P for the group F, such that $J(P) = P$. The fixed points of j are represented by the fixed point of J (the center of P), the vertices of P, and the midpoints of exactly $2g-4\gamma$ sides. This result for hyperelliptic Riemann surfaces can be also found in the thesis of D. Gallo [5].

These types of polygons, for γ greater than zero, are different than those studied by Fricke and Keen [12]. However, for $\gamma = 0$, such a polygon is a double of a Fricke polygon.

In addition, we recall an algebraic description of γ–hyperelliptic Riemann surface of genus $g \geq 2$. That is classical, but we include as a matter of completeness. Let us say that there exist a lot of representations of γ-hyperelliptic Riemann surfaces and we are only considering a few of them.

1. Definitions

Definition 1. *A pair (S, j) is called a γ–hyperelliptic Riemann surface of genus g provided S is a closed Riemann surface of genus g and $j : S \to S$ is a conformal involution with exactly $2g + 2 - 4\gamma$ fixed points.*

Classically, 0–hyperelliptic Riemann surfaces are called hyperelliptic. In this case the 0–hyperelliptic involution is the hyperelliptic one and it is unique. For such a reason, we may just call S a hyperelliptic Riemann surface. For $\gamma > 0$ the uniqueness of j does not hold in general and a fixed closed Riemann surface may have different γ–hyperelliptic structures. In fact, let S be a hyperelliptic Riemann surface of genus g which admits a conformal involution $k : S \to S$ with exactly $2g - 2$ fixed points. Denote by h the hyperelliptic involution on S. Then $t = h \circ k$ is again a 1–hyperelliptic involution on S different from k. In this way, (S, k) and (S, t) are two different 1–hyperelliptic Riemann surfaces. If the genus g is sufficiently large in comparison with γ, then one has the uniqueness of the γ–hyperelliptic involution [4]

Definition 2 (Schottky groups). *Let C_k, C'_k, $k = 1, \cdots, g$, be $2g$ Jordan curves on the Riemann sphere, $\hat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$, such that they are mutually disjoint and bound a $2g$-connected domain. Call D the common exterior of all these curves, and suppose that for each k there exists a fractional linear transformation A_k with the following properties*

i) $A_k(C_k)=C'_k$;

ii) A_k *maps the exterior of* C_k *onto the interior of* C'_k.

The group G generated by the transformations A_k is a Kleinian group. (The transformations A_k are necessarily loxodromic). The region D is a fundamental domain for G, called a standard fundamental domain for G with respect to the generators A_k. The group G is called a Schottky group *of genus g. The trivial group is called a Schottky group of genus zero. If G is a Schottky group of genus g and $A_1, \cdots, A_g$ is a set of (free) generators, then the pair $(G; \{A_1, \cdots, A_g\})$ is called a marked Schottky group of genus g.*

In [3] it is proved that for any set of free generators $A_1, \cdots, A_g$, for a Schottky group G of genus g, there exists a standard fundamental domain D for G with respect to the given generators.

We say that a Schottky group G is classical if we can find a set of free generators with a standard fundamental domain bounded by circles. In the Schottky Space, the classical ones form a open set. It is known that there are Schottky groups which are not classical [15]. In the literature there is only one explicit example of such a Schottky group [10]

If $\Omega = \cup_{A \in G} A(cl(D))$, where $cl(D)$ denotes the Euclidean closure of D, then Ω is the region of discontinuity of G. The limit set of G is by definition the complement of Ω in the Riemann sphere. We will denote this set by $\wedge(G)$. This set is closed and totally disconnected. If $g \geq 2$, then the limit set is also a perfect set. For $g = 1$ the limit set $\wedge(G)$ consists of two points, and for $g = 0$ the set $\wedge(G)$ is empty.

Definition 3. *A uniformization of a Riemann surface S is a triple $(\Omega, G, \pi : \Omega \to S)$, where G is a Kleinian group with invariant component Ω ([13]) and $\pi : \Omega \to S$ is a regular covering map with G as covering group. If G is a Schottky or a Fuchsian group, then it is called a Schottky or a Fuchsian uniformization, respectively.*

It is a very well known fact that a closed Riemann surfaces has a Schottky uniformization. This is known as the Retrosection Theorem [2]. The following still an open problem. Does every closed Riemann surface has a Schottky uniformization given by a classical Schottky group?

Definition 4. *Let (S, j) be a γ–hyperelliptic Riemann surface, and $(\Omega, G, \pi : \Omega \to S)$ an uniformization of S. We say that $(\Omega, G, \pi : \Omega \to S)$ is a γ–hyperelliptic uniformization of (S, j) (or that it reflects the γ–hyperellipticity of S given by j), if there exists a conformal automorphism $J : \Omega \to \Omega$ such that $\pi \circ J = j \circ \pi$. In the particular case that the above is a Schottky uniformization of S, we call it a γ–hyperelliptic Schottky uniformization of (S, j) and the group G a γ–hyperelliptic Schottky group.*

Definition 5 (γ–hyperelliptic polygons). *Let P be a strictly convex $4g$–sided hyperbolic polygon, $g \geq 2$, in the unit disc U. Then P is called a γ-hyperelliptic polygon if the following holds:*

i) P *is invariant under the transformation* $J(z) = -z$*;*

ii) *The sum of all interior angles of* P *is exactly* 2π*;*

iii) *If* $s_1, \cdots, s_{4g}$ *are the sides of* P*, in counterclockwise order, then there exist linear fractional transformations* $T_1, \cdots, T_{2g}$*, fixing the unit disc* U*, that is, isometries of* U *with the hyperbolic metric, such that:*

(iii.1) $T_i(s_i) = s_{2g+i}^{-1}$, *for* $i = 1, \cdots, 2g - 4\gamma$;

(iii.2) $T_{2g-4\gamma+2t-1}(s_{2g-4\gamma+4t-3}) = s_{2g-4\gamma+4t-1}^{-1}$, *for* $t = 1, \cdots, \gamma$.

(iii.3) $T_{2g-4\gamma+2t}(s_{2g-4\gamma+4t-2}) = s_{2g-4\gamma+4t}^{-1}$, *for* $t = 1, \cdots, \gamma$.

(iii.4) $T_{2g-2\gamma+2t-1} = JT_{2g-4\gamma+2t-1}J$, *for* $t = 1, \cdots, \gamma$.

(iii.5) $T_{2g-2\gamma+2t} = JT_{2g-4\gamma+2t}J$, *for* $t = 1, \cdots, \gamma$.

Remark 1. *If* F *is the group generated by the side pairings of the polygon* P *in Definition 5, then Poincare's theorem [13] asserts that* F *is a Fuchsian group with* P *as a fundamental domain. The surface* U/F *is a closed Riemann surface of genus* g *with an involution* j*, induced by* J*, with* $2g+2-4\gamma > 0$ *fixed points. The fixed points of* j *are the projection of the vertices of* P*, the origin and the midpoints of the sides* s_i *for* $i = 1, \cdots, 2g - 4\gamma$*. In particular,* $(U/F, j)$ *is a* γ*–hyperelliptic Riemann surface of genus* g*.*

Definition 6 (γ–hyperelliptic (N, M) hyperbolic polygons). *Let* P *be a strictly convex* $4g$*–sided hyperbolic polygon ,* $g \geq 2$*, in the unit disc* U*. Let* N *and* M *be non–negative integers satisfying* $0 \leq N, M \leq \gamma$*. Then* P *is called a* γ*-hyperelliptic* (N, M) *hyperbolic polygon provided the following holds:*

(i) P *is invariant under the transformation* $J(z) = -z$*;*

(ii) *The sum of all interior angles of* P *is exactly* 2π*;*

(iii) *If* $s_1, \cdots, s_{4g}$ *are the sides of* P*, in counterclockwise order, then there exist* $i_1, \cdots, i_N \in \{1, \cdots, \gamma\}$*,* $j_1, \cdots, j_N \in \{1, \cdots, \gamma\}$*, where* $i_t \neq i_s$ *if* $t \neq s$ *and* $j_t \neq j_s$ *if* $t \neq s$*, and there exist linear fractional transformations* $T_1, \cdots, T_{2g}$*, keeping the unit disc* U *invariant, such that:*

(iii.1) $T_i(s_i) = s_{2g+i}^{-1}$, *for* $i = 1, \cdots, 2g - 4\gamma$;

(iii.2) $T_{2g-4\gamma+2t-1}(s_{2g-4\gamma+4t-3}) = s_{2g-4\gamma+4t-1}^{-1}$, *if* t *does not belong to the set* $\{i_1, \cdots, i_N\}$;

(iii.3) $T_{2g-4\gamma+2t-1}(s_{2g-4\gamma+4t-3}) = s^{-1}_{4g-4\gamma+4t-1}$, *if* t *belongs to the set* $\{i_1, \cdots, i_N\}$;

(iii.4) $T_{2g-4\gamma+2t}(s_{2g-4\gamma+4t-2}) = s^{-1}_{2g-4\gamma+4t}$, *if* t *does not belong to the set*
$\{j_1, \cdots, j_M\}$;

(iii.5) $T_{2g-4\gamma+2t}(s_{2g-4\gamma+4t-2}) = s^{-1}_{4g-4\gamma+4t}$, *if* t *belongs to the set* $\{j_1, \cdots, j_M\}$;

(iii.6) $T_{2g-2\gamma+2t-1} = JT_{2g-4\gamma+2t-1}\ J$, *for* $t = 1, \cdots, \gamma$.

(iii.7 $T_{2g-2\gamma+2t} = JT_{2g-4\gamma+2t}\ J$, *for* $t = 1, \cdots, \gamma$.

Remark 2. *Note that* γ*–hyperelliptic* $(0,0)$ *hyperbolic polygons are in fact* γ*–hyperelliptic hyperbolic polygon in Definition 5. If* F *is the group generated by the side pairings of the polygon* P *in Definition 6, then by Poincare's theorem [13]* F *is a Fuchsian group with* P *as fundamental domain. The surface* U/F *is a closed Riemann surface of genus* g *with an involution* j, *induced by* J, *with* $2g + 2 - 4\gamma > 0$ *fixed points. The fixed points of* j *are the projection of the vertices of* P, *the origin and the midpoints of the sides* s_i, *for* $i = 1, \cdots, 2g - 4\gamma$.

We need some of the basics from quasiconformal maps. The main tool is the Ahlfors–Bers theorem about solutions of the Beltrami equation and continuity of parameters.

Definition 7 (Quasiconformal homeomorphisms). *Let* $\mu(z)$ *be a measurable function defined on the Riemann sphere* $\hat{\mathbf{C}}$, *with* $\| \mu \|_\infty < 1$, *and* $f : \hat{\mathbf{C}} \to \hat{\mathbf{C}}$ *an orientation preserving homeomorphism of the Riemann sphere. We say that* f *is* μ*-quasiconformal homeomorphism if it satisfies the following equation:*

$$\partial f/\partial \bar{z} = \mu(z)\ \partial f/\partial z, \quad \text{a.e.}$$

2. Auxiliary Results

Theorem 1 (Ahlfors–Bers Theorem [2]).

