ON k^* -RADICAL IN BCI-ALGEBRAS

Hamza A. S. Abujabal
King Abdulaziz University, Saudi Arabia.

Abstract

In [2] the notion of nil radical in BCI-algebras was introduced and various properties are developed in [3]. Further, several results on nil ideals were obtained in [4] and [5]. The aim of this paper is to study and investigate some properties of k^ -radicals in BCI-algebras. The notion of k^* -semiprime ideal is introduced.*

Key words and phrases. BCI-algebra, ideals, subalgebra, k -radical, nil ideal, k^* -semiprime ideal. 1991 AMS Subject classification. 03G25, 06F35

A BCI-algebra is an abstract system $X = \langle X; *, 0 \rangle$, where X is a non-empty set, $*$ a binary operation on X and 0 is a constant in X satisfying:

$$\text{BCI-1 } ((x * y) * (x * z)) * (z * y) = 0,$$

$$\text{BCI-2 } (x * (x * y)) * y = 0,$$

$$\text{BCI-3 } x * x = 0,$$

$$\text{BCI-4 } x * y = 0 \text{ and } y * x = 0 \text{ imply } x = y.$$

A binary relation $\leq$ on X can be defined by $x \leq y$ if and only if $x * y = 0$. Then $\langle X; \leq \rangle$ is a partially ordered set. The BCK-part of a BCI-algebra X is $X_+ = \{x \in X | x \geq 0\}$. The p -radical of a BCI-algebra is the BCK-part of X . A BCI-algebra is called p -semisimple if its p -radical is $\{0\}$.

In any BCI-algebra X the following hold:

$$(1) x * 0 = x,$$

$$(2) (x * y) * z = (x * z) * y,$$

$$(3) x * (x * (x * y)) = x * y,$$

$$(4) 0 * (x * y) = (0 * x) * (0 * y),$$

$$(5) x \leq 0 \text{ implies } x = 0.$$

$$(6) (x * z) * (y * z) \leq x * y,$$

for all $x, y, z \in X$.

A non-empty subset S of a BCI-algebra X is called a subalgebra of X if for any x and y in S , $x * y \in S$. A non-empty subset I of a BCI-algebra X is called an ideal of X if $0 \in I$ and if $x * y \in I$, $y \in I$ imply that $x \in I$. An ideal in a BCK-algebra is a subalgebra, but this is not true in general in a BCI-algebra. An ideal I of a BCI-algebra X is a closed ideal of X if $0 * x \in I$ for every $x \in I$. We note that in a BCI-algebra X , every closed ideal is a subalgebra of X . An ideal I of a BCI-algebra X is called weakly implicative if, whenever $(x * y) * z \in I$ and $y * z \in I$, then $(x * z) * z \in I$. An ideal I of a BCI-algebra X is called implicative if, whenever $(x * y) * z \in I$ and $y * z \in I$, then $x * z \in I$. An ideal I of a BCI-algebra X is called strongly implicative if, whenever $(x * y) * z \in I$ and $y * z \in I$, then $x \in I$. An ideal I of a BCI-algebra X is called commutative if, whenever $x \wedge y \in I$, then $y \wedge x \in I$, where $x \wedge y = y * (y * x)$ for $x, y \in X$. A non-empty subset A in a BCI-algebra X is called a p -ideal of X if $0 \in A$, $(x * z) * (y * z) \in A$ and $y \in A$ imply $x \in A$, for $x, y, z \in X$. A mapping $f : X \rightarrow Y$ of BCI-algebras is called a homomorphism if $f(x * y) = f(x) * f(y)$ for all $x, y \in X$.

For any element x, y in a BCI-algebra X , we use $x * y^k$ to denote the element $(\dots ((x * y) * y \dots) * y)$, where y occurs k -times. An element x in a BCI-algebra X is said to be a nilpotent element of index k , if $0 * x^k = 0$ for some positive integer k . An ideal I of a BCI-algebra X is called a nil ideal of X if every element of I is nilpotent (see [2]).

In this paper, we introduced the notion of k^* -radical and k^* -semiprime ideal for any positive integer k .

Let S be a non-empty subset of a BCI-algebra X . For any positive integer k , the set

$$N_k(S) = \{x \in S \mid 0 * x^k = 0\}$$

has been studied with its properties in [2], [3], [4] and [5].