(1) If $\mu(z)$ *is a measurable function defined on the Riemann sphere* $\hat{\mathbf{C}}$ *and it satisfies* $\| \mu \|_\infty < 1$, *then there exists a unique quasi-conformal homeomorphism* W^μ *of the Riemann sphere satisfying the equation*

$$\partial W^\mu/\partial \bar{z} = \mu(z)\ \partial W^\mu/\partial z,$$

with $W^\mu(0) = 0$, $W^\mu(1) = 1$, *and* $W^\mu(\infty) = \infty$.

(2) *If $\mu(z)$ is a measurable function defined on the unit disc U, and $\| \mu \|_{\infty} < 1$, then there exists a unique quasiconformal homeomorphism W_{μ} of the unit disc U satisfying the equation*

$$\partial W_{\mu}/\partial \bar{z} = \mu(z)\, \partial W_{\mu}/\partial z,$$

with $W^{\mu}(0) = 0$, $W^{\mu}(1) = 1$, and $W^{\mu}(i) = i$.

(3) *The solutions $W^{t\mu}$ and $W_{t\mu}$ in (1) and (2) vary continuously with the parameter $t \in [0, 1]$.*

Lemma 1. *Let G be a Kleinian group with invariant component $\triangle$ of the regular region Ω of G. Suppose there exist μ a measurable function on $\triangle$ and $f : \triangle \to \tilde{\triangle}$ an μ–quasiconformal homeomorphism such that*

$$\mu(g(z))\overline{g'(z)} = \mu(z) g'(z), \quad \text{for all} \quad g \in G,$$

then $f \circ g \circ f^{-1}$ is again a fractional linear transformation if $\tilde{\triangle}$ has the property that every conformal automorphism of it is a fractional linear transformation.

The proof of this lemma is a direct computation and we do not do it here.

Definition 8. *If $\| \mu \|_{\infty} < 1$ in Lemma 1, then we call μ a Beltrami coefficient for the group G in $\triangle$.*

Let $\pi : S \to \tilde{S}$ be a two sheeted holomorphic branched covering, where S and $\tilde{S}$ are closed Riemann surfaces of genus g and γ, respectively. Let q_i, $i = 1, \cdots, K$, be the branched values on $\tilde{S}$ of the above covering. The Riemann–Hurwitz formula [4] implies the equality

$$g = 2(\gamma - 1) + 1 + K/2$$

In particular, we obtain that K must be even. Let $\eta_{1,1}$, $\eta_{1,2}$, $\eta_{2,1}$, $\eta_{2,2}, \cdots, \eta_{\gamma,1}, \eta_{\gamma,2}, \eta_1, \cdots, \eta_K$ be simple loops on $\tilde{S}$ satisfying the following properties:

(1) $\eta_{1,1}, \eta_{1,2}, \eta_{2,1}, \eta_{2,2}, \cdots, \eta_{\gamma,1}, \eta_{\gamma,2}$ are homologically independent loops on $\tilde{S}$;

(2) η_i bounds a small disc around q_i, for $i = 1, \cdots, K$;

(3) $\eta_i \cap \eta_j = \phi$, if $i \neq j$;

(4) $\eta_i \cap \eta_{k,t} = \phi$, for $i = 1, \cdots, K$; $k = 1, \cdots, \gamma$; $t = 1, 2$;

(5) $\eta_{k,1} \cap \eta_{k,2} = \{x_k\}$, for $k = 1, \cdots, \gamma$; and

(6) $\eta_{k,t} \cap \eta_{s,l} = \phi$, if $k \neq s$.

Consider a point x_0 in $\tilde{S}^*$, where

$$\tilde{S}^* = \tilde{S} - \{\eta_{k,t}, \eta_i : k = 1, \cdots, \gamma; t = 1, 2; i = 1, \cdots, K\},$$

and consider simple disjoint path $\alpha_{k,t}, \alpha_i$ joining $\eta_{k,t}$ and η_i to x_0, respectively.

The loops $\alpha_{k,t}\eta_{k,t}\alpha_{k,t}^{-1}$, $\alpha_i\eta_i\alpha_i^{-1}$ form a basis for the fundamental group of $\tilde{S}^0 = \tilde{S} - \{q_1, \cdots, q_K\}$, based at x_0. Consider the two sheeted unbranched holomorphic covering $\pi^0 : S^0 \to \tilde{S}^0$, where S^0 is the lifting under π of $\tilde{S}^0$. Since the loops $\alpha_i\eta_i\alpha_i^{-1}$ lift to a path, the above covering is totally determined by how the loops $\alpha_{k,t}\eta_{k,t}\alpha_{k,t}^{-1}$ lift. We can see that $\alpha_{k,t}\eta_{k,t}\alpha_{k,t}^{-1}$ lifts to a loop if and only if $\eta_{k,t}$ lifts to a loop. So, the covering $\pi : S \to \tilde{S}$ is totally determined by knowing how the loops $\eta_{k,t}$ lift to S.

We define the symbol $(n_{1,1}, n_{1,2}, n_{2,1}, n_{2,2}, \cdots, n_{\gamma,1}, n_{\gamma,2})$ associated to the above loops $\eta_{1,1}$, $\eta_{1,2}$, $\eta_{2,1}$, $\eta_{2,2}, \cdots, \eta_{\gamma,1}$, $\eta_{\gamma,2}$, where $n_{k,t} \in \{0, 1\}$. This symbol has the following meaning

$$n_{k,t} = \begin{cases} 0 & \text{if } n_{k,t} \text{ lifts to a loop} \\ 1 & \text{otherwise} \end{cases}$$

Lemma 2. *Let S, $\tilde{S}$ and π as above. Then we can find simple loops $\eta_{1,1}$, $\eta_{1,2}$, $\eta_{2,1}$, $\eta_{2,2}, \cdots, \eta_{\gamma,1}$, $\eta_{\gamma,2}$, $\eta_1, \cdots, \eta_K$, satisfying the conditions (1) to (6) above , with associated symbol*

(i) $(1, 0, \cdots, 0)$, *if* $K = 0$;

(ii) $(0, \cdots, 0)$, *if* $K > 0$.

Proof : Start with a set of loops on $\tilde{S}$, $\eta_{1,1}$, $\eta_{1,2}$, $\eta_{2,1}$, $\eta_{2,2}, \cdots, \eta_{\gamma,1}$, $\eta_{\gamma,2}$, $\eta_1, \cdots, \eta_K$, satisfying the conditions (1) to (6) and associated symbol

$$(n_{1,1}, n_{1,2}, n_{2,1}, n_{2,2}, \cdots, n_{\gamma,1}, n_{\gamma,2}).$$

If we look at the pair $(n_{i,1}, n_{i,2})$, for each $i = 1, \cdots, \gamma$, we can see that the possibilities for this pair are the following:

(1) $(n_{i,1}, n_{i,2}) = (0, 0)$;

(2) $(n_{i,1}, n_{i,2}) = (1, 0)$;

(3) $(n_{i,1}, n_{i,2}) = (0, 1)$;

(4) $(n_{i,1}, n_{i,2}) = (1, 1)$.

In case (3), we change our loops $\eta_{i,1}, \eta_{i,2}$ by $\eta_{i,2}, \eta_{i,1}^{-1}$, respectively, as shown in figure 1. Now, we are in the case (2).

In case (4), we change our loops $\eta_{i,1}, \eta_{i,2}$ by simple loops freely homotopic to $\eta_{i,1}, \eta_{i,2}\eta_{i,1}$, respectively, as shown in figure 2. Now, we are in case (2).

After all these changes and some permutations, we obtain a set of simple loops satisfying the conditions (1) to (6) and associated symbol $(1, 0, 1, 0, \cdots, 1, 0, 0, 0, \cdots, 0, 0)$.

Now, we make more changes of the following type: look at $\eta_{i,1}$, $\eta_{i,2}$, $\eta_{i+1,1}$, $\eta_{i+1,2}$ with associated symbol (1,0,1,0). In this case, we change our loops $\eta_{i,1}, \eta_{i,2}$, $\eta_{i+1,1}$, $\eta_{i+1,2}$ by simple loops free homotopics to $\eta_{i,1}$, $\eta_{i,2}\eta_{i+1,2}$, $\eta_{i,1}\ \eta_{i+1,1}^{-1}$ *and* $\eta_{i+1,2}^{-1}$, respectively, as shown in figure 3.

We continue with these changes to obtain a set of simple loops satisfying the conditions (1) to (6) and associated symbol $(1, 0, 0, \cdots, 0, 0)$.

If $K = 0$, we are done. If $K \neq 0$, then we change $\eta_{1,1}$ and η_1 by simple loops freely homotopic to $\eta_{1,1}\eta_1$ and η_1, respectively, as shown in figure 4, and we get the symbol $(0, \cdots, 0)$ as desired.

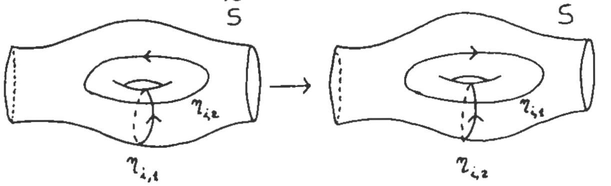

FIGURE 1. Change of loops in case (3)

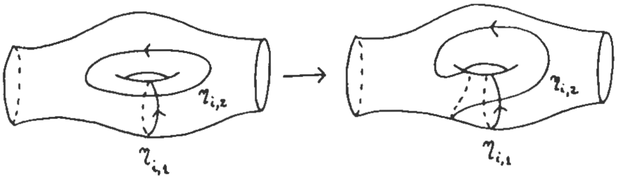

FIGURE 2. Change of loops in case (4)

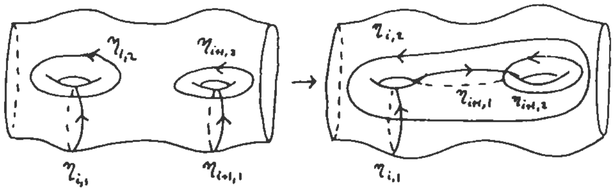

FIGURE 3

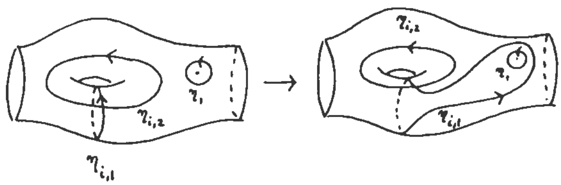

FIGURE 4

Lemma 3. *Let S and S' be two closed Riemann surfaces of genus γ, with K distinguished points each one, say $p_1, \cdots, p_K$ on S and $q_1, \cdots, q_K$ on S'. Let $\eta_{1,1}$, $\eta_{1,2}$, $\eta_{2,1}$, $\eta_{2,2}, \cdots, \eta_{\gamma,1}$, $\eta_{\gamma,2}$, $\eta_1, \cdots, \eta_K$, and $\beta_{1,1}$, $\beta_{1,2}$, $\beta_{2,1}$, $\beta_{2,2}, \cdots, \beta_{\gamma,1}$, $\beta_{\gamma,2}$, $\beta_1, \cdots, \beta_K$ be simple closed loops on S and S', respectively, satisfying the conditions (1) to (6) as before. Then there exists a quasiconformal diffeomorphism $f : S \to S'$, with the following properties:*

(i) $f(\eta_{i,j}) \approx \beta_{i,j}$, *for* $i = 1, \cdots, \gamma$; *and* $j = 1, 2$;

(ii) $f(\eta_t) \approx \beta_t$, *for* $t = 1, \cdots, K$;

(iii) $f(p_l) = q_l$, *for* $l = 1, \cdots, K$.

($\approx$ means freely homotopic (isotopic)).

Proof : By the topological classification of closed orientable surfaces, we can find $g : S \to S'$ an orientation preserving homeomorphism, such that

(i) $g(\eta_{i,j}) = \beta_{i,j}$, for $i = 1, \cdots, \gamma$; and $j = 1, 2$;

(ii) $g(\eta_t) = \beta_t$, for $t = 1, \cdots, K$;

(iii) $g(p_l) = q_l$, for $l = 1, \cdots, K$.

Since S is compact, we can approach g by C^{∞}–diffeomorphisms, which turn to be quasiconformal maps. Since, two homeomorphisms which are "near" each other are necessarily homotopic, we are done.

Remark 3. *Let $\pi : S \to \tilde{S}$ be a two sheeted holomorphic branched covering, and let $\eta_{1,1}, \eta_{1,2}, \eta_{2,1}, \eta_{2,2}, \cdots, \eta_{\gamma,1}, \eta_{\gamma,2}, \eta_1, \cdots, \eta_K$ be simple loops on $\tilde{S}$ satisfying conditions (1) to (6). If we assume these loops to be smooth loops, then in Lemma 3 we can assume equality instead of homotopy. From now on, our set of loops will be assumed to be smooth.*

Proposition 1. *Let K be a group of fractional linear transformations that contains a Schottky group as subgroup of finite index. Let h be any elliptic element of K and let x and y be its fixed points. Then either x and y are in the region of discontinuity of K or there is a loxodromic element g in G commuting with h.*

Proof : If K is torsion free, then there is nothing to check. Let us assume K has torsion and let h be any elliptic element of K with x and y as fixed points. If both points are in the region of discontinuity of K, we are done. Assume y is a limit point of the group K and let j be a primitive elliptic element in K fixing y.

The point x is a fixed point of j. In fact, if $j(x) \neq x$, then the commutator $[j, h] = jhj^{-1}h^{-1}$ is a parabolic element in K with y as fixed point. This is a contradiction to the fact That K has no parabolic elements.

If $g(y) = y$, for some g in K, then either g is conjugate in K to a power of j or g is a loxodromic element with x and y as fixed points. In fact, let g in K be such that $g(y) = y$. The only possibility is for g to be elliptic or loxodromic. By our assumption on y, we obtain that necessarily $g(x) = x$; otherwise $[g, j]$ will be a parabolic element of K fixing the point y. At this point, g is either a power of j, or a loxodromic element with x and y as fixed points.

Let L be the geodesic in $\mathbf{H}^3$ with x and y as end points. Necessarily, the transformation j acts as the identity on L.

Let P be a convex fundamental polyhedron for K. Since y is a limit point, which is not a parabolic fixed point, it must be a point of approximation for K (see page 128 in [13]). This implies that y cannot be in the closure of P (see page 122 in [13]).

We can find a sequence of points $y_n \in L$, converging to y, all of them non–equivalent points under K, and a sequence $g_n \in K$, all of them different, such that $g_n(y_n) = z_n \in cl(P)$, where $cl(P)$ denotes the Euclidean closure of P. Consider a subsequence such that z_n converges, say to z, $g_n(y)$ converges, say to u, and $g_n(x)$ converges, say to t. In this way, the points u and t are limit points for the group K. Since $z_n \in cl(P)$, we have z$\in cl(P)$. There are two possibilities for the point z, that is, either z is in the region of discontinuity of K, or z is a parabolic fixed point (see page 128 in [13]). Since K does not have parabolic elements, z belongs to the region of discontinuity of K. It is clear that the points z_n are elliptic fixed points, in fact $z_n = g_n \circ j \circ g_n^{-1}(z_n)$. This implies that z_n belongs to some edge of P. Since P has only a finite number of edges, we may assume all the points z_n to be on the same edge of P. Let M be the geodesic in $\mathbf{H}^3$ containing that edge. In particular, z belongs to the closure of M. Let us consider the geodesics $L_n = g_n(L)$ through z_n, and having end points $g_n(x)$ and $g_n(y)$. Since we have supposed $g_n(x)$ and $g_n(y)$ to converge to t and u, respectively, the sequence L_n converges either to a point or to the geodesic with end points u and t. If L_n converges to a point, then we necessarily have $u = t = z$. This is a contradiction to the fact that z is regular point and u is a limit point. The other possibility is that L_n converges to a geodesic γ, with end points u and t. In this case, since the end points of γ are limit points and z is a regular point, we must have z in $\gamma \cap \mathbf{H}^3$. Any neighborhood of z contains z_n, for n sufficiently large. Since z is a regular point, there exists a neighborhood of z which is precisely invariant by the elements of K fixing z, which is known to be finite. We can then assume without lost of generality that $g_n \circ j \circ g_n^{-1}(z) = z$, and $g_n \circ j \circ g_n^{-1} = h$. In other words, $(g_m^{-1} \circ g_n) \circ j \circ (g_m^{-1} \circ g_n)^{-1} = j$. Since $g_n \circ j \circ g_n^{-1}(z_n) = z_n$, $g_n \circ j \circ g_n^{-1}(z) = z$, and $z_n \neq z$, for all n, we have $g_n \circ j \circ g_n^{-1}(w) = w$, for all w in γ. In particular, $g_n \circ j \circ g_n^{-1}(t) = t$ and $g_n \circ j \circ g_n^{-1}(u) = u$. It follows that $\{g_n(x), g_n(y)\} = \{t, u\}$. The facts that $t \neq u$ and that $g_n(x)$ converges to t imply that $g_n(x) = t$ and $g_n(y) = u$, for n sufficiently large. We may assume it holds for every n. The last observation implies that $g_m^{-1} \circ g_n(x) = x$ and $g_m^{-1} \circ g_n(y) = y$, for all n, m. The transformations $g_m^{-1} \circ g_n$ also keep L invariant, and for $n \neq m$ this transformation cannot be the identity on L. This implies that $g_m^{-1} \circ g_n$ is a loxodromic element of K with x and y as fixed points. Since G has finite index in K, the result follows.

A consequence of Proposition 1 is the following. Let K be a group of fractional linear transformations containing a Schottky group G as a subgroup of index two. Let Ω be the region of discontinuity of G (K). Let us denote by $S = \Omega/G$, $X = \Omega/K$, the quotient Riemann surfaces obtained by the action of G and K, respectively, and by $\pi : \Omega \to S$, $\pi_1 : S \to X$ the respective holomorphic (branched) coverings. On S there exists a conformal involution j such that:

(i) $X = S/ < j >$;

(ii) $K =< J, G >$, where J is a lifting of j to Ω;

(iii) The branching of π_1 is exactly at the fixed points of j.

Proposition 1 asserts that there is a natural pairing of the fixed points of j (compare to Condition (A) in [9] for general groups of conformal automorphisms). This pairing is given as follows: Let p be any fixed point of j and let P be any lifting of p. Let T be the unique lifting of j fixing P. Denote by Q the other fixed point of T. Since P is a point in the region of discontinuity of K, Q is also a point in the region of discontinuity of K. The point Q projects on S to a fixed point q of j. If $q = p$, then there is an element g in G with $g(P) = Q$. Since G has no elliptic elements, $g(Q)$ is different from P; otherwise, g has order two. Now the commutator of T and $g \circ T \circ g^{-1}$ is a parabolic element in K fixing Q, a contradiction. So we must have $p \neq q$. Similar arguments show q is uniquely determined by p. In that way we obtain a pairing of the fixed points of j and at the same time a pairing of the branch values of π_1.

The above shows that if l is any simple loop on X bounding a topological disc R, where R contains all the branch values of π_1, then l must lift to a loop on $\Omega(G)$. Clearly, l lifts to two disjoint simple loops, l_1 and l_2, on S. We will use this information to prove the second part of Theorem B.

Let S be a closed Riemann surface of genus $g \geq 2$, and let $\omega_1, \omega_2, \cdots, \omega_g$ be a basis for the space $H^{1,0}(S, \mathbf{C})$ of holomorphic 1–forms on S. As a consequence of the Riemann–Roch Theorem [4], for any point p on S there exists some ω_i which is non-zero at p. Let us consider the canonical map

$$\varphi : S \to \mathbf{CP}_{g-1},$$

where $\varphi(p)$ is given by $(\omega_1(p), \omega_2(p), \cdots, \omega_g(p))$, in homogeneous coordinates. Let z be any local chart on S, vanishing at p, then we can write the differential ω_i, in this chart, as $f_i(z)dz$, where $f_i(z)$ is an holomorphic map in some neighborhood of the origin. In this local chart φ is given by

$$\varphi(z) = (f_1(z), \cdots, f_g(z)).$$

If we change our local chart, we do not change our point in $\mathbf{CP}_{g-1}$, as can be easily computed. Moreover, this map is analytic (see [4]).

Theorem 2 ([4]). *The holomorphic map $\varphi : S \to \mathbf{CP}_{g-1}$ is one–to–one if S is a non–hyperelliptic Riemann surface. $\varphi(S)$ is an irreducible, non-singular algebraic curve of degree $2g-2$.*

Lemma 4. *Let S be a hyperelliptic Riemann surface of genus $g \geq 2$, with hyperelliptic involution $h : S \to S$. If $\pi : S \to \hat{\mathbb{C}}$ is a degree two holomorphic (branched) covering and A is a group of conformal automorphisms of S, then*

(i) $A/ < h >$ is isomorphic to H, if h belongs to A; otherwise,

(ii) A is isomorphic to H, where H is a finite group of fractional linear transformations, fixing the set $\{p_1, \cdots, p_{2g+2}\}$, obtained as the projection of the fixed points of h (the Weierstrass points of S) under π.

Proof : Since the hyperelliptic involution is unique, we can define a homomorphism

$$\Phi : A \to Aut(\hat{\mathbf{C}}),$$

as follow:

Since h is in the center of the group of conformal automorphisms of S, for g in A we can find a fractional linear transformation $\Phi(g)$ such that $\pi \circ g = \Phi(g) \circ \pi$. The transformation $\Phi(g)$ is unique and satisfies $\Phi(g \circ f) = \Phi(g) \circ \Phi(f)$, for f and g in A. Moreover, the kernel of Φ is given by

$$Kernel\Phi = \begin{cases} < h > & \text{if } h \text{ is in } A, \\ < 1 > & \text{otherwise} \end{cases}$$

3. Main Theorems

In this section we show some different representations of γ–hyperelliptic Riemann surfaces. The proof of the following theorems will be given in the next sections.

Theorem A . *Let (S, j) be a γ–hyperelliptic Riemann surface of genus $g \geq 2$.*

(i) If j has no fixed points, that is, $g = 2\gamma - 1$, then there exist a pair of disjoint simple loops, say α and β, satisfying:

(i.1) *Neither α nor β divide S;*

(i.2) $j(\alpha) = \beta$;

(i.3) *$\alpha \cup \beta$ divides S into two surfaces with boundary, say S_1 and S_2, each one of genus $\gamma - 1$ with two deleted discs;*

(i.4) $j(S_1) = S_2$.