Now, let S be a non-empty subset of a BCI-algebra X . In this paper, we consider the set

$$\sqrt[k]{S} = \{x \in S \mid 0 * x^k \in S\},$$

which is called the k^* -radical of S , in X , for any positive integer k . Some properties of k^* -radical are investigated. It is clear that $0 \in \sqrt[k]{S}$ and $\sqrt[k]{S} \subseteq S$. But it is not necessary $S = \sqrt[k]{S}$. Let I be an ideal of a BCI-algebra X . Then I is called a k^* -semiprime ideal of X if $I = \sqrt[k]{I}$ for any positive integer k .

Lemma 1 [4]. *For any BCI-algebra X and any positive integer k , we have*

$$0 * (x * y)^k = (0 * x^k) * (0 * y^k)$$

for all $x, y \in X$.

Lemma 2 [4]. *For any BCI-algebra X and any positive integer k , we have*

$$0 * (0 * x)^k = 0 * (0 * x^k)$$

for all $x \in X$.

Theorem 1 *If S is a subalgebra of a BCI-algebra X , and $x \in \sqrt[k]{S}$, for any positive integer k , then $0 * x \in \sqrt[k]{S}$.*

Proof Let $x \in \sqrt[k]{S}$. Then $x \in S$, $0 * x^k \in S$ and $0 * x \in S$. But S is a subalgebra of X . Thus $0 * (0 * x^k) \in S$. By Lemma 2, we have $0 * (0 * x)^k = 0 * (0 * x^k) \in S$. Therefore, $0 * x \in \sqrt[k]{S}$.

Theorem 2 *Let S be a subalgebra of a BCI-algebra X . Then $\sqrt[k]{S}$ is a subalgebra of X and $S = \sqrt[k]{S}$, for any positive integer k .*

Proof. Let $x, y \in \sqrt[k]{S}$. Then $x \in S$, $y \in S$, $0 * x^k \in S$ and $0 * y^k \in S$. Then $(0 * x^k) * (0 * y^k) \in S$ because S is a subalgebra of X . By Lemma 1, we have $0 * (x * y)^k = (0 * x^k) * (0 * y^k) \in S$. Therefore, $x * y \in \sqrt[k]{S}$. Thus $\sqrt[k]{S}$ is a subalgebra of X and $\sqrt[k]{S} \subseteq S$.

Now, let $x \in S$. We claim that $0 * x^k \in S$. But $0 * x^k = (\dots((0 * x) * x \dots) * x)$, where y occurs k -times and $0 * x \in S$. Hence $(0 * x) * x \in S$.

Thus, $(\dots((0 * x) * x \dots) * x) \in S$, because S is a subalgebra, $x \in S$ and so $0 * x^k \in S$ and $x \in \sqrt[k]{S}$. Therefore, $S = \sqrt[k]{S}$.

Theorem 3. *Let X be a BCI-algebra, and let I be an ideal of X . Then $\sqrt[k]{I}$ is an ideal of X , for any positive integer k .*

Proof. We have that $0 \in \sqrt[k]{I}$. If y and $x * y \in \sqrt[k]{I}$, then $y \in I$, $x * y \in I$, $0 * y^k \in I$, $0 * (x * y)^k \in I$ and $x \in I$ because I is an ideal. By Lemma 1, $0 * (x * y)^k = (0 * x^k) * (0 * y^k)$. Then $(0 * x^k) * (0 * y^k) \in I$. But I is an ideal and $0 * y^k \in I$, so $0 * x^k \in I$. Therefore, $x \in \sqrt[k]{I}$ and so $\sqrt[k]{I}$ is an ideal of X .

Theorem 4. *Let I be a closed ideal of a BCI-algebra X . Then $\sqrt[k]{I}$ is a closed ideal of X , and $I = \sqrt[k]{I}$ for any positive integer k .*

Proof. If I is a closed ideal of X . Then I is a subalgebra of X . Let $x \in \sqrt[k]{I}$. Then by Theorem 1, $0 * x \in \sqrt[k]{I}$. By Theorem 3, $\sqrt[k]{I}$ is an ideal. Therefore, $\sqrt[k]{I}$ is a closed ideal of X . Therefore, $\sqrt[k]{I}$ is a subalgebra of X and hence $I = \sqrt[k]{I}$ by Theorem 2.