(ii) *If j has fixed points, say $p_1, \cdots, p_{2N}$ ($N = g - 2\gamma + 1$), then there exist disjoint simple loops, say $\alpha_1, \alpha_2, \cdots, \alpha_N$, satisfying:*

(ii.1) *α_i contains exactly two fixed points of j;*

(ii.2) $j(\alpha_i) = \alpha_i^{-1}$;

(ii.3) *$S - \cup\{\alpha_{i_j}; j = 1, \cdots, l\}$ is connected, $i_j \in \{1, \cdots, N\}$, if $l < N$;*

(ii.4) *$S - \cup\{\alpha_i; i = 1, \cdots, N\}$ has two components, S_1 and S_2, each one a surface of genus γ with N deleted discs;*

(ii.5) $j(S_1) = S_2$.

Theorem B . *Let (S, j) be a γ–hyperelliptic Riemann surface of genus $g \geq 2$. Then there exists a Schottky uniformization $(\Omega, G, \pi : \Omega \to S)$ of S reflecting the γ–hyperellipticity of S, that is, a γ–hyperelliptic Schottky uniformization of (S, j). Moreover, if j has fixed points, then there exist free generators for G, say $A_1, \cdots, A_g$, and a lifting J of j of order two such that the fixed points of j are the projections of the fixed points of the fractional linear transformations of order two J and $J \circ A_i$, for $i = 2\gamma + 1, \cdots, g$.*

Remark 4. *The first part of Theorem B, for γ=0, that is, the hyperelliptic case can also be found in the paper of L. Keen [11]. We can also construct all the $\gamma-$hyperelliptic Schottky groups, up to conjugation. This is done after the proof of Theorem B in the next section.*

Corollary 1. *If (S, j) is a 2–hyperelliptic Riemann surface of genus three, then S is necessarily hyperelliptic. Moreover, the hyperelliptic involution on S is the lifting of the hyperelliptic involution of the quotient Riemann surface of genus two.*

Remark 5. *There are many different proofs of Corollary 1 in the literature. We give a different one.*

Theorem C . *Let (S, j) be a γ–hyperelliptic Riemann surface of genus $g \geq 2$. Assume j acts with fixed points on S. Then there there exists a γ–hyperelliptic polygon P with associated Fuchsian group F, uniformizing S, such that $J(z) = -z$ is a lifting of the involution j to the unit disc U and $J(P) = P$. Moreover, the fixed points of j are represented in the polygon P by the origin (the center of P), the vertices of P and the midpoints of the sides $s_1, \cdots, s_{2g-4\gamma}$ (as in definition 5).*

Remark 6. *The case $\gamma = 0$ was already thought of by E. Whittaker [16] and solved by D. Gallo [5]. This case can be also obtained as an easy application of Fricke polygons.*

Theorem D . *Let (S, j) be a γ–hyperelliptic Riemann surface of genus $g \geq 2$, and assume j to have fixed points. Let $\tilde{S}$ be the quotient Riemann surface of genus γ obtained by the action of j on S.*

(i) If $0 \leq N, M \leq \gamma$ are fixed integers, then there exists a γ–hyperelliptic (N, M) hyperbolic polygon P with associated Fuchsian group F, uniformizing S, such that $J(z) = -z$ is a lifting of j to the unit disc U and $J(P) = P$. Moreover, the fixed points of j are represented in the polygon P by the origin (the center of P), the vertices of P and the midpoints of the sides $s_1, \cdots, s_{2g-4\gamma}$.

(ii) Let R be any γ–hyperelliptic Riemann surface of the same genus g as S, and let $r : R \to R$ be a γ–hyperelliptic involution on R (hence with fixed points). Suppose the quotient Riemann surface $R/ < r >$ is conformally equivalent to $\tilde{S}$ respecting the branch points. Let P be a γ–hyperelliptic hyperbolic polygon for S with side pairing transformations $T_1, T_2, \cdots, T_{2g}$. Then there exist integers $0 \leq N, M \leq \gamma$ and a set of indices $B = \{i_1, \cdots, i_N, j_1, \cdots, j_M\}$ contained in $\{1, \cdots, \gamma\}$, such that $i_k \neq i_t$ (respectively $j_l \neq j_h$), if $k \neq t$ (respectively $l \neq h$), such that the polygon P with side pairing transformations $L_1, L_2, \cdots, L_{2g}$, is a γ–hyperelliptic (N, M) hyperbolic polygon for R, where

$$L_i = \begin{cases} T_i\,, & \text{if } \; i \in \{1, \cdots, 2g-4\gamma\} \text{ or } i \text{ does not belong to } B; \\ J \circ T_i\,, & \text{if } \; i \; \text{ belongs to the set} B \end{cases}$$

In this case $J(z) = -z$ is a common lifting of j and r to the unit disc U.

Corollary 2. *Let (S, j) be a γ–hyperelliptic Riemann surface of genus $g \geq 2$, and assume $j : S \to S$ acts with fixed points. Let F be any torsion–free group uniformizing S, and let x_0 be a lifting of any fixed point of j to the unit disc U. If J is the lifting of j to U with x_0 as fixed point, and N, M are integers, $0 \leq N, M \leq \gamma$, then there exists a $4g$–sided hyperbolic polygon P satisfying all the conditions of a γ–hyperelliptic (N, M) hyperbolic polygon (Definition 6) except the invariance under the transformation $z \to -z$ which is replaced by J–invariance. The polygon P is a fundamental domain for F, and the fixed points of j are represented by x_0 (the center of P), the vertices of P and by the midpoints of some $2g - 4\gamma$ sides of P.*

Theorem E . *Algebraic Characterization Non-Hyperelliptic). Let (S, j) be a γ–hyperelliptic Riemann surface of genus $g \geq 2$ such that, S is non-hyperelliptic. Then S can be realized as a non–singular, irreducible algebraic curve C of degree $2g-2$ in $\mathbf{CP}_{g-1}$, invariant under $J \in Aut(\mathbf{CP}_{g-1})$, such that j corresponds to the restriction of J to C, with*

$$J = \begin{pmatrix} I_\gamma & 0 \\ 0 & -I_{(g-\gamma)} \end{pmatrix}$$

where I_n means the identity matrix of rank n.

Theorem F . *(Algebraic Characterization Non-Hyperelliptic). Let (S, j) be a γ–hyperelliptic Riemann surface of genus $g \geq 2$, such that S is hyperelliptic and j is different from the hyperelliptic involution. Then S corresponds to the Riemann surface associated to one of the following hyperelliptic curves:*

(1) $W^2 = Z(Z^2 - 1)(Z^2 - a_1) \cdots (Z^2 - a_{g-1})$; or

(2) $W^2 = (Z^2 - 1)(Z^2 - b_1) \cdots (Z^2 - b_g)$, where $a_i \neq a_j$ if $i \neq j$, $b_t \neq b_s$ if $s \neq t$, and $a_i, b_j \neq 0, 1, \infty$.

The γ–hyperelliptic involution j corresponds to the lifting of $J(z) = -z$ via the natural projection $\pi : S \to \hat{\mathbf{C}}$ given by the action of the hyperelliptic involution.

4. Proof of Theorems A and B

Proof of Theorem A. Let us apply Lemma 2 to the two sheeted holomorphic branched covering

$$\pi : S \to S/ < j > .$$

Case 1. The involution j acts freely

Let $\eta_{1,1},\ \eta_{1,2}, \cdots, \eta_{\gamma,1},\ \eta_{\gamma,2}$ be simple loops, as given in Lemma 2, with associated symbol $(1, 0, \cdots, 0)$. Now, proceed to cut the surface $S/ < j >$ along $\eta_{1,2}$ to obtain a surface of genus $\gamma - 1$ with two deleted discs. The boundary is given by $\eta_{1,2}^{+}$ and $\eta_{1,2}^{-}$. Consider two copies of such surface and proceed to glue them by gluing $\eta_{1,2}^{+}$ (respectively $\eta_{1,2}^{-}$) of one of them to $\eta_{1,2}^{-}$ (respectively $\eta_{1,2}^{+}$) of the other. This gives an explicit construction of the surface S, and the action of j is given by interchange of the two glued surfaces above.

Case 2. The involution j has fixed points

Let $\eta_{1,1},\ \eta_{1,2}, \cdots, \eta_{\gamma,1},\ \eta_{\gamma,2},\ \eta_1,\ \eta_2, \cdots, \eta_{2N}$ be simple loops, as given by Lemma 2, with associated symbol $(0, \cdots, 0)$. Consider simple paths (all of them disjoint) $\alpha_1,\ \alpha_2, \cdots, \alpha_N$, satisfying the following properties:

(i) α_i connects two different branch points;

(ii) $\alpha_i \cap \alpha_j = \phi$, if $i \neq j$;

(iii) $\alpha_i \cap \eta_{k,t} = \phi$, for all i, k, t.

Now, proceed to cut $S/ < j >$ along the paths $\alpha_1,\ \alpha_2, \cdots, \alpha_N$ to get a surface, say S_1, with boundary, say $\alpha_1^{+},\ \alpha_2^{+}, \cdots, \alpha_N^{+}$. Consider another copy, say S_2, and denote its boundary by $\alpha_1^{-},\ \alpha_2^{-}, \cdots, \alpha_N^{-}$, and glue them together by gluing α_i^{+} to α_i^{-} in a such way that the points where they are glue corresponds to the same point in $s/ < j >$. In this way we obtain an explicit construction of the surface S, and the action of j is given by interchange of the two glued surfaces above.

Proof of Theorem B. Let S be a closed Riemann surface of genus g and let $j : S \to S$ be a conformal involution with $2g + 2 - 4\gamma$ fixed points. Denote by $\pi : S \to S/H$ the natural holomorphic (branched) covering induced by the action of the cyclic group H of order two generated by the transformation j.

Assume first j acts freely, that is, $g = 2\gamma - 1$. In this case $\pi : S \to S/H$ is unbranched and the genus of S/H, that is γ, is greater or equal to 1. Consider on S/H a simple loop α_1 given by the projection of the loops α (β) of Theorem A. This loop is a non-dividing simple loop. Complete to a set of γ homologically independent disjoint simple loops $\alpha_1, \cdots, \alpha_\gamma$

on S/H. Now, consider the loops obtained by the liftings of these loops to S. If we remove the loop β from this family, then we obtain a set of g homologically independent disjoint simple loops, $\eta_1 = \alpha, \cdots, eta_g$, such that $\pi(\eta_{2l}) = \pi(\eta_{2l+1}) = \alpha_{l+1}$, for $l = 1, \cdots, \gamma - 1$. This family of loops define a Schottky uniformization $(\Omega, G, \pi : \Omega \to S)$ of S (Retrosection theorem), for which the involution j lifts. It is easy to see that the Schottky group G has free generators $A_1, \cdots, A_g$, such that A_1 is a lifting of j and $A_1 \circ A_{2l} \circ A_1^{-1} = A_{2l+1}$, for $l = 1, \cdots, \gamma - 1$. In figure 5(a) it is shown the case $g = 3$, $\gamma = 2$. Observe that we also can construct another Schottky uniformizations by consider a set of homologically independent simple loops $\delta_1, \cdots, \delta_\gamma$ on S/H with the property that δ_j is disjoint from the loop α_1 above, for $j = 2, \cdots, \gamma$, and δ_1 intersects α_1 transversaly at exactly one point. Now, the liftings of the loops δ_i, $i = 1, \cdots, \gamma$, are exactly g homologically independent disjoint simple loops β_j, $j = 1, \cdots, g$, with the property that $\pi(\beta_{2l}) = \pi(\beta_{2l+1}) = \delta_{l+1}$, for $l = 1, \cdots, \gamma - 1$, and $j(\beta_1) = \beta_1$. The loops β_j, $j = 1, \cdots, g$, define a Schottky uniformization $(\Omega, G, \pi : \Omega \to S)$ of S, for which the involution j lifts. In this case we can find a lifting J of order two and a set of free generators for G, say $B_1, \cdots, B_g$, satisfying $J \circ B_1 = B_1 \circ J$, $J \circ B_{2l} \circ J = B_{2l+1}$, for $l = 1, \cdots, \gamma - 1$. In figure 5(b) it can be seen the case $g = 3$, $\gamma = 2$.

Assume the involution j has fixed points. Denote by p_i, $i = 1, \cdots, 2g + 2 - 4\gamma$, the fixed points of j. Set $P_i = \pi(p_i)$ and $H =< j >$. By Theorem A, we can find, on S/H, a set of γ homologically independent disjoint simple loops, $\alpha_1, \cdots, \alpha_\gamma$, all of them disjoint from the points P_j, for all j, such that they lift to loops on S via the branched covering $\pi : S \to S/H$. Now consider disjoint simple paths, also disjoint from the above loops, say $\beta_1, \cdots, \beta_{g+1-2\gamma}$, such that β_j connects the points P_{2j-1} and P_{2j}, for $j = 1, \cdots, g + 1 - 2/gamma$. If we consider the liftings of the loops α_i and the paths β_j^2, for $i = 1, \cdots, \gamma$ and $j = 1, \cdots, g - 2\gamma$, we obtain on S a set of g homologically independent disjoint simple loops, $\alpha_{i,k}$, $\beta_{j,1}$, for $i = 1, \cdots, \gamma$, $k = 1, 2$ and $j = 1, \cdots, g - 2\gamma$, such that $\pi(\alpha_{i,1}) = \pi(\alpha_{i,2})$ and $j(\beta_{j,1}) = \beta_{j,1}$. This set of loops defines a Schottky uniformization $(\Omega, G, \pi : \Omega \to S)$ of S, for which j lifts. Moreover, there is a lifting J of order two and a set of free generators $A_1, \cdots, A_{2\gamma}$, $B_1, \cdots, B_{g-2\gamma}$, for G such that $J \circ A_{2i-1} \circ J = A_{2i}$ and $J \circ B_j \circ = B_j^{-1}$, for $i = 1, \cdots, \gamma$ and $j = 1, \cdots, g - 2\gamma$. In figure 5(c) can be seen the case $g = 2$, $\gamma = 1$. From our construction it is easy to see our claim on the fixed points of j.

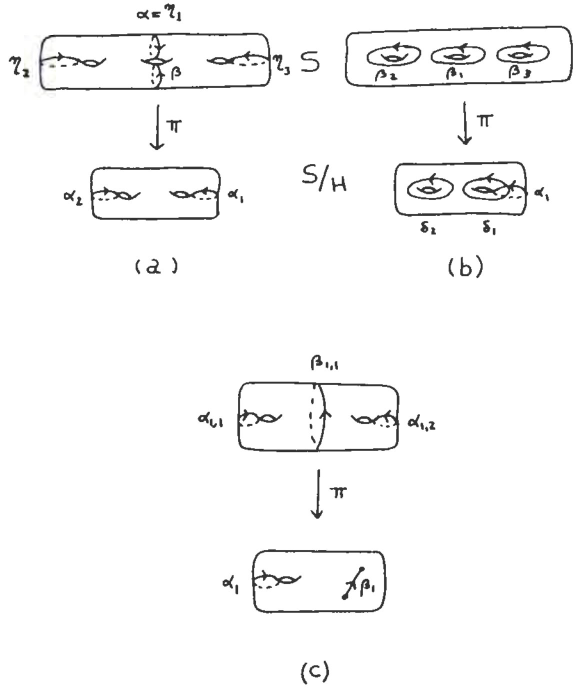

FIGURE 5

Proof of Corollary 1 By Theorem B, we can uniformize S by a Schottky group G with generators A_1, A_2 and A_3 satisfying the following properties:

(i) $J \circ A_1 \circ J = A_2$,

(ii) $J \circ A_3 \circ J = A_3$,

where J is a fractional linear transformation of order two. The quotient Riemann surface of genus two, obtained by the action of the 2–hyperelliptic involution, is uniformized by the group $K =< G, J >$.

Normalize the groups in such a way that the fixed points of A_3 are 1 and -1, and the fixed points of A_1 are r and $-r$, for some r. To see this normalization, consider the Lie bracket of A_3 and A_1, that is, $E = A_3 \circ A_1 - A_1 \circ A_3$. Simple computations show that $E^2 = I$ and $E \circ A_i \circ E = A_i^{-1}$, for $i = 1, 3$. Now, normalize such that the fixed points of E are 0 and ∞. In this case $E(z) = -z$, the fixed points of A_3 are p and $-p$, and the fixed points of A_1 are q and $-q$, for some p and q. Now conjugate with the transformation $T(z) = z/p$ to obtain the desired normalization. Since the fixed points of J are the same as for A_3, J becomes under this normalization the transformation $J(z) = 1/z$.

The general form of a fractional linear transformation with t and $-t$ as fixed points is given by the matrix

$$1/2 \begin{pmatrix} u & -Ut \\ -U/t & u \end{pmatrix},$$

where $U^2 = u^2 - 4$, and u^2 is the square of the trace of such an element.

Since the fixed points of A_2 are 1/r and -1/r, we have:

$$\begin{aligned} A_1 &= 1/2 \begin{pmatrix} x & -Xr \\ -X/r & x \end{pmatrix}, \\ A_2 &= 1/2 \begin{pmatrix} x & -X/r \\ -Xr & x \end{pmatrix}, \\ A_3 &= 1/2 \begin{pmatrix} y & -Y \\ -Y & y \end{pmatrix}, \end{aligned}$$

where $X^2 = x^2 - 4$ and $Y^2 = y^2 - 4$.

This is an hyperelliptic Schottky group [11], with the hyperelliptic involution represented by $E(z) = -z$.

Observe that $E = A_i \circ A_j - A_j \circ A_i$, for $i, j \in \{1, 2, 3\}$, $i \neq j$. Since the group $< G, J > = < A_1, A_3, J >$ uniformizes the quotient Riemann surface of genus two, is easy to see that E is a lifting of the hyperelliptic involution of such a surface. This fact implies the second part of the corollary.

5. Explicit Construction of γ–Hyperelliptic Schottky Groups

Let $(\Omega, G, \pi : \Omega \to S)$ be a γ–hyperelliptic Schottky uniformization of the γ–hyperelliptic Riemann surface (S, j) of genus g. It is a well known fact that the conformal automorphisms of the region of discontinuity of a Schottky group are linear fractional transformations [1]. Denote by K the group obtained by lifting the cyclic group H generated by j. Then G is a subgroup of K of index two and, in particular, it follows that K has no parabolic elements, is geometrically finite (so finitely generated) function group, and has the same region of discontinuity as G.

Denote by X the surface S/H and by $\pi_1 : S \to X$ the natural two sheeted covering induced by H. We consider two cases, that is, when the involution j either has fixed points or not.

5.1. The involution j has fixed points

By Theorem A, we can choose a simple loop l on X bounding a topological disc R containing all the branch values of π_1, satisfying the following.

(1) The loop l lifts to two disjoint simple loops l_1 and l_2 on S;

(2) $S - \{l_1, l_2\} = S_1 \cup S_2 \cup S_3$, where $j(S_1) = S_2$ and $j(S_3) = S_3$.

As a consequence of Proposition 1, we have that the loop l lifts to a simple loop on Ω. Fix one connected component $\tilde{l}$ of $(\pi_1 \circ \pi)^{-1}(l)$, and denote by Y the complement of the topological disc R, that is, $X = R \cup Y$ and $R \cap Y = l$. Observe that the liftings of Y on S are S_1 and S_2, and the lifting of R is S_3. Let Ω_1 and Ω_2 be connected components of $(\pi_1\pi)^{-1}(Y)$ and $(\pi_1\pi)^{-1}(R)$, respectively, with $\tilde{l}$ as common boundary. Denote by G_1 and G_2 the stabilizers of Ω_1 and Ω_2 in K, respectively. One can see that K is necessarily the free product of G_1 and G_2 (as in Klein-Maskit Combination theorem I in [13]). Since $\pi_1 : S_1 \to Y$ is one-to-one, we have that G_1 is a subgroup of G. In particular, G_1 is a Schottky group of genus γ. Let A_k, $k = 1, \cdots, \gamma$, be free-generators for G_1 and choose a standard fundamental domain for it, inside Ω_1, with respect to these generators.

Proposition 1 gives us a natural pairing of the branch values of π_1, as we observed long before. We may assume that the branch values of π_1 are P_{2i-1}, P_{2i}, for $i = 1, \cdots, g + 1 - 2\gamma$, where P_{2i-1} is paired to P_{2i}. Let α_i, $i = 1, \cdots, g + 1 - 2\gamma$, be disjoint simple paths contained in R such that, α_i connects P_{2i-1} and P_{2i}. The liftings of α_i to the region Ω are disjoint simple loops, each one invariant under an elliptic element of order two which interchange the two topological discs bounded by such a loop. We can choose exactly one loop for each i in Ω_2 such that $\tilde{l}$ and these loops bounds a common region of connectivity $g + 2 - 2\gamma$. Denote such loops by C_s and by J_s the elliptic element of order two as above,

$s = 1, \cdots, g+1-2\gamma$. The group G_2 is a free product of the cyclic groups of order two generated by those elliptic transformations. In this way we obtain an explicit construction of K (see figure 6 for $g = 2$, $\gamma = 1$). Algebraically, K has the presentation

$$K =< A_1 > * \cdots * < A_\gamma > * < J_1; J_1^2 = 1 > * \cdots * < J_{g+1-2\gamma}; J_{g+1-2\gamma}^2 = 1 >,$$

where A_i belongs to G. Since G has index two in K and J_i does not belong to G (since G is torsion free), we have that $J_1 \circ J_s$ belongs to G, for $s = 2, \cdots, g+1-2\gamma$. The group G is generated by A_i, $J_1 \circ A_i \circ J_1$ and $J_1 \circ J_s$, $i = 1, \cdots, \gamma$ and $s = 2, \cdots, g+1-2\gamma$. These are in fact free generators and there is a standard fundamental domain for this generators invariant under J_1 as it is shown in figure 7.

5.2. The involution j has no fixed points

We have two possibilities for K; either it is torsion free or has elliptic elements.