Remark 1. If I is an ideal of a BCI-algebra X , then is not necessary, $I = \sqrt[k]{I}$ for every positive integer k .

The following example demonstrate Remark 1:

Example 1 Let $X = \{0, 1, 2, 3\}$ in which $*$ is given by the table:

$*$	0	1	2	3
0	0	0	2	2
1	1	0	3	2
2	2	2	0	0
3	3	2	1	0

clearly, $(X, *, 0)$ is a BCI-algebra, $C = \{0, 1\}$, is a subalgebra of X , which is an ideal and $I = \{0, 3\}$ is an ideal of X , but is not a subalgebra of X . Further, $\sqrt[3]{I} = \{0\} \subset I$ and $I \neq \sqrt[3]{I}$, because $3 \in I$, and $0 * 3^3 = 2 \notin I$ implies $3 \notin \sqrt[3]{I}$. Thus I is not a 3^* -semiprime ideal. But $\sqrt[2]{I} = \{0, 3\} = I$. Hence I is a 2^* -semiprime ideal. The ideal C is closed and hence a subalgebra of X . Then $C = \sqrt[2]{C}$. Thus C is a 2^* -semiprime ideal. Also $\sqrt[2]{I}$ is a closed ideal. Further, $\sqrt[3]{C} = \{0, 1\} = C$ which is a closed ideal of X and 3^* -semiprime ideal.

Lemma 3 [4]. *In a BCI-algebra, every p -ideal is an ideal.*

Lemma 4. *For any BCI-algebra X , we have*

$$x * [x * (x * y)] = x * y$$

for all $x, y \in X$.

Lemma 5 [4]. *An ideal A of a BCI-algebra X is a p -ideal if and only if $0 * (0 * x) \in A$ implies $x \in A$, where $x \in A$.*

Theorem 5. *Let A be a p -ideal of a BCI-algebra X , and let k be a positive integer. Then $\sqrt[k]{A}$ is a p -ideal of X .*

Proof By Lemma 3, A is an ideal of X . Thus $\sqrt[k]{A}$ is an ideal of X by Theorem 3. Let $0 * (0 * x) \in \sqrt[k]{A}$. Then $0 * (0 * x) \in A$, and $x \in A$, because A is a p -ideal by Lemma 5. Hence $0 * (0 * (0 * x))^k \in A$, and $0 * (0 * (0 * x)^k) \in A$ by Lemma 2. Thus $0 * (0 * (0 * x^k)) \in A$ by Lemma 2 and $0 * x^k \in A$ by Lemma 4. But $x \in A$. Hence $x \in \sqrt[k]{A}$. Therefore, $\sqrt[k]{A}$ is a p -ideal of X .

Alternative Proof. Let $(x * z) * (y * z) \in \sqrt[k]{A}$ and $y \in \sqrt[k]{A}$. Thus $(x * z) * (y * z) \in A$ and $y \in A$. Therefore, $x \in A$ because A is a p -ideal. Hence $0 * ((x * z) * (y * z))^k \in A$ and $0 * y^k \in A$ and by Lemma 1, we have $(0 * (x * z)^k) * (0 * (y * z)^k) \in A$ and $0 * y^k \in A$. Thus $((0 * x^k) * (0 * z^k)) * ((0 * y^k) * (0 * z^k)) \in A$ and $0 * y^k \in A$. Since A is a p -ideal of a BCI-algebra X , we get $0 * x^k \in A$ and $x \in A$ imply $x \in \sqrt[k]{A}$. Thus $\sqrt[k]{A}$ is a p -ideal of X .

Theorem 6 *Let A be a weakly implicative ideal in a BCI-algebra X . Then $\sqrt[k]{A}$ is weakly implicative, for a positive integer k .*

Proof Let $(x * y) * z \in \sqrt[k]{A}$ and $y * z \in \sqrt[k]{A}$. Then $(x * y) * z \in A$, $y * z \in A$, imply $(x * z) * z \in A$ because A is weakly implicative. Furthermore, $0 * ((x * y) * z)^k \in A$ and $0 * (y * z)^k \in A$. By Lemma 1, we have $(0 * (x * y)^k) * (0 * z^k) \in A$, $(0 * y^k) * (0 * z^k) \in A$, $((0 * x^k) * (0 * y^k)) * (0 * z^k) \in A$ and $(0 * y^k) * (0 * z^k) \in A$. Since A is a weakly implicative ideal, we have $((0 * x^k) * (0 * z^k)) * (0 * z^k) \in A$. By Lemma 1, we have $(0 * (x * z)^k) * (0 * z^k) \in A$ and $0 * ((x * z) * z)^k \in A$. Thus $(x * z) * z \in A$ and so $(x * z) * z \in \sqrt[k]{A}$. Therefore, $\sqrt[k]{A}$ is a weakly implicative ideal of X .