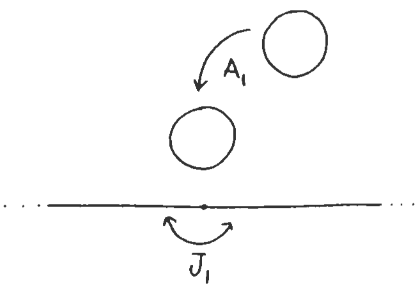

FIGURE 6. The group K

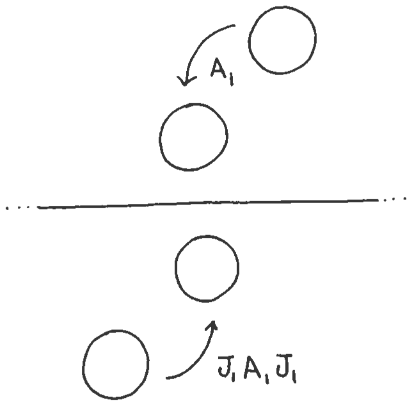

FIGURE 7. The group G

5.2.1. The group K is torsion free

Since K cannot have parabolic elements, K is necessarily purely loxodromic, geometrically finite, finitely generated function group, with the same region of discontinuity as a Schottky group. As a consequence of the classification of finitely generated function groups [14], we obtain that K is necessarily a Schottky group of genus γ. Let $A_1, \cdots, A_\gamma$ be any set of free generators for K. Since G has order two in K, then either A_i belongs to G or A_i^2 does it. Clearly, some of the A_i cannot be in G, otherwise $K = G$. We may assume A_γ does not belong to G. If some A_j does not belong to G ($j < \gamma$), then we replace A_j by $A_j \circ A_\gamma$ which now belongs to G. After these changes we still having a set of free generators for K, but now A_i belongs to G, for $i = 1, \cdots, \gamma - 1$. Let us consider the group L generated by the elements $A_1, \cdots, A_{\gamma-1}$, $A_\gamma A_1 A_\gamma^{-1}, \cdots, A_\gamma A_{\gamma-1} A_\gamma^{-1}$ and A_γ^2. The group L is a normal subgroup of index two of K and contained in G. As a consequence, the groups G and L are the same. Choose a standard fundamental domain D for K with respect to the above generators. We obtain a standard fundamental domain for G with respect to the above generators as $D \cup A_\gamma(D)$ (see figure 8 in the case $g = 3$, $\gamma = 2$).

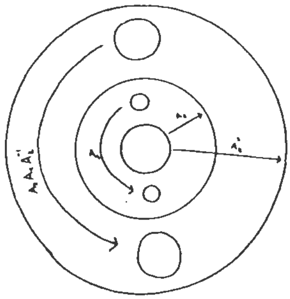

FIGURE 8. The groups K and G in the torsion free case

5.2.2. The group K has torsion

In this case, every elliptic element has order two, the product of any two of them is an element of G, and they commute with some loxodromic elements of G.

The region of discontinuity of K is connected (the same as for G) and K is a finitely generated, geometrically finite function group. By the classification of finitely generated function groups, K is constructed from cyclic loxodromic groups and cyclic groups generated by involutions, in such a way that every such involution commutes with some loxodromic element of the resulting group. One can check that such a group K is obtained in the following way. let $J_1, I_1, \cdots, I_k$, be a finite family of index set, such that $\gamma = ord(J_1) + ord(I_1) + 2ord(I_2) + \cdots + kord(I_k)$, where $ord(B)$ denotes the cardinality of B. Write $J_1 = \{j_1, \cdots, j_J\}$, $I_s = \{i_{s,1}, \cdots, i_{s,I(s)}\}$. We consider a loxodromic element A_{j_n}, for each $j_n \in J_1$, s involutions $T_{i_{s,m},1}, \cdots, T_{i_{s,m},s}$, and s loxodromic elements $C_{i_{s,m},1}, \cdots, C_{i_{s,m},s}$, such that $C_{i_{s,m},r} \circ T_{i_{s,m},r} \circ C_{i_{s,m},r}^{-1} = T_{i_{s,m},r+1}$, $r+1$ modulo s, for each $i_{s,m} \in I_s$, and such that there is a fundamental domain P, for the group generated by these transformations, as shown in figure 9.

By changing $C_{i_{s,m},r}$ by $C_{i_{s,m},r} \circ T_{i_{s,m},r+1}$, if necessary, we may assume it belongs to G. We may also change A_{j_n} by $A_{j_n} \circ T_{i_{s,1},1}$, some s, to assume that it also belongs to G. Observe that for each of the above changes we can modify our domain P in such a way that we get a new fundamental domain as in figure 9 for the new generators.

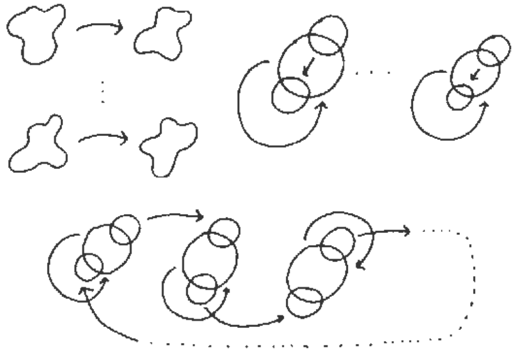

FIGURE 9. The group K

Let t be the smallest s for which $ord(I_s)$ is not zero. Now it is easy to see that G is generated (and free generated) by the elements: A_{j_n}, $C_{i_{t,1},1} \circ A_{j_n} \circ C_{i_{t,1},1}$, C_{i_s,m_s,r_s}, $T_{i_{t,1},1} \circ T_{i_{t,f_t},1}$ $T_{i_{t,1},1} \circ T_{i_{l,f_l},g_l}$, $n = 1, \cdots, J$, $m_s = 1, \cdots, s$, $r_s = 1, \cdots, I_s$, $f_t = 2, \cdots, t$, $f_l = 1, \cdots, l$, $g_l = 1, \cdots, I_l$, $s = t, \cdots, k$, $l = t+1, \cdots, k$. (See figure 10 in the case $g = 3$, $\gamma = 2$).

In this way we get all possible γ–hyperelliptic Schottky groups, up to conjugation.

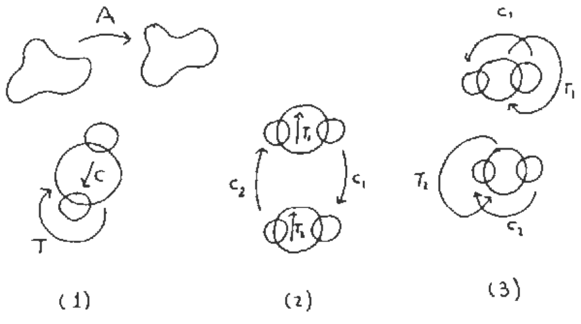

FIGURE 10. The groups K and G for $g = 3$ and $\gamma = 2$

6. Proof of the Theorems C, D, E and F

In this section, we give the proofs of the resting theorems.

Remark 7. *Observe that the second part of the above Corollary is also a consequence of Theorem A (part (i) with $\gamma = 2$) and a theorem of Haas and Susskind [6].*

Proof of Theorem C We construct a particular $4g$–sided γ–hyperelliptic polygon in the unit disc U. We apply Lemmas 2 and 3, and Ahlfors–Bers theorem to obtain a torsion free Fuchsian group F, uniformizing S, and such that $J(z) = -z$ is a lifting of the γ-hyperelliptic involution j to U. Using the continuity arguments of $t\mu$, for a Beltrami coefficient μ (Ahlfors–Bers theorem), we show that F has a γ–hyperelliptic polygon as fundamental domain.

Construction of a γ–hyperelliptic polygon with $4g$ sides

Subdivide the unit disc U by lines $R_1, \cdots, R_{2g}$ through the origin, such that the angle between R_i and R_{i+1} is $2\pi/4g$. Set R_1 be the real axis. See figure 11 in the case of $g = 2$.

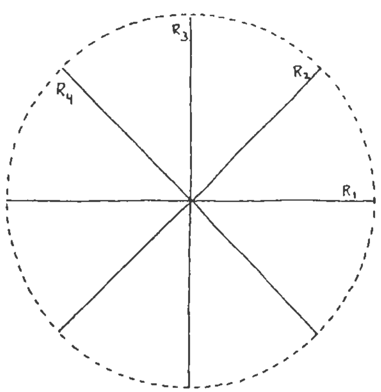

FIGURE 11. Configuration of the lines R_1, R_2, R_3, R_4, in genus two

For each r, $0 < r < 1$, choose points r_i, $-r_i$ in R_i such that $| r_i |= r$, and construct the strictly convex non-Euclidean polygon P_r with vertices r_i and $-r_i$, for $i = 1, \cdots, 2g$. See figure 12.

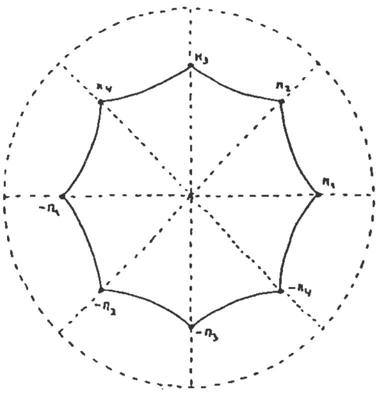

FIGURE 12. The polygon P_r in genus two

The interior angle α_r at any vertex of P_r strictly increases from 0 to $\pi - 2\pi/4g$ as r decreases from 1 to 0. In fact, for given r, consider one of the triangles $\Delta = (0,\ r_i,\ r_{i+1})$. The area of Δ is equal to $\pi - 2\pi/4g - \alpha_r$. Since the area of Δ goes to zero in a decreasing way as r approaches zero, the angle α_r must approach in an increasing way $\pi - 2\pi/4g$.

As a consequence, the sum of all the interior angles of P_r, $\theta_r = 4g\alpha_r$, strictly increases from 0 to $4g(\pi - 2\pi/4g)$ as r decreases from 1 to 0. Since $g \geq 2$, we have that $4g(\pi - 2\pi/4g) > 2\pi$. By continuity, there exists a unique r_0 for which $\theta_{r_0} = 2\pi$. Set $P_{r_0} = P^*$. We label the sides of P^* by $s_1^*, \cdots, s_{4g}^*$ (in counterclockwise order).

Let M_i^* be the elliptic element of order two keeping invariant the geodesic containing the side s_i^* of P^* and fixing the midpoint of s_i^*, for $i = 1, \cdots, 2g$, and set $L(z) = e^{\pi i/g}z$. Define fractional linear transformations T_i^*, for $i = 1, \cdots, 2g$, as follow (see figure 13):

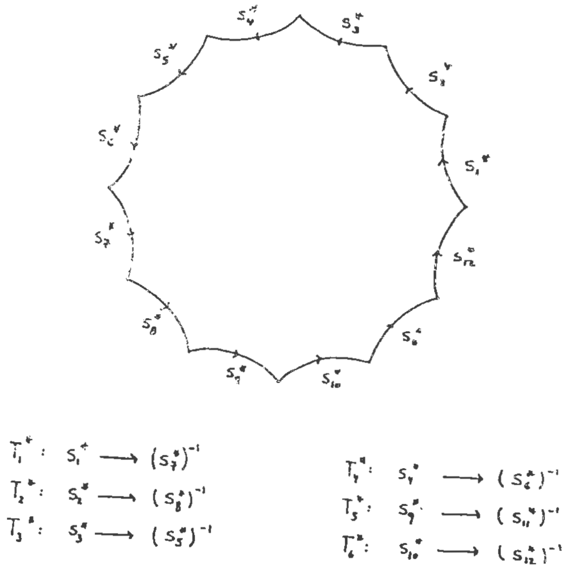

FIGURE 13. The side pairings for $g = 3$ and $\gamma = 1$

$$\begin{array}{rcllll}
T_i^* & = & J \circ M_i^*, & \text{for} & i = 1, \cdots, 2g - 4\gamma. \\
T_{2g-4\gamma+2t-1}^* & = & L \circ M_{2g-4\gamma+2t-3}^*, & \text{for} & t = 1, \cdots, \gamma. \\
T_{2g-4\gamma+2t}^* & = & L \circ M_{2g-4\gamma+2t-2}^*, & \text{for} & t = 1, \cdots, \gamma. \\
T_{2g-2\gamma+2t-1}^* & = & J \circ T_{2g-4\gamma+2t-1}^* \circ J, & \text{for} & t = 1, \cdots, \gamma. \\
T_{2g-2\gamma+2t}^* & = & J \circ T_{2g-4\gamma+2t}^* \circ J, & \text{for} & t = 1, \cdots, \gamma.
\end{array}$$

Continuity Arguments

Let F^* be the group generated by the fractional linear transformations T_i^*, for $i = 1, \cdots, 2g$. Poincare's theorem [13], asserts that F^* is a torsion free Fuchsian group with P^* as fundamental domain and with the following presentation:

$$\begin{aligned}
F^* &= < T_1^*, \cdots, T_{2g}^*; l_1 u_1 l_2 u_2 = 1 >, \quad \text{where} \\
l_1 &= \prod_{s=1}^{\gamma} [T_{2g-2\gamma+2s}^*, \; T_{2g-2\gamma+2s-1}^*], \\
l_2 &= \prod_{s=1}^{\gamma} [T_{2g-4\gamma+2s}^*, \; T_{2g-4\gamma+2s-1}^*], \\
u_1 &= T_{2g-4\gamma}^*(T_{2g-4\gamma-1}^*)^{-1}, \cdots, (T_3^*)^{-1} T_2^* (T_1^*)^{-1}, \\
u_2 &= (T_{2g-4\gamma}^*)^{-1} T_{2g-4\gamma-1}^*, \cdots, T_3^* (T_2^*)^{-1} T_1^*,
\end{aligned}$$

where $[A, B] = A \circ B \circ A^{-1} \circ B^{-1}$.

From the above construction, we see that F^* is an index two (so normal) subgroup of the Fuchsian group $< J, F^* >$, where $J(z) = -z$. A fundamental domain for such group is given by the convex non-Euclidean polygon P_0^* determined by the sides $s_1^*, \cdots, s_{2g}^*$, and $\mathbf{R} \cap P^*$ (see figure 14).

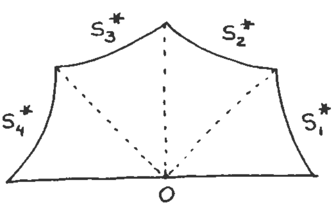

FIGURE 14. The polygon P_0^* in genus two

The transformations $J \circ T_1^*, \cdots, J \circ T_{2g-4\gamma}^*, T_{2g-4\gamma+1}^*, \cdots, T_{2g-2\gamma}^*$ and J generate the group $< J, F^* >$. These transformations are the side pairing of the polygon P_0^* (see figure 15).

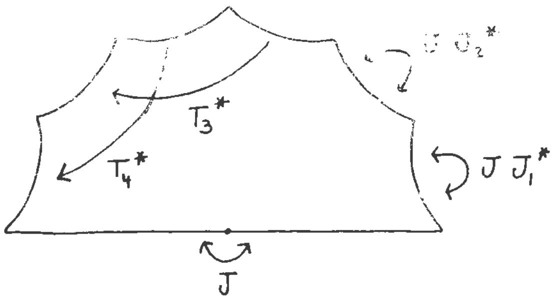

FIGURE 15. The side pairings of P_0^* for $g = 3$ and $\gamma = 1$

The transformation $J(z) = -z$ induces a γ–hyperelliptic involution j^* on the closed Riemann surface U/F^*, of genus g, whose quotient by j^* is the closed Riemann surface $U/ < J, F^* >$, of genus γ, with branch values the projection of the fixed points of j^*.

By Lemma 2, we can find a set of smooth simple loops on $U/ < J, F^* >$ and $\tilde{S} = S/ < j >$, respectively, satisfying the properties described in Lemma 2. As a consequence of Lemma 3, we can find a quasiconformal homeomorphism

$$f : U/ < J, F^* > \longrightarrow \tilde{S},$$

sending the special loops on $U/ < J, F^* >$ onto those on $\tilde{S}$, and the branch points on $U/ < J, F^* >$ onto the branch points on $\tilde{S}$.

The map f defines on P_0^* a function $\mu(z)$ (locally, $\mu(z) = f_{\bar{z}}/f_z$), and we can extend it to all of the unit disc U using the group $< J, F^* >$. The Beltrami coefficient μ constructed in this way is a Beltrami coefficient for $< J, F^* >$ and, in particular, for F^*. Let W_μ be the unique solution of the Beltrami equation

$$\partial W_\mu / \partial \bar{z} = \partial W_\mu / \partial z,$$

normalized by $W_\mu(0) = 0$, $W_\mu(1) = 1$ and $W_\mu(i) = i$ (Ahlfors–Bers theorem). We obtain the following:

(i) $W_\mu \circ F^* \circ (W_\mu)^{-1} = F$ is a Fuchsian group;

(ii) $W_\mu \circ J \circ (W_\mu)^{-1} = J$;

(iii) F uniformizes the Riemann surface S;

(iv) J is a lift of j to the unit disc U.

Let us rename the vertices of P^* by $a_1(0), \cdots, a_{4g}(0)$, in counterclockwise order such that, $a_1(0) = r_0$. Let $0 \leq t \leq 1$, and consider $t\mu$, which is again a Beltrami coefficient for the group $< J, F^* >$. Define $F_t = W_{t\mu} \circ F^* \circ (W_{t\mu})^{-1}$ which is a Fuchsian group, isomorphic to F^*, and uniformizing a closed Riemann surface S_t of genus g. The map $W_{t\mu}$ corresponds to the unique solution of the Beltrami equation for the Beltrami coefficient $t\mu$. In this notation $F_0 = F^*$ and $F_1 = F$. Define P_t as follows.

(i) Set $a_i(t) = W_{t\mu}(a_i(0))$, for $i = 1, \cdots, 4g$.

(ii) Let $s_i(t)$ be the geodesic arc joining $a_i(t)$ to $a_{i+1}(t)$, if $i = 1, \cdots, 4g-1$, and let $s_{4g}(t)$ be the geodesic arc joining $a_{4g}(t)$ to $a_1(t)$.

(iii) The side $s_i(t)$ is oriented from $a_i(t)$ to $a_{i+1}(t)$.

his way we obtain a closed polygonal curve $\gamma_t = s_1(t) \cup \cdots \cup s_{4g}(t)$.

(iv) Define the angle $\alpha_i(t)$ at $a_i(t)$ as follows.

(iv.1) $\alpha_i(t) = 0$ if $a_i(t) \in s_{i-1}(t)$ (see figure 16).

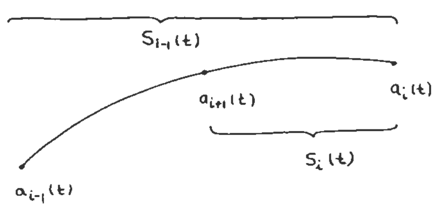

FIGURE 16

(iv.2) $\alpha_i(t) = \pi$ if $a_{i-1}(t)$, $a_i(t)$ and $a_{i+1}(t)$ lie in the same geodesic and $a_{i-1}(t) < a_i(t) < a_{i+1}(t)$ (see figure 17).

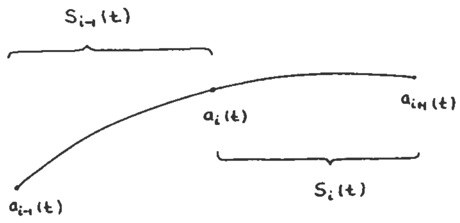

FIGURE 17

(iv.3) If we are not in one of the above cases, we consider $\gamma_i(t) = s_{i-1}(t) \cup s_i(t)$ with the orientation given above. Then define $\alpha_i(t)$ to be the angle measured from $s_{i-1}(t)$ to $s_i(t)$ at $a_i(t)$ at the left side of $\gamma_i(t)$ (see figure 18).

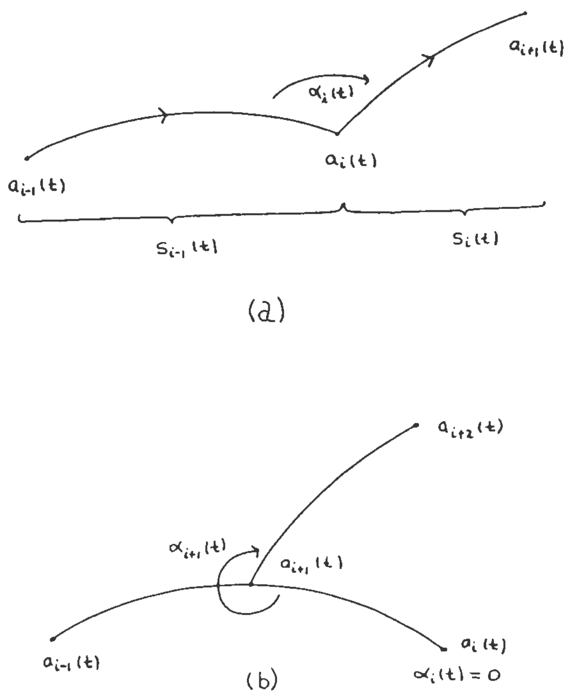

FIGURE 18

Claim.

(1) *The limit of any convergent sequence of strictly convex polygons is either a convex polygon or a geodesic.*

(2) *The closed polygonal curve γ_t defined above is J-invariant, for $J(z) = -z$.*

This a classical result and we do not prove it here. Define by P_t the polygon bounded by the curve γ_t (not necessarily a regular one) set

$$A = \{t \in [0,1]; \gamma_t \text{ is the boundary of a } \gamma \text{ –hyperelliptic polygon } \}.$$

(i) The set A is not empty since $t = 0$ belongs to A.

(ii) Let $t_0 \in A$. If t is close t_0, then the continuity of $W_{t\mu}$ on t implies that γ_t is close to γ_{t_0}. Since γ_{t_0} bounds an strictly convex hyperbolic polygon, If we chose t close enough to t_0, then γ_t also will bound a strictly convex hyperbolic polygon. It is easy to see, by the construction, that P_t is J–invariant, and that the fractional linear transformations

$$T_i(t) = W_{t\mu} \circ T_i^* \circ (W_{t\mu})^{-1}$$

identify the sides of P_t in the same combinatorial way as the fractional linear transformations T_i^* identify the sides of P_t^*. This implies that the sum of the angles of P_t must be of the form $2k\pi$, for some integer k. Using the fact that $\sum_{i=1}^{4g} \alpha_i(t)$ is a continuous function on t, and that $\sum_{i=1}^{4g} \alpha_i(t_0) = 2\pi$, we obtain that $\sum_{i=1}^{4g} \alpha_i(t) = 2\pi$. The same continuity argument shows that for t very near to t_0, we must have $0 < \alpha_i(t) < \pi$. Thus, γ_t is a γ-hyperelliptic polygon. As a consequence, the set A is an open subset of the closed interval $[0,1] = \{t; 0 \leq t \leq 1\}$.