Theorem 7 *Let A be an implicative ideal in a BCI-algebra X . Then $\sqrt[k]{A}$ is an implicative ideal, where k is any positive integer.*

Proof Let $(x * y) * z \in \sqrt[k]{A}$ and $y * z \in \sqrt[k]{A}$, where $(x * y) * z \in A$ and $y * z \in A$. Hence $x * z \in A$. Thus $0 * ((x * y) * z)^k \in A$ and $0 * (y * z)^k \in A$. By Lemma 1, $(0 * (x * y)^k) * (0 * z^k) \in A$ and $(0 * y^k) * (0 * z^k) \in A$. Hence $((0 * x^k) * (0 * y^k)) * (0 * z^k) \in A$ and $(0 * y^k) * (0 * z^k) \in A$. But A is an implicative ideal, $(0 * x^k) * (0 * z^k) \in A$, and $0 * (x * z)^k \in A$. But $x * z \in A$. Hence $x * z \in \sqrt[k]{A}$. Therefore, $\sqrt[k]{A}$ is an implicative ideal.

Theorem 8. *Let A be a strongly implicative ideal in a BCI-algebra X . Then $\sqrt[k]{A}$ is strongly implicative, where k is any positive integer.*

Proof Let $(x * y) * z \in \sqrt[k]{A}$ and $y * z \in \sqrt[k]{A}$. Then $(x * y) * z \in A$ and $y * z \in A$. So $x \in A$ because A is strongly implicative. Hence $0 * ((x * y) * z)^k \in A$ and $0 * (y * z)^k \in A$. Thus $(0 * (x * y)^k) * (0 * z^k) \in A$ and $(0 * y^k) * (0 * z^k) \in A$ by Lemma 1. Hence $((0 * x^k) * (0 * y^k)) * (0 * z^k) \in A$ and $(0 * y^k) * (0 * z^k) \in A$ by Lemma 1. Since A is strongly implicative and $x \in A$, we have $0 * x^k \in A$ and so $x \in \sqrt[k]{A}$. Therefore, $\sqrt[k]{A}$ is strongly implicative.

Lemma 6 [6]. For any BCI-algebra X , we have

$$0 * (0 * (0 * x)) = 0 * x$$

for each x in X .

Lemma 7. [1]. Let I be a closed ideal in a BCI-algebra X . Then the following are equivalent.

- (i) X/I is a p -semisimple BCI-algebra.
- (ii) I contains the p -radical of X ,
- (iii) $x \leq y$ and $x \in I$ implies that $y \in I$,
- (iv) $x * y \in I$ implies that $y * x \in I$,
- (v) $0 * x \in I$ implies that $x \in I$,

for $x, y \in X$.

Theorem 9 Let A be a closed ideal in a BCI-algebra X . Then $X/\sqrt[k]{A}$ is a p -semisimple BCI-algebra, where k is any positive integer.

Proof. We have A is a closed ideal. Then $\sqrt[k]{A}$ is a closed ideal. Suppose $x * y \in \sqrt[k]{A}$. But $\sqrt[k]{A}$ is closed. So $0 * (x * y) \in \sqrt[k]{A}$. Hence $0 * (x * y) \in A$. Thus $0 * (0 * (x * y))^k \in A$, and $0 * (0 * (x * y)^k) \in A$ by Lemma 2. So $0 * ((0 * x^k) * (0 * y^k)) \in A$ by Lemma 1, and $(0 * (0 * x^k)) * (0 * (0 * y^k)) \in A$. Hence $(0 * (0 * (0 * y^k))) * (0 * x^k) \in A$ and $(0 * y^k) * (0 * x^k) \in A$ by Lemma 6. So $0 * (y * x)^k \in A$ by Lemma 1. Hence $y * x \in \sqrt[k]{A}$. Thus $x * y \in \sqrt[k]{A}$ imply $y * x \in \sqrt[k]{A}$. By Lemma 7, we get $X/\sqrt[k]{A}$ is a p -semisimple BCI-algebra.

Lemma 8. For any BCI-algebra X and any positive integer n , we have

$$0 * (x \wedge y)^n = (0 * x^n) \wedge (0 * y^n)$$

for all $x, y \in X$.

Proof Let $x, y \in X$. By using Lemma 1, we obtain

$$\begin{aligned} 0 * (x \wedge y)^n &= 0 * (y * (y * x))^n \\ &= (0 * y^n) * (0 * (y * x)^n) \\ &= (0 * y^n) * ((0 * y^n) * (0 * x^n)) \\ &= (0 * x^n) \wedge (0 * y^n) \end{aligned}$$

Theorem 10 *If I is a commutative ideal of a BCI-algebra X , then $\sqrt[k]{I}$ is a commutative ideal of X , for any positive integer k .*

Proof Let $x \wedge y \in \sqrt[k]{I}$. Then $x \wedge y \in I$. Thus $y \wedge x \in I$, because I is commutative. Then $0*(x \wedge y)^k \in I$. By Lemma 8, we have $(0*x^k) \wedge (0*y^k) \in I$. But I is a commutative ideal, so $(0*y^k) \wedge (0*x^k) \in I$. Again, by Lemma 8, we have $0*(y \wedge x)^k \in I$. Hence $y \wedge x \in I$ and thus $y \wedge x \in \sqrt[k]{I}$. Therefore, $\sqrt[k]{I}$ is commutative.

Theorem 11 *If I and J are ideals of a BCI-algebra X , then*

$$\sqrt[k]{I \cap J} = \sqrt[k]{I} \cap \sqrt[k]{J},$$

for any positive integer k .

Proof Let $x \in \sqrt[k]{I \cap J}$. Then $x \in I \cap J$ and $0*x^k \in I \cap J$. Thus $x \in I$, $x \in J$, $0*x^k \in I$, and $0*x^k \in J$. So $x \in \sqrt[k]{I}$, and $x \in \sqrt[k]{J}$. Hence, $x \in \sqrt[k]{I} \cap \sqrt[k]{J}$. Thus $\sqrt[k]{I \cap J} \subseteq \sqrt[k]{I} \cap \sqrt[k]{J}$. Let $x \in \sqrt[k]{I} \cap \sqrt[k]{J}$. Then $x \in \sqrt[k]{I}$, and $x \in \sqrt[k]{J}$. Therefore, $x \in I$, $0*x^k \in I$, $x \in J$, and $0*x^k \in J$. Thus $x \in I \cap J$ and $0*x^k \in I \cap J$. Hence $x \in \sqrt[k]{I \cap J}$. Thus $\sqrt[k]{I} \cap \sqrt[k]{J} \subseteq \sqrt[k]{I \cap J}$. Therefore, $\sqrt[k]{I \cap J} = \sqrt[k]{I} \cap \sqrt[k]{J}$.

Theorem 12 *If I and J are k^* -semiprime ideals of a BCI-algebra X , then $I \cap J$ is k^* -semiprime ideal, for any positive integer k .*

Proof From Theorem 11, we get $\sqrt[k]{I \cap J} = \sqrt[k]{I} \cap \sqrt[k]{J}$. But $\sqrt[k]{I} = I$ and $\sqrt[k]{J} = J$, because I and J are k^* -semiprime ideals. Thus $\sqrt[k]{I \cap J} = I \cap J$. Therefore, $I \cap J$ is k^* -semiprime ideal.

Theorem 13 *If $f : X \rightarrow Y$ is a homomorphism of BCI-algebras, then for every subalgebra S of Y , $f^{-1}(\sqrt[k]{S})$ is a subalgebra of X containing $\sqrt[k]{f^{-1}(S)}$ for any positive integer k .*

Proof Now, we prove that $\sqrt[k]{f^{-1}(S)} \subseteq f^{-1}(\sqrt[k]{S})$. Let $x \in \sqrt[k]{f^{-1}(S)}$. Then $0*x^k \in f^{-1}(S)$ and $x \in f^{-1}(S)$. Since f is a homomorphism, we have $0*f^k(x) = f(0)*f(x^k) = f(0*x^k) \in f(S)$ and $f(x) \in S$. Therefore, $f(x) \in \sqrt[k]{S}$, and so $x \in f^{-1}(\sqrt[k]{S})$. Hence $\sqrt[k]{f^{-1}(S)} \subseteq f^{-1}(\sqrt[k]{S})$.

If $x, y \in f^{-1}(\sqrt[k]{S})$, then $f(x), f(y) \in \sqrt[k]{S}$. By Theorem 2, $\sqrt[k]{S}$ is a subalgebra and so $f(x)*f(y) \in \sqrt[k]{S}$. Hence $f(x*y) = f(x)*f(y) \in \sqrt[k]{S}$, because f is a homomorphism. Thus $x*y \in f^{-1}(\sqrt[k]{S})$. Therefore, $f^{-1}(\sqrt[k]{S})$ is a subalgebra of X .

Lemma 9 *If $f : X \rightarrow Y$ is a BCI-homomorphism of BCI-algebras and I is an ideal of Y , then $f^{-1}(I)$ is an ideal of X .*

Proof Let I be an ideal of Y . Then $f^{-1}(I)$ is non-empty as I is non-empty. Let $y \in f^{-1}(I)$ and $x * y \in f^{-1}(I)$. Then $f(y) \in I$ and $f(x * y) \in I$. Hence $f(x) * f(y) \in I$, because f is a BCI-homomorphism. But I is an ideal of Y . Hence $f(x) \in I$, and so $x \in f^{-1}(I)$. Therefore, $f^{-1}(I)$ is an ideal of X .

Theorem 14 *If $f : X \rightarrow Y$ is a homomorphism of BCI-algebras, then for every ideal I of Y , $f^{-1}(\sqrt[k]{I})$ is an ideal of X containing $\sqrt[k]{f^{-1}(I)}$ for any positive integer k .*

Proof If I is an ideal of Y , then $\sqrt[k]{I}$ is an ideal of Y by Theorem 3 and $f^{-1}(\sqrt[k]{I})$ is an ideal of X by Lemma 9.

Now, let $x \in \sqrt[k]{f^{-1}(I)}$. Then $x \in f^{-1}(I)$ and $0 * x^k \in f^{-1}(I)$. Thus $f(x) \in I$ and $f(0 * x^k) \in I$. So $f(x) \in I$ and $0 * f^k(x) \in I$, for f is a BCI-homomorphism. Thus $f(x) \in \sqrt[k]{I} \subseteq Y$. Hence $x \in f^{-1}(\sqrt[k]{I})$. Therefore, $\sqrt[k]{f^{-1}(I)} \subseteq f^{-1}(\sqrt[k]{I})$.

Lemma 10 *Let $f : X \rightarrow Y$ be an onto homomorphism of BCI-algebras. If I is an ideal of X such that $\text{Ker } f \subseteq I$, then $f^{-1}(f(I)) = I$.*

Proof Clearly, $I \subseteq f^{-1}(f(I))$. Now, assume that $x \in f^{-1}(f(I))$. Then $f(x) = f(y)$ for some $y \in I$. So $f(x) * f(y) = 0$. But $f(x * y) = f(x) * f(y)$, for f is a homomorphism. Hence $f(x * y) = 0$ and so $x * y \in \text{Ker } f \subseteq I$. Thus $x * y \in I$ and $y \in I$. Hence $x \in I$, for I is an ideal of X . So $f^{-1}(f(I)) \subseteq I$. Therefore, $f^{-1}(f(I)) = I$.

Theorem 15 *If $f : X \rightarrow Y$ is a homomorphism of BCI-algebras, and I is an ideal of Y , then*

$$(i) f(\sqrt[k]{I}) \subset \sqrt[k]{f(I)}.$$

(ii) *If $\text{Ker } f \subseteq I$, and f is an onto homomorphism, then $\sqrt[k]{f(I)} = f(\sqrt[k]{I})$, where k is any positive integer.*

Proof. Let I be an ideal of X .

(i) If $y \in f(\sqrt[k]{I})$, then there exists $x \in \sqrt[k]{I}$ such that $y = f(x)$. Hence $x \in I$ and $0 * x^k \in I$. Thus $y = f(x) \in f(I)$ and $f(0 * x^k) \in f(I)$. Hence $0 * f^k(x) = 0 * f(x^k) = f(0 * x^k) \in f(I)$. So $y \in f(I)$ and $0 * y^k \in f(I)$. Therefore $y \in \sqrt[k]{f(I)}$ and so $f(\sqrt[k]{I}) \subset \sqrt[k]{f(I)}$.

(ii) Let $x \in \sqrt[k]{f(I)}$. Then $x = f(y) \in f(I)$, for $y \in I$ and $0 * x^k \in f(I)$. Thus $0 * f^k(y) \in f(I)$ and so $f(0 * y^k) \in f(I)$, because f is a BCI-homomorphism. Therefore, $0 * y^k \in f^{-1}(f(I))$. But $\text{Ker } f \subseteq I$ and f is onto. Hence $y \in I$ and $0 * y^k \in I$. So $y \in \sqrt[k]{I}$. Thus $x = f(y) \in f(\sqrt[k]{I})$. Hence $\sqrt[k]{f(I)} \subseteq f(\sqrt[k]{I})$. Using (i), we obtain $\sqrt[k]{f(I)} = f(\sqrt[k]{I})$.

Let X and Y be BCI-algebras. Define $*$ on $X \times Y = \{(x, y) \mid x \in X, y \in Y\}$ by $(x * y) * (u, v) = (x * u, y * v)$, for every $(x, y), (u, v) \in X \times Y$. Then $(X \times Y, *, (0, 0))$ is a BCI-algebra.

Lemma 11 [4, Proposition 5]. *Let X and Y be BCI-algebras. For any $(x, y) \in X \times Y$, we have $(0, 0) * (x, y)^k = (0 * x^k, 0 * y^k)$, for any positive integer k .*

Lemma 12 [4, Theorem 8]. *If I and J are ideals of BCI-algebras X and Y respectively, then $I \times J$ is an ideal of $X \times Y$.*

Theorem 16. *Let S and T be non-empty subsets of BCI-algebras X and Y respectively. Then $\sqrt[k]{S} \times \sqrt[k]{T} = \sqrt[k]{S \times T}$, where k is any positive integer.*

Proof. Let S and T be non-empty subsets of BCI-algebras X and Y respectively. Then

$$\begin{aligned} \sqrt[k]{S \times T} &= \{(s, t) \in S \times T \mid (0, 0) * (s, t)^k \in S \times T\} \\ &= \{(s, t) \in S \times T \mid (0 * s^k, 0 * t^k) \in S \times T\} \\ &= \{(s, t) \in S \times T \mid (0 * s^k \in S, 0 * t^k \in T)\} \\ &= \{(s, t) \mid s \in \sqrt[k]{S}, t \in \sqrt[k]{T}\} \\ &= \sqrt[k]{S} \times \sqrt[k]{T} \end{aligned}$$

Theorem 17 *Let I and J be ideals of BCI-algebras X and Y respectively. Then*

$$\frac{X \times Y}{\sqrt[k]{I \times J}} \cong \frac{X}{\sqrt[k]{I}} \times \frac{Y}{\sqrt[k]{J}},$$

for any positive integer k .

Proof If I and J are ideals of X and Y respectively, then $\sqrt[k]{I}$ and $\sqrt[k]{J}$ are ideals of X and Y respectively. By Lemma 12, $\sqrt[k]{I} \times \sqrt[k]{J}$ is an ideal of $X \times Y$. Consider the natural homomorphisms

$$\pi_X : X \rightarrow \frac{X}{\sqrt[k]{I}}, \text{ by } \pi_X(x) = [x], x \in X,$$

$$\pi_Y : Y \rightarrow \frac{Y}{\sqrt[k]{J}}, \text{ by } \pi_Y(y) = [y]', y \in Y.$$

Clearly, $[x * x'] = \pi_X(x * x') = \pi_X(x) * \pi_X(x') = [x] * [x']$, for $x, x' \in X$ and $[y * y']' = \pi_Y(y * y') = \pi_Y(y) * \pi_Y(y') = [y]' * [y']'$, for $y, y' \in Y$. Define

$$f : X \times Y \rightarrow \frac{X}{\sqrt[k]{I}} \times \frac{Y}{\sqrt[k]{J}}$$

by

$$f((x, y)) = (\pi_X(x), \pi_Y(y)) = ([x], [y]'),$$

for every $(x, y) \in X \times Y$.

Let $(x, y), (u, v) \in X \times Y$ such that $(x, y) = (u, v)$. Then $([x], [y]') = ([u], [v]')$ and so $f((x, y)) = f((u, v))$. Therefore, f is a well-defined map. Also,

$$\begin{aligned} f((x, y) * (u, v)) &= f((x * u, y * v)) \\ &= ([x * u], [y * v]') \\ &= ([x] * [u], [y]' * [v]') \\ &= ([x], [y]') * ([u], [v]') \\ &= f((x, y)) * f((u, v)) \end{aligned}$$

Therefore, f is a BCI-homomorphism. Clearly, f is an onto map. By the homomorphism Theorem, we have

$$\frac{X \times Y}{\text{Ker } f} \cong \frac{X}{\sqrt[k]{I}} \times \frac{Y}{\sqrt[k]{J}}.$$

Furthermore,

$$\begin{aligned} \text{Ker } f &= \{(x, y) \in X \times Y \mid f((x, y)) = ([0], [0]')\} \\ &= \{(x, y) \in X \times Y \mid ([x], [y]') = ([0], [0]')\} \\ &= \{(x, y) \in X \times Y \mid [x] = [0], [y]' = [0]'\} \\ &= \{(x, y) \in X \times Y \mid x \in \sqrt[k]{I}, y \in \sqrt[k]{J}\} \\ &= \sqrt[k]{I} \times \sqrt[k]{J} \\ &= \sqrt[k]{I \times J} \end{aligned}$$

Therefore,

$$\frac{X \times Y}{\sqrt[k]{I \times J}} \cong \frac{X}{\sqrt[k]{I}} \times \frac{Y}{\sqrt[k]{J}}.$$

Theorem 18 *An ideal I of a BCI-algebra X is a k^* -semiprime ideal if and only if $\frac{X}{\sqrt[k]{I}}$ has no non-zero nilpotent elements of index k , for any positive integer k .*

Proof. Let I be a k^* -semiprime ideal and let $[a] \in \frac{X}{\sqrt[k]{I}} = \frac{X}{\sqrt[k]{I}}$ be a nilpotent element of index k . Then $[0] * [a]^k = [0]$. Thus $[0 * a^k] = [0]$ and so $0 * a^k \in I$. Therefore, $a \in \sqrt[k]{I}$. But $I = \sqrt[k]{I}$, because I is a k^* -semiprime ideal. Hence $a \in I$ and so $[a] = [0]$. Therefore, any nilpotent element of index k in $\frac{X}{\sqrt[k]{I}}$ is zero.

Conversely, we have $\sqrt[k]{I} \subseteq I$. Let $[a] \in \frac{X}{I}$ be a nilpotent element of index k . Then $[a]^k = [0]$ and hence $a \in I$. Moreover, $[0] * [a]^k = [0]$ and so $0 * a^k = [0]$. Thus $0 * a^k \in I$ and so $a \in \sqrt[k]{I}$. Hence $I \subseteq \sqrt[k]{I}$. Therefore, $I = \sqrt[k]{I}$ and hence I is a k^* -semiprime ideal.

References

- [1] C. S. Hoo, *Closed ideals and p-semisimple BCI-algebras*, Math. Japonica, 35, pp. 1103-1112, (1990).
- [2] W. Huang, *Nil-radical in BCI-algebras*, Math. Japonica, 37, 363-366, (1992).
- [3] Y. B. Jun, *A note on nil ideals in BCI-algebras*, Math. Japonica, 38, 1017-1021, (1993).
- [4] Y. B. Jun, J. Meng and E. H. Roh, *On nil ideals in BCI-algebras*, Math. Japonica, 38, 1051- 1056, (1993).
- [5] Y. B. Jun, and E. H. Roh, *Nil ideals in BCI-algebras*, Math. Japonica, 41, 297- 302, (1995).
- [6] C. Z. Mu and W. H. Xiong, *On ideals in BCI-algebras*, Math. Japonica, 36, 497- 501, (1991).
- [7] L. Ti and Changchang Xi, *p-radical in BCI- algebras*, Math. Japonica, 30, 511-517, (1985).

Received : February 26, 1996.

Hamza A S Abujabal

Department of Mathematics

Faculty of Science

King Abdulaziz University

Saudi Arabia