(iii) Let t_n a sequence of points in A converging to t. For n sufficiently large t_n is very near to t, so by similar continuity argument as in (i), P_{t_n} is very near to P_t. The above Claim asserts that P_t is either a convex polygon or a geodesic. Assume P_t is a geodesic, then the group $F_t = W_{t\mu} \circ F^* \circ (W_{t\mu})^{-1}$ keeps invariant such a geodesic. In particular, the limit set of F_t is contained in the set of end points of this geodesic. Thus, F_t is an elementary group. This a contradiction, since by construction the group F_t uniformizes a closed Riemann surface of genus g. It follows that P_t is in fact a convex hyperbolic polygon. Since the sides of this polygon are paired in the same combinatorial way as the sides of P_0, the sum of the angles of P_t must be of the form $2k\pi$, for some integer k. The continuity of $\sum_{i=1}^{4g} \alpha_i(t)$ and the fact that $\sum_{i=1}^{4g} \alpha_i(t_n) = 2\pi$ implies that $\sum_{i=1}^{4g} \alpha_i(t) = 2\pi$. The J–invariance of P_t, the fact that $\alpha_i(t) > 0$, for $i = 1, \cdots, 4g$, and the above observation on the sum of the interior angles imply that $0 < \alpha_i(t) < \pi$. As a consequence, P_t is a γ–hyperelliptic polygon, and A is also a closed subset of $[0, 1]$.

Now, (i), (ii) and (iii) imply that $A = [0, 1]$. In particular, F has a γ-hyperelliptic polygon as a fundamental domain.

Proof of Theorem D

(i) The proof is the same as for theorem C.

(ii) Let P be a γ-hyperelliptic polygon for a Fuchsian group F uniformizing the surface S as in the hypotheses. Choose integers N, M such that, $0 \leq N, M \leq \gamma$, and integers $i_1, \cdots, i_N$, $j_1, \cdots, j_M$ such that $i_k \neq i_l$ (respectively $j_r \neq j_s$) if $k \neq l$ (respectively if $r \neq s$). Define the side pairings $T_i(N, M, i_1, \cdots, i_N, j_1, \cdots, j_M) = T'_i$ as follow:

(a) $T'_i = T_i$, for $i = 1, \cdots, 2g - 4\gamma$.

(b) $$T'_{2g-4\gamma+2t-1} = \begin{cases} T_{2g-4\gamma+2t-1} & \text{if } t \text{ does not belong to } \{i_1, \cdots, i_N\}, \\ J \circ T_{2g-4\gamma+2t-1} & \text{otherwise} \end{cases}$$

(c) $$T'_{2g-4\gamma+2t} = \begin{cases} T_{2g-4\gamma+2t} & \text{if } t \text{ does not belong to } \{j_1, \cdots, j_M\}, \\ J \circ T_{2g-4\gamma+2t} & \text{otherwise} \end{cases}$$

(d) $T'_{2g-2\gamma+l(t)} = J \circ T_{2g-4\gamma+l(t)} \circ J$, for $l(t)$ of the form $2t - 1$ or $2t$.

Apply Poincare's theorem to the polygon P and the side pairing defined above to obtain a torsion free Fuchsian group $\mathrm{F}(i_1, \cdots, i_N, j_1, \cdots, j_M)$, which is a normal subgroup, of index two, of $< J, F >$, uniformizing a closed Riemann surface of genus g. We obtain in this way $2^{2\gamma}$ non-conjugated (in $< J, F >$) torsion free Fuchsian groups . This is exactly the number of non-equivalent two sheeted coverings of a closed Riemann surface of genus γ with the same branch locus.

Proof of Corollary 2 Let S, $j : S \to S$, F, $J : U \to U$ and x_0 be as in the hypotheses and let $\pi_F : U \to S$ be the universal covering of the surface S with F as covering group. Denote by p_0 the point $\pi_F(x_0)$. By Theorem D asserts we can find a γ–hyperelliptic (N, M) hyperbolic polygon P with associated Fuchsian group G, uniformizing S, such that 0 is a lifting of p_0 and with $B(z) = -z$ a lifting to U of j. Since G and F uniformize the same Riemann surface S, there exists a fractional linear transformation H, fixing the unit disc U such that, $\pi_F \circ H = \pi_G$. Since $H(0)$ must be equivalent with x_0, we can find an element f in F such that, $f^{-1}(x_0) = H(0)$. As a consequence, $f \circ H(0) = x_0$ and $f \circ H$ also makes the diagram, in figure 18, commutative. Thus, we may assume that $H(0) = x_0$.

(i) $H \circ B \circ H^{-1}$ is a fractional linear transformation of order two fixing x_0 and keeping the unit disc U invariant. Since there is only one such a transformation, we obtain $H \circ B \circ H^{-1} = J$.

(ii) $H(P)$ is a hyperbolic polygon as desired.

Proof of Theorem E If V is finite dimensional vector space and $h : V \to V$ is an isomorphism of order two, then we have a decomposition $V = W^+ \oplus W^-$, where

$$W^+ = \{v \in V; h(v) = v\} \text{ and } W^- = \{v \in V; h(v) = -v\}.$$

In our case, $V = H^{1,0}(S, \mathbf{C})$ and h is the isomorphism induced by the γ–hyperelliptic involution $j : S \to S$. We need to check the right dimensions of W^+ and W^-. The subspace W^+ corresponds to the holomorphic differential forms on $S/ < j >$, that is, $H^{1,0}(S/ < j >, \mathbf{C})$, which has dimension γ. Since the space $H^{1,0}(S, \mathbf{C})$ has dimension g, the space W^- has dimension $g - \gamma$. Choose basis $w_1, \cdots, w_\gamma$ for W^+ and $w_{\gamma+1}, \cdots, w_g$ for W^-. Now we can apply Theorem 2.

Proof of Theorem F Apply Lemma 4 to the group A generated by the γhyperelliptic involution $j : S \to S$ and $h : S \to S$ is the hyperelliptic one. If $j \neq h$, then A is the lift of a cyclic group of order two. Up to normalization, we may assume that group to be generated by $J(z) = -z$.

The Weierstrass points of S are projected onto a set R, consisting of exactly $2g+2$ points, invariant under the involution J. We have the following possibilities:

(i) If 0 and ∞ belong to the set R, then the surface S is represented by the hyperelliptic curve

$$W^2 = Z(Z^2-1)(Z^2-a_1)\cdots(Z^2-a_{g-1}),$$

where $a_i \neq a_j$ if $i \neq j$ and $a_i \neq 0, 1, \infty$. Here R consists of the points 0, ∞, 1, -1, $a_1, \cdots, a_{g-1}$.

(ii) If 0 and ∞ are not in this set of points, then the surface S is represented by the hyperelliptic curve

$$W^2 = (Z^2-1)(Z^2-b_1)\cdots(Z^2-b_g),$$

where $b_t \neq b_s$ if $s \neq t$ and $b_j \neq 0, 1, \infty$. Here R consists of the points 1, -1, $b_1, \cdots, b_g$.

The hyperelliptic involution is given by

$$\left\{ \begin{array}{lcr} W & \rightarrow & -W \\ Z & \rightarrow & Z \end{array} \right\}.$$

The Marked Schottky Space of Genus g and the Γ–Hyperelliptic Schottky locus

In this section, we define the marked Schottky space. We describe a locus of those marked Schottky groups which uniformize γ–hyperelliptic Riemann surfaces and reflect their γ–hyperellipticity. This part is direct consequence of Theorem B.

Definition 9. *Let $(G_1; A_1, \cdots, A_g)$ and $(G_2; B_1, \cdots, B_g)$ be two marked Schottky groups of genus g. We say that they are equivalent if there exists a linear fractional transformation H satisfying*

$$H \circ A_i = B_i \circ H, \quad \text{for}\, i = 1, \cdots, g.$$

We define the Marked Schottky Space of genus g, denoted by S_g, as the set of equivalence classes of marked Schottky groups of genus g.

To a loxodromic transformation A we have three related objects; its repelling fixed point $r(A)$, its attracting fixed point $a(A)$ and the square of its trace (up to a sign) $t(A)$. The transformation A is uniquely determinated by such a triple.

We have a map

$$\Phi : \mathcal{S}_g \to C^{3g-3}.$$

This map is given as follow. Let $(G; A_1, \cdots, A_g)$ be a marked Schottky group of genus g. We can find a unique equivalent marked Schottky group such that $r(A_1) = 1$, $a(A_1) = -1$ and $r(A_2) = -a(A_2)$. In this way, the map Φ is defined by

$$\begin{aligned} \Phi(G; A_1, \cdots, A_g) &= \begin{array}{r} (t(A_1), \cdots, t(A_g), a(A_2), r(A_3), a(A_3), ..., \\ r(A_g), a(A_g)) \end{array} \\ &= (t_1, \cdots, t_g, a_2, r_3, a_3, \cdots, r_g, a_g). \end{aligned}$$

It is very well known that the image under Φ of the marked Schottky space of genus g is an connected open set of C^{3g-3}.

D. Hejhal [7] have shown that the marked Schottky space of genus g is also a domain of holomorphy.

Using the above normalization and Theorem C, we get the following parametrization of γ–hyperelliptic Schottky groups in the Schottky marked space. Observe that in the particular case of $\gamma = 0$ we obtain the same parametrization given by L. Keen in [11].

Theorem 3.

(1) *The sublocus of the marked Schottky space defined by the $(3g-3)/2$ equations*

$$\begin{aligned} t_{2i} &= t_{2i+1}; \\ r_{2i}r_{2i+1} &= 1; \\ a_{2i}a_{2i+1} &= 1; \end{aligned}$$

for $i = 2, \cdots, (g-1)/2$, consists of Schottky groups which represent closed Riemann surfaces having a conformal automorphism of order two acting freely. The involution above is represented by

$$J = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

(2) *The sublocus of the marked Schottky space of genus g defined by the $g+\gamma-2$ equations*

$$\begin{aligned} r_i &= -a_i, &&\text{for} \quad i = 3, \cdots, g-2\gamma; \\ t_{g-2\gamma+i} &= t_{g-\gamma+i}, &&\text{for} \quad i = 1, \cdots, \gamma; \\ r_{g-2\gamma+i} &= -r_{g-\gamma+i}, &&\text{for} \quad i = 1, \cdots, \gamma; \end{aligned}$$

$a_{g-2\gamma+i} = -a_{g-\gamma+i}$, *for* $i = 1, \cdots, \gamma$;
consists precisely of those marked Schottky groups which represent closed Riemann surfaces having a conformal automorphism of order two with $2(g-2\gamma+1) \geq 4$ fixed points. The above involution is represented by

$$J = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

(3) *The sublocus of the marked Schottky space of genus g defined by the $3\gamma-2$ equations*

$$\begin{aligned} t_{2i} &= t_{2i-1}, &&\text{for} \quad i = 1, \cdots, \gamma; \\ r_{2i}r_{2i-1} &= r, &&\text{for} \quad i = 1, \cdots, \gamma; \\ a_{2i}a_{2i-1} &= r, &&\text{for} \quad i = 1, \cdots, \gamma; \end{aligned}$$

consists precisely of those marked Schottky groups which represent closed Riemann surfaces having a conformal automorphism of order two with two fixed points.

The above involution is represented by

$$J = \begin{pmatrix} 0 & ir^{1/2} \\ i/r^{1/2} & 0 \end{pmatrix}.$$

